H

ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

CANDIDATE NAME	

TUTORIAL/ FORM CLASS		INDEX NUMBER				

MATHEMATICS 9758/01

Paper 1 **29 August 2019**

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of __ printed pages.

Anglo-Chinese Junior College **[Turn Over**

1 The points $A(2,-3)$ and $B(-3,1)$ are on a curve with equation $y = \mathrm{f}(x)$. The corresponding points on the curve $y = \mathrm{f}\big(a(x-b)\big)$ are $A'(7,-3)$ and $B'(-1,1)$. Find the values of a and b. [3]

2 Use differentiation to find the area of the largest rectangle with sides parallel to the coordinate axes, lying above the x-axis and below the curve with equation $y = 44 + 4x - x^2$. [5]

3 Solve the equation $\left|\dfrac{x^2+3x}{x-1}\right| = 2x+3$ exactly. [4]

Hence, by sketching appropriate graphs, solve the inequality $\left|\dfrac{x^2+3x}{x-1}\right| < 2x+3$ exactly. [2]

4 A kite 50 m above ground is being blown away from the person holding its string in a direction parallel to the ground at a rate 5 m per second. Assuming that the string is taut, at the instant when the length of the string already let out is 100 m, find, leaving your answers in exact form,

(i) the rate of change of the angle between the string and the ground, [3]

(ii) the rate at which the string of the kite should be let out, [4]

5 Given that $y = \tan\left(1 - \mathrm{e}^{3x}\right)$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = k\mathrm{e}^{3x}\left(1 + y^2\right)$, where k is a constant to be determined. By further differentiation of this result, or otherwise, find the first three non-zero terms in the Maclaurin series for $\tan\left(1 - \mathrm{e}^{3x}\right)$. [5]

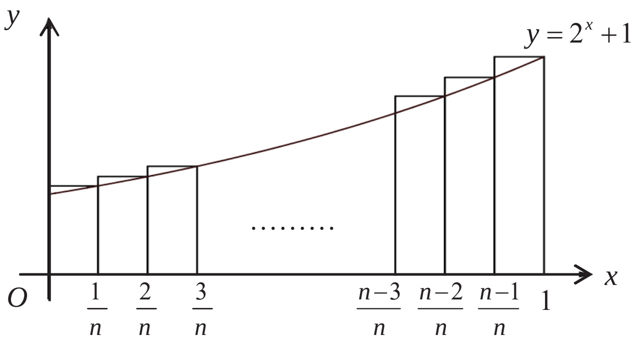
The first two terms in the Maclaurin series for $\tan\left(1 - \mathrm{e}^{3x}\right)$ are equal to the first two non-zero terms in the series expansion of $\dfrac{x}{a+bx}$. Find the constants a and b. [3]

6 The diagram below shows the graph of $y = 2^x + 1$ for $0 \le x \le 1$. Rectangles, each of width $\dfrac{1}{n}$, are also drawn on the graph as shown.

Show that the total area of all n rectangles, S_n, is given by

$$S_n = \frac{2^{\frac{1}{n}}}{n(2^{\frac{1}{n}} - 1)} + 1 .$$ [3]

Find the exact value of $\lim\limits_{n\to\infty} S_n$. [2]



7 **(a)** Find $\int \sin px \cos qx \, dx$ where p and q are positive integers such that $p \neq q$. [2]

(b) Show that $\int x \sin nx \, dx = -\dfrac{x \cos nx}{n} + \dfrac{\sin nx}{n^2} + c$ where n is a positive integer and c is an arbitrary constant. [1]

Hence find

(i) $\int_0^{\pi} x \sin nx \, dx$, giving your answers in the form $\dfrac{k\pi}{n}$ where the possible values of k are to be determined, [2]

(ii) $\int_0^{\frac{\pi}{2}} |x \sin 3x| \, dx$ in terms of π. [3]

8 **Do not use a calculator in answering this question.**

(a) The complex numbers z and w satisfy the following equations

$$w-2z=9,$$
$$3w-wz^{*}=17-30\text{i}.$$

Find w and z in the form $a+b\text{i}$, where a and b are real and $\text{Re}(z)<0$. [4]

(b) **(i)** Given that $-\text{i}$ is a root of the equation

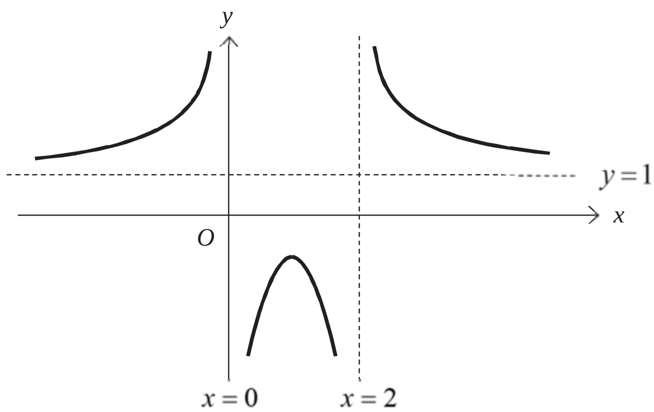
$$z^3+kz^2+\left(8+2\sqrt{2}\,\text{i}\right)z+8\,\text{i}=0,$$

where k is a constant to be determined, find the other roots, leaving your answers in exact cartesian form $x+y\text{i}$, showing your working. [3]

(ii) Hence solve the equation $\text{i}z^3+kz^2+\left(2\sqrt{2}-8\,\text{i}\right)z-8\,\text{i}=0$, leaving your answers in exact cartesian form. [2]

(iii) Let z_0 be the root in **(i)** such that $\arg(z_0)>0$. Find the smallest positive integer value of n such that $(\text{i}z_0)^n$ is a purely imaginary number. [2]

9 **(a)** The diagram below shows the graph of $y=\dfrac{1}{\text{f}(x)}$ with asymptotes $x=0$, $x=2$, and $y=1$, and turning point $(1,-2)$.



(i) Given that $\text{f}(0)=\text{f}(2)=0$, sketch the graph of $y=\text{f}(x)$, stating clearly the coordinates of any turning points and points of intersection with the axes, and the equations of any asymptotes. [3]

(ii) The function f is now defined for $x>k$ such that f^{-1} exists.
State the smallest value of k. On the same diagram, sketch the graphs of $y=\text{f}(x)$ and $y=\text{f}^{-1}(x)$, showing clearly the geometrical relationship between the two graphs. [3]

(b) The function g is defined for $x > 0$ as

$$\mathrm{g}: x \mapsto 2^n x - 1, \quad \frac{1}{2^n} \le x < \frac{1}{2^{n-1}}, \text{ where } n \in \square .$$

(i) Fill in the blanks.

$$\mathrm{g}(x) = \begin{cases} \boxed{}, & \frac{1}{4} \le x < \frac{1}{2} \\ \boxed{}, & \frac{1}{2} \le x < 1 \end{cases}$$

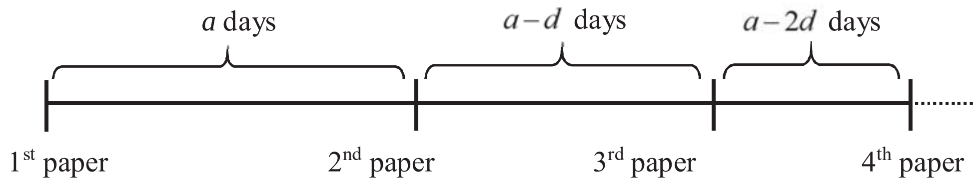
Hence sketch the graph $y = \mathrm{g}(x)$ for $\frac{1}{4} \le x < 1$. [3]

(ii) Show that $\mathrm{g}(x) = \mathrm{g}\left(\frac{x}{2}\right)$. [2]

(iii) Find the number of solutions of $\mathrm{g}(x) = x$ for $0.001 < x < 1$. [2]

10 David is preparing for an upcoming examination with 9 practice papers to complete in 90 days. The examination is on the 91st day. He is planning to spread out the practice papers according to the following criteria, and illustrated in the diagram below.

- He only completes 1 practice paper a day.
- He attempts the first practice paper on the first day.
- The duration between the first and the second practice paper is a days.
- The duration between each subsequent paper decreases by d days.
- He completes the last practice paper as close to the examination date as possible.



(i) By first writing down two inequalities in terms of a and d, determine the values of a and d. [4]

The mark for his n-th practice paper, u_n, can be modelled by the formula

$$u_n = 92 - 65(b)^n \text{ where } 0 < b < 1.$$

(ii) What is the significance of the number 92 in the formula? [1]

(iii) Find m, his average mark, for the nine practice papers he completed, leaving your answer in terms of b. [3]

(iv) Given that he scored higher than m from his fourth practice paper onwards, find the range of values of b. [2]

11 A toy paratrooper is dropped from a building and the attached parachute opens the moment it is released. The toy drops vertically and the distance it drops after t seconds is x metres. The motion of the toy can be modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+k\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2=10,$$

where k is a constant.

By substituting velocity, $v=\frac{\mathrm{d}x}{\mathrm{d}t}$, write down a differential equation in v and t. [1]

Given that $\frac{\mathrm{d}v}{\mathrm{d}t}=6$ when $v=\sqrt{10}$, and that the initial velocity of the toy is zero, show that

$$v=\frac{5(1-\mathrm{e}^{-4t})}{1+\mathrm{e}^{-4t}},$$

and deduce the velocity of the toy in the long run. [6]

The toy is released from a height of 10 metres. Find the time it takes for the toy to reach the ground. [5]

12 In air traffic control, coordinates (x, y, z) are used to pinpoint the location of an aircraft in the sky within certain air space boundaries. In a particular airfield, the base of the control tower is at $(0,0,0)$ on the ground, which is the x-y plane. Assuming that the aircrafts fly in straight lines, two aircrafts, F_1 and F_2, fly along paths with equations

$$\mathbf{r}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-4\mathbf{j}+\mathbf{k}) \text{ and } x+2=\frac{y-1}{m}=\frac{3-z}{7}$$

respectively.

(i) What can be said about the value of m if the paths of the two aircrafts do not intersect? [3]

(ii) The signal detecting the aircrafts is the strongest when an aircraft is closest to the controller, who is in the control tower 3 units above the base. Find the distance of F_1 to the controller when the signal detecting it is the strongest. [3]

In a choreographed flying formation, the aircraft F_3 takes off from the point $(1,1,0)$ and flies in the direction parallel to $\mathbf{i}-\mathbf{k}$. The path taken by another aircraft, F_4, is the reflection of the path taken by F_2 along the path taken by F_3.

For the case when $m=5$, find

(iii) the cartesian equation of the plane containing all three flight paths. [2]

(iv) the vector equation of the line that describes the path taken by F_4. [4]

2019 ACJC H2 Math Prelim P1 Marker's Report

<table>
<tr><th>Qn</th><th>Solutions</th><th>Comments</th></tr>
<tr><td>1</td><td>

$y=\mathrm{f}(x)\xrightarrow[\text{by factor }\frac{1}{a}]{\text{scaling // }x\text{-axis}}y=\mathrm{f}(ax)$

$\xrightarrow[\text{the positive }x\text{ direction}]{\text{translate }b\text{ units in}}y=\mathrm{f}(a(x-b))$

$A:(2,-3)\to\left(\frac{2}{a},-3\right)\to\left(\frac{2}{a}+b,-3\right)=(7,-3)$

$B:(-3,1)\to\left(-\frac{3}{a},1\right)\to\left(-\frac{3}{a}+b,1\right)=(-1,1)$

$\left.\begin{aligned}\frac{2}{a}+b&=7\\-\frac{3}{a}+b&=-1\end{aligned}\right\}$ solving gives $a=\frac{5}{8},\ b=\frac{19}{5}$

</td><td>

Badly done:
Common errors:
(1) f(a(x – b)) is taken as translate b units in the positive x-direction then scale // x-axis by factor 1/a.
Thus (2+b)/a = 7 and (-3 + b)/a = -1 were commonly seen.
(2) Equations
a(2-b) = 7 and
a(-3-b) = -1 commonly seen.
The correct equations should be a(7-b) = 2 and a(-1-b) = -3

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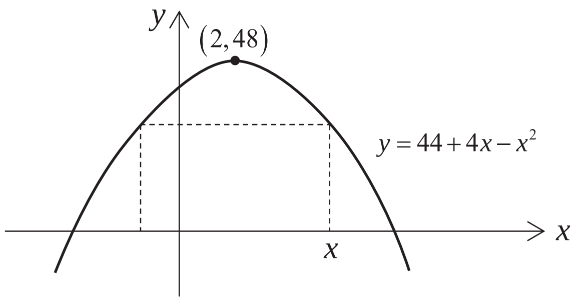
<tr><td>2</td><td>



Area of rectangle, $A=2(x-2)y$

$=2(x-2)(44+4x-x^2)$

$\frac{\mathrm{d}A}{\mathrm{d}x}=0\ \Rightarrow\ 2(x-2)(4-2x)+(44+4x-x^2)(2)=0$

i.e. $-6x^2+24x+72=0$

i.e. $x^2-4x-12=0$

Hence $x=-2$ or 6.

Check that $\left.\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\right|_{x=6}=-48<0$,

therefore A is maximum when $x=6$.

Maximum $A=2(6-2)(44+24-36)=256$ sq. units.

</td><td>

Badly done
(1) Many assume that the area of the rectangle is xy or 2xy.
(2) Some even differentiate y = 44 + 4x – x^2 to find the maximum value of y = 48
Thus area = 48 x 2 = 96
(3) Quite a number of candidates did not check that the area is maximum by checking the 2nd derivative

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<tr><td>3</td><td>

$\left|\frac{x^2+3x}{x-1}\right|=2x+3$

Note that for the equation to have any solution,

$2x+3\geq 0\Rightarrow x\geq-\frac{3}{2}$.

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Squaring both sides would lead to tedious working unless students were able to apply a^2-b^2.

Another tedious method was to consider 4 different regions according to $x=-3$, 0, 1.

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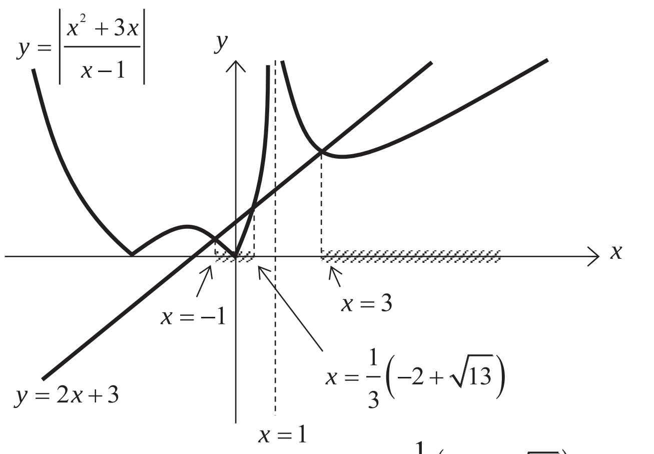
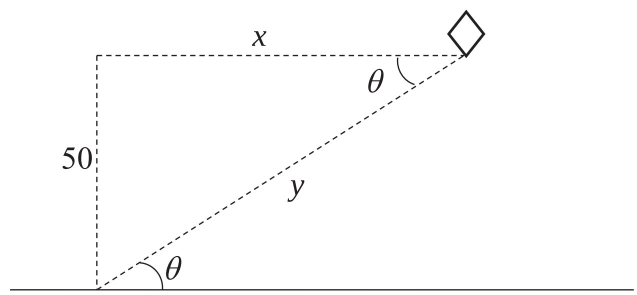
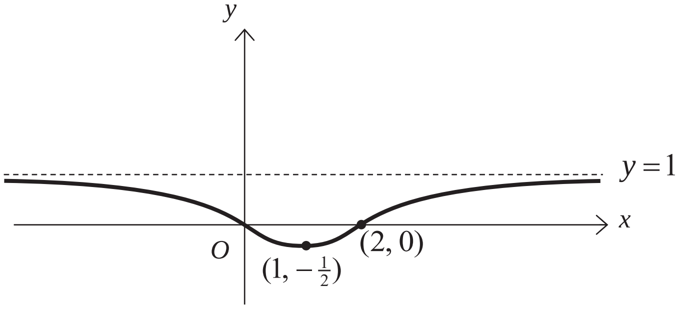
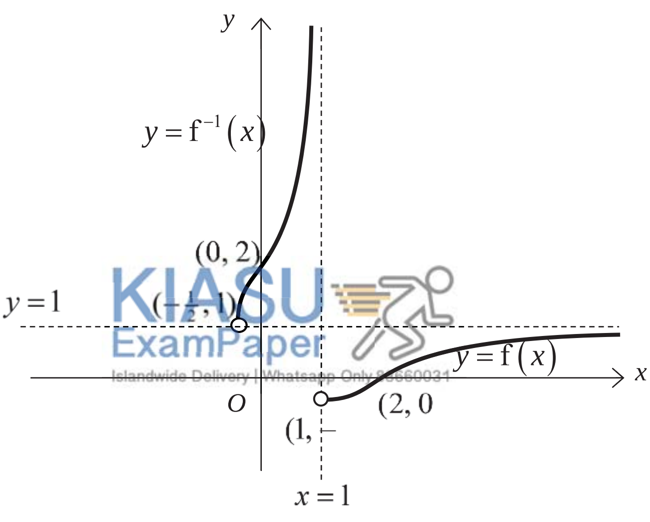
	$\dfrac{x^2+3x}{x-1}=2x+3$ $x^2+3x=(2x+3)(x-1)$ $x^2-2x-3=0$ $(x-3)(x+1)=0$ $\therefore x=3,-1$ $\dfrac{x^2+3x}{x-1}=-(2x+3)$ $x^2+3x=-(2x+3)(x-1)$ $3x^2+4x-3=0$ $x=\dfrac{-4\pm\sqrt{16-4(3)(-3)}}{2(3)}$ $=\dfrac{1}{3}\left(-2\pm\sqrt{13}\right)$ Reject $\dfrac{1}{3}\left(-2-\sqrt{13}\right)$(since $\approx -1.87<-1.5$) $\therefore x=-1, \dfrac{1}{3}\left(-2+\sqrt{13}\right)$ or 3.	Final answers need to be simplified: $\dfrac{\sqrt{52}}{6}=\dfrac{\sqrt{13}}{3}$ & $\dfrac{4}{6}=\dfrac{2}{3}$. Many students were not aware of the need to reject $\dfrac{1}{3}\left(-2-\sqrt{13}\right)$. Of those who rejected this negative root, few were able to provide reason that $x\geq-\dfrac{3}{2}$.
	 From graph, solution is $-1<x<\dfrac{1}{3}\left(-2+\sqrt{13}\right)$ or $x>3$	"Hence, **by sketching appropriate graph<u>S</u>**, solve…" was not followed: • $y=\left\|\dfrac{x^2+3x}{x-1}\right\|-2x-3$ was drawn instead. • Sign test was seen instead. • No graphs were seen in some scripts. Missing asymptote of $x=1$ resulted in a pointed graph as shown in the GC.
4	 Given that $\dfrac{dx}{dt}=5$.	**Very badly done:** - wrong understanding of question. The rate of change given in the question is not the length of the string but the horizontal distance of the kite and the person. - many assume that the length of the string is constant at 100 and resulted in $\cos\theta=\dfrac{x}{100}$ and similar expressions.
(i)	From the diagram, $\tan\theta=\dfrac{50}{x}$. Hence, $x=\dfrac{50}{\tan\theta}=50\cot\theta$. Differentiating with respect to θ,	- There were many who did average rate of change, rather than instantaneous rate of change - when doing differentiation or integration, the angle is always in radians, a significant number left the answer in °/sec.

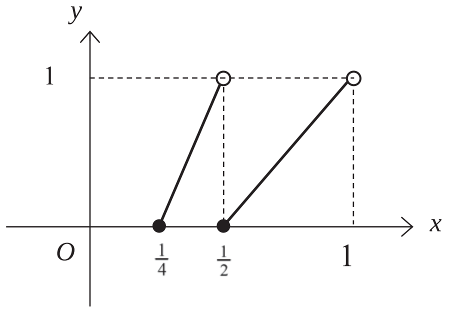
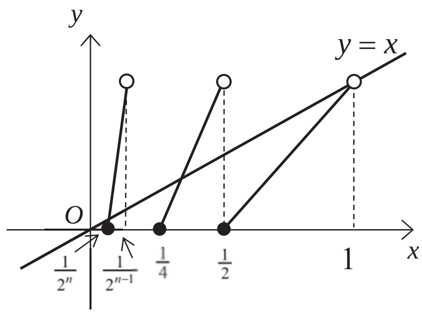
	$\frac{dx}{d\theta} = -50\,\text{cosec}^2\theta$. When $y = 100$, $\sin\theta = \frac{1}{2}$ and $\frac{dx}{dt} = 5$, hence $\frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} \quad \Rightarrow \quad 5 = \left(-50\,\text{cosec}^2\theta\right)\cdot\frac{d\theta}{dt}$ $\Rightarrow \quad 5 = \left(-50\cdot(2)^2\right)\cdot\frac{d\theta}{dt}$ $\Rightarrow \quad \frac{d\theta}{dt} = \frac{5}{-50(4)} = -\frac{1}{40}$ rad s^{-1}. ALTERNATIVELY, $\tan\theta = \frac{50}{x}$ At the instant when $y = 100$, $x = \sqrt{100^2 - 50^2} = \sqrt{7500} = 50\sqrt{3}$, and $\theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$. Differentiating (*) with respect to *t*, $\sec^2\theta\frac{d\theta}{dt} = -\frac{50}{x^2}\frac{dx}{dt}$. When $x = 50\sqrt{3}$, $\theta = \frac{\pi}{6}$ and $\frac{dx}{dt} = 5$, $\sec^2\left(\frac{\pi}{6}\right)\cdot\frac{d\theta}{dt} = -\frac{50}{\left(50\sqrt{3}\right)^2}\cdot(5)$ $\Rightarrow \quad \frac{d\theta}{dt} = -\frac{250}{(7500)\left(\frac{2}{\sqrt{3}}\right)^2} = -\frac{1}{40}$ rad s^{-1}.	- there are some who used $\theta = \cos^{-1}\frac{x}{\sqrt{x^2+50^2}}$ or $\theta = \sin^{-1}\frac{50}{\sqrt{x^2+50^2}}$ which will result in very tedious differentiation. - many did not use chain rule when differentiating $\theta = \tan^{-1}\frac{50}{x}$ and similar expressions, resulting in $\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{50}{x}\right)^2}$ which is wrong - many students still do not know how to differentiate sec, cot, cosec directly, resulting in unnecessary working
(ii)	Want to find $\frac{dy}{dt}$ when $y = 100$. Now $x^2 + 50^2 = y^2 \quad \Rightarrow \quad y = \sqrt{x^2+50^2}$ $\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(x^2+50^2\right)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+50^2}}$ By chain rule, $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$. Hence when $y = 100$, $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$ $\frac{dy}{dt} = \left(\frac{x}{\sqrt{x^2+50^2}}\right)\cdot 5 = \left(\frac{50\sqrt{3}}{\sqrt{\left(50\sqrt{3}\right)^2+50^2}}\right)\cdot 5 = \frac{5\sqrt{3}}{2}$ cm s^{-1}.	This part was left empty in most scripts. For those who did this, many did not simplify the final answer.

5	$y=\tan\left(1-e^{3x}\right) \Rightarrow \tan^{-1} y=1-e^{3x}$. Differentiating with respect to x, $\frac{1}{1+y^2}\frac{dy}{dx}=-3e^{3x}$ $\frac{dy}{dx}=-3e^{3x}\left(1+y^2\right)$. Hence $k=-3$. Differentiating again with respect to x, $\frac{d^2y}{dx^2}=-3\left(2y\frac{dy}{dx}\right)e^{3x}-3\left(1+y^2\right)\left(3e^{3x}\right)$ $=-6y\frac{dy}{dx}e^{3x}-9e^{3x}\left(1+y^2\right)$ $=-3e^{3x}\left(2y\frac{dy}{dx}+3+3y^2\right)$ $\frac{d^3y}{dx^3}=-3e^{3x}\left(2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+6y\frac{dy}{dx}\right)-9e^{3x}\left(2y\frac{dy}{dx}+3+3y^2\right)$ When $x=0$, $y=0$, $\frac{dy}{dx}=-3$, $\frac{d^2y}{dx^2}=-9$ and $\frac{d^3y}{dx^3}=-81$. Hence, $y=\tan\left(1-e^{3x}\right)$ $=0-3x+\frac{(-9)}{2!}x^2+\frac{(-81)}{3!}x^3+\dots$ $\approx -3x-\frac{9}{2}x^2-\frac{27}{2}x^3$.	Most approach it this way: differentiate $\tan\left(1-e^{3x}\right)$ to get $-3e^{3x}\sec^2\left(1-e^{3x}\right)$ and use trigo identity to show k. A lot of complete and accurate work, as many of them make careless mistakes/slips: • $\frac{d}{dx}\left(1+y^2\right)=1+2y\frac{dy}{dx}$ • $\frac{d}{dx}\left(y\frac{dy}{dx}\right)=y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)$ And some did not use product rule: • $\frac{d}{dx}\left(y\frac{dy}{dx}\right)=\frac{dy}{dx}\frac{d^2y}{dx^2}$ Clearly taught to avoid quotient rule for implicit differentiation but some students still proceed in that direction.
	$\frac{x}{a+bx}=x(a+bx)^{-1}$ $=\frac{x}{a}\left(1+\frac{bx}{a}\right)^{-1}$ $=\frac{x}{a}\left(1-\frac{bx}{a}+\left(\frac{bx}{a}\right)^2+\dots\right)$ $\approx\frac{x}{a}-\frac{bx^2}{a^2}$ Hence $-3x-\frac{9}{2}x^2=\frac{x}{a}-\frac{bx^2}{a^2}$. Comparing coefficients, x: $-3=\frac{1}{a} \Rightarrow a=-\frac{1}{3}$ x^2: $-\frac{9}{2}=-\frac{b}{a^2} \Rightarrow b=\frac{1}{2}$	Instead of using binomial series to expand, more students differentiated twice, use Maclaurins again, compared coefficient of x with $f'(0)$, but many made mistake in comparing the coefficient of x^2 where they equated $f''(0)=-\frac{9}{2}$.

6	Total area of n rectangles $=\frac{1}{n}\left[(2^{\frac{1}{n}}+1)+(2^{\frac{2}{n}}+1)+...(2^{\frac{n-1}{n}}+1)+(2^{1}+1)\right]$ $=\frac{1}{n}\left[2^{\frac{1}{n}}+2^{\frac{2}{n}}+...2^{\frac{n-1}{n}}+2^{1}\right]+\frac{1}{n}[1+1+...1+1]$ $=\frac{1}{n}\sum_{r=1}^{n}2^{\frac{r}{n}}+1$ $=\frac{1}{n}\left(\frac{2^{\frac{1}{n}}(1-(2^{\frac{1}{n}})^{n})}{1-2^{\frac{1}{n}}}\right)+1=\frac{2^{\frac{1}{n}}(-1)}{n(1-2^{\frac{1}{n}})}+1=\frac{2^{\frac{1}{n}}}{n(2^{\frac{1}{n}}-1)}+1$	Most students could identify the area of the r[th] rectangle having width $\frac{1}{n}$ and length $(2^{\frac{r}{n}}+1)$. Many fail to recognize that they should **split** the following sum: $\frac{1}{n}\sum_{r=1}^{n}(2^{\frac{r}{n}}+1)$ and proceed with S_{GP} and sum of constant. As this is an answer given question, students have to show the **full details** when applying the GP formula.
	$\lim_{n\to\infty}S_n=\int_0^1 2^x+1\,\mathrm{d}x=\left[\frac{2^x}{\ln 2}+x\right]_0^1=\left(\frac{2^1}{\ln 2}+1\right)-\frac{1}{\ln 2}=\frac{1}{\ln 2}+1$	Most students aren't aware that the limit of the sum of the area of n rectangles is the area under curve. Many students concluded that $n(2^{\frac{1}{n}}-1)$ tends to zero when the table in the GC indicates that $n(2^{\frac{1}{n}}-1)$ tends to 0.69339. But since the question asked for exact answers, students can't use this method.
7(a)	$\int\sin px\cos qx\,\mathrm{d}x=\frac{1}{2}\int\sin(p+q)x+\sin(p-q)x\,\mathrm{d}x$ $=-\frac{\cos(p+q)x}{2(p+q)}-\frac{\cos(p-q)x}{2(p-q)}+c$	Half of the cohort not aware the need to use factor formula. They attempted to solve by parts.
(b)	$\int x\sin nx\,\mathrm{d}x=x(-\frac{\cos nx}{n})-\int-\frac{\cos nx}{n}\,\mathrm{d}x$ $=-\frac{x\cos nx}{n}+\int\frac{\cos nx}{n}\,\mathrm{d}x=-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}+c$	Most able to use by parts correctly.
(b)(i)	$\int_0^{\pi}x\sin nx\,\mathrm{d}x=\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi}=-\frac{\pi\cos n\pi}{n}=\pm\frac{\pi}{n}$ $\therefore k=\pm 1$	Many have no idea what to do with $\cos n\pi$.

(b)(ii)	$\int_0^{\frac{\pi}{2}} \lvert x\sin 3x\rvert \, dx = \int_0^{\frac{\pi}{3}} x\sin 3x \, dx - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} x\sin 3x \, dx$ $= \left[-\frac{x\cos 3x}{3}+\frac{\sin 3x}{9}\right]_0^{\frac{\pi}{3}} - \left[-\frac{x\cos 3x}{3}+\frac{\sin 3x}{9}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$ $= -\frac{\pi\cos\pi}{9} - \left(-\frac{\pi\cos\frac{3\pi}{2}}{6}+\frac{\sin\frac{3\pi}{2}}{9}\right)+\left(-\frac{\pi\cos\pi}{9}\right)$ $= \frac{2\pi+1}{9}$	Some students used 1 instead of $\frac{\pi}{3}$. They gotten the number from GC.
8(a)	From $w-2z=9$, $w=9+2z$. Substitute into $3w-wz^*=17-30\mathrm{i}$: $3(9+2z)-(9+2z)z^*=17-30\mathrm{i}$ $\Rightarrow \quad 27+6z-9z^*-2zz^*=17-30\mathrm{i}$ Let $z=a+b\mathrm{i}$, then $27+6(a+b\mathrm{i})-9(a-b\mathrm{i})-2(a+b\mathrm{i})(a-b\mathrm{i})=17-30\mathrm{i}$ i.e. $27+6a-9a-2a^2-2b^2+6b\mathrm{i}+9b\mathrm{i}=17-30\mathrm{i}$ i.e. $27-3a-2a^2-2b^2+15b\mathrm{i}=17-30\mathrm{i}$. Comparing coefficients, Imaginary: $15b=-30 \quad \Rightarrow \quad b=-2$ Real: $27-3a-2a^2-2b^2=17 \quad \Rightarrow \quad 2-3a-2a^2=0$ solving for a, $a=\frac{1}{2}$ or -2. Since $\mathrm{Re}(z)<0$, $a=-2$. Therefore, $z=-2-2\mathrm{i}$, and $w=9+2(-2-2\mathrm{i})=5-4\mathrm{i}$.	It is good practice to work to eliminate one variable when solving simultaneous equations before substituting $z=a+b\mathrm{i}$. Many students who started by using $z=a+b\mathrm{i}$ and $w=c+d\mathrm{i}$ made careless mistakes in their computation and were not successful in arriving at the correct answer.
8(b)(i)	$-\mathrm{i}$ is a root of the equation $z^3+kz^2+\left(8+2\sqrt{2}\,\mathrm{i}\right)z+8\,\mathrm{i}=0$ hence $(-\mathrm{i})^3+k(-\mathrm{i})^2+\left(8+2\sqrt{2}\,\mathrm{i}\right)(-\mathrm{i})+8\,\mathrm{i}=0$ $\mathrm{i}-k-8\mathrm{i}+2\sqrt{2}+8\,\mathrm{i}=0$ $\mathrm{i}-k+2\sqrt{2}=0$ $\therefore \quad k=2\sqrt{2}+\mathrm{i}$. $z^3+\left(2\sqrt{2}+\mathrm{i}\right)z^2+\left(8+2\sqrt{2}\,\mathrm{i}\right)z+8\,\mathrm{i}=0$ $\Rightarrow \quad (z+\mathrm{i})\left(z^2+bz+8\right)=0$ Comparing coefficients of z, $8+2\sqrt{2}\,\mathrm{i}=8+b\mathrm{i}$. Hence $2\sqrt{2}=b$. $(z+\mathrm{i})\left(z^2+2\sqrt{2}\,z+8\right)=0$ $\therefore \quad z=-\mathrm{i}$ or $z=\frac{-2\sqrt{2}\pm\sqrt{8-32}}{2} = -\sqrt{2}\pm\sqrt{6}\,\mathrm{i}$. The other roots are $-\sqrt{2}+\sqrt{6}\,\mathrm{i}$ and $-\sqrt{2}-\sqrt{6}\,\mathrm{i}$.	The given equation is NOT a real polynomial as the coefficients are not all real numbers. Hence conjugate of $-\mathrm{i}$ is NOT a root. k is a constant does not mean it is a real number. In this case it is a complex constant. So when $\mathrm{i}-k+2\sqrt{2}=0$ $\Rightarrow \quad k=2\sqrt{2}+\mathrm{i}$ Instead, some students **wrongly** proceeded to equate real and imaginary parts.

(b)(ii)	$\mathrm{i}z^3+kz^2+\left(2\sqrt{2}-8\,\mathrm{i}\right)z-8\,\mathrm{i}=0$ $\Rightarrow\quad -(\mathrm{i}z)^3-k(\mathrm{i}z)^2-\left(8+2\sqrt{2}\,\mathrm{i}\right)(\mathrm{i}z)-8\,\mathrm{i}=0$ i.e. $\quad (\mathrm{i}z)^3+k(\mathrm{i}z)^2+\left(8+2\sqrt{2}\,\mathrm{i}\right)(\mathrm{i}z)+8\,\mathrm{i}=0$. Hence from **(i),** $\mathrm{i}z=-\mathrm{i},\quad -\sqrt{2}+\sqrt{6}\,\mathrm{i},\quad$ or $\quad -\sqrt{2}-\sqrt{6}\,\mathrm{i}$ $\therefore\quad z=-1,\quad -\sqrt{6}+\sqrt{2}\,\mathrm{i},\quad$ or $\quad \sqrt{6}+\sqrt{2}\,\mathrm{i}$.	Not well done. A substitution is needed here.
(b)(iii)	$z_0=-\sqrt{2}+\sqrt{6}\,\mathrm{i}$ $\arg z_0=\dfrac{2\pi}{3}$ For $(\mathrm{i}z_0)^n$ to be purely imaginary, $\arg(\mathrm{i}z_0)^n=\pm\dfrac{\pi}{2}\quad\Rightarrow\quad n\arg(\mathrm{i}z_0)=\pm\dfrac{k\pi}{2}$ where k is odd i.e. $\quad n\left[\arg\mathrm{i}+\arg z_0\right]=\pm\dfrac{k\pi}{2}$ i.e. $\quad n\left[\dfrac{\pi}{2}+\dfrac{2\pi}{3}\right]=\pm\dfrac{k\pi}{2}$ i.e. $\quad n\left[-\dfrac{5\pi}{6}\right]=\pm\dfrac{k\pi}{2}$ Hence smallest positive integer value of n is 3.	Not well done. Some errors in the method to find argument of a complex number.
9(ai)		Generally well done, except some students who totally do not know how to sketch reciprocal graph. Common errors are - both tails tend to infinity - left tail tends to infinity - wrong y-value for minimum point
(a)(ii)	$k=1$ 	Some did not get the mark $k=1$ even though their (i) is correct. Weird! Common mistakes for $y=\mathrm{f}^{-1}(x)$ graph - draw $y=\mathrm{f}'(x)$ instead - the left end points are on the respective asymptotes - missing labelling of points, especially the y-intercept - the point $(1,-\frac{1}{2})$ becomes $(-1,\frac{1}{2})$ - missing vertical asymptote

(b)(i)	$g(x)=\begin{cases}\boxed{4x-1}, & \frac{1}{4}\le x<\frac{1}{2}\\ \boxed{2x-1}, & \frac{1}{2}\le x<\frac{1}{4}\end{cases}$ 	Most students get the 2 marks if they attempt it Common mistakes in graph - the *y*-value of both end points are at 1, so they should reach the same height - scale on the x-axis, there were a significant number who drew the same width for both pieces - swap the rule for the domains
(b)(ii)	$g(x)=2^n x-1,\ \frac{1}{2^n}\le x<\frac{1}{2^{n-1}}$ $g\left(\frac{x}{2}\right)=2^n\left(\frac{x}{2}\right)-1,\ \frac{1}{2^n}\le\frac{x}{2}<\frac{1}{2^{n-1}}$ $=2^{n-1}x-1,\frac{1}{2^{n-1}}\le x<\frac{1}{2^{n-2}}$ $=2^k x-1,\frac{1}{2^k}\le x<\frac{1}{2^{k-1}}$ (optional) $=g(x)$	About half did not attempt this. Most who attempted it got partial marks. Partial marks were given if - Obtain the rule for the general case in simplified form - obtain the rule for the case in **(i)** with the correct domain
(b)(iii)	Consider $\frac{1}{2^n}<0.001<\frac{1}{2^{n-1}}$ $n>\frac{\ln 0.001}{\ln\left(\frac{1}{2}\right)}=9.97$ $\therefore \frac{1}{2^{10}}<0.001<\frac{1}{2^9}$  When $n=10$, $g(x)=2^{10}x-1,\ \frac{1}{2^{10}}\le x<\frac{1}{2^9}$. Solving $g(x)=x$ when $n=10$, $2^{10}x-1=x \Rightarrow x=0.000978<0.001$. Hence there is no solution when $n=10$. There are therefore **8** solutions (since *n* =1 also has has no solution).	Most students assume that the solution is in the region of the graph drawn in **b(i)**. Few who attempted this got the correct answer, which is fine.
10(i)	As there are 9 papers, there are 8 durations in between the papers.	Very few students manage to write down **both** inequalities correctly. Many students wrote $S_8\le 90$ as they may have miss out on the fact that the 1st practice paper is

	$S_8 = \frac{8}{2}\left(2a + (8-1)(-d)\right) < 90$ $\Rightarrow 8a - 28d < 90$ $\Rightarrow a < 11.25 + 3.5d$ $T_8 = a + (8-1)(-d) > 0$ $\Rightarrow a > 7d$ For the last paper to be as close to the exam date as possible, a and S_8 must be as large as possible, By trial and error, $d = 1, a < 14.75 \Rightarrow a = 14(> 7(1))$ and $S_8 = 84$ $d = 2, a < 18.25 \Rightarrow a = 18(> 7(2))$ and $S_8 = 88$ $d = 3, a < 21.75 \Rightarrow a = 21(\not> 7(3))$ Therefore $d = 2, a = 18$.	already attempted on the 1st day. By writing $S_8 \le 90$, this implies that the 9th practice paper could be on the 91st day (examination day) which is not what the question wants. Quite a number of student also wrote $T_8 \ge 0$. This is incorrect as $T_8 \ge 0$ means that it is possible for the 8th and 9th practice paper to be done on the same day which is also not what the question wants. Even fewer students realised of the need to determine the values of a and d by trial and error. Quite a number of students attempt to "solve" the 2 inequalities by treating them as "equations" and attempting to "solve" them "simultaneously" which is incorrect.
10(ii)	92 is the highest (theoretical) mark that he will get even if he practise many many times. OR 92 is the highest (theoretical) mark that he will get based on his aptitude and ability.	This part was quite well attempted as students generally know the significance of the number 92. However, their answer can be improved on by being clearer in stating the reason why 92 is the highest mark David will get.
10(iii)	$m = \frac{1}{9}\sum_{n=1}^{9} u_n = \frac{1}{9}\sum_{n=1}^{9}\left(92 - 65(b^n)\right)$ $= \frac{1}{9}\left(92(9) - 65\sum_{n=1}^{9} b^n\right)$ $= 92 - \frac{65}{9}\left(\frac{b(1-b^9)}{1-b}\right)$	Many students assume wrongly that $\sum_{n=1}^{9} u_n$ is an AP and took the first term as $\left(92 - 65(b)\right)$ and the last term as $\left(92 - 65(b^9)\right)$ which are incorrect. Quite a number of student could not recall the S_n of GP correctly.
10(iv)	As he scored higher than m from his 4th paper onwards,	Many students could write the inequality but could not solve as they did not realise that GC can be used to help them solve.

	$92-\frac{65}{9}\left(\frac{b(1-b^9)}{1-b}\right)<92-65(b^4)$ $\Rightarrow 1-b^9>9b^3(1-b)$ $\Rightarrow b^9-9b^4+9b^3-1<0$ From GC, $0<b<0.726$	
11	$\frac{d^2x}{dt^2}+k\left(\frac{dx}{dt}\right)^2=10$ Substitute $v=\frac{dx}{dt}$ and $\frac{dv}{dt}=\frac{d^2x}{dt^2}$ into DE, $\therefore \frac{dv}{dt}+kv^2=10$	Majority are able to get the differential equation. A handful of students left blank. Many students misinterpreted the question as $v=\frac{x}{t}$.
	when $v=\sqrt{10}$, $\frac{dv}{dt}=6$ $6+k(\sqrt{10})^2=10 \therefore k=0.4$ $\frac{dv}{dt}=10-0.4v^2$ $\int \frac{1}{10-0.4v^2}\,dv=\int dt$ $\frac{1}{0.4}\int \frac{1}{25-v^2}\,dv=t+c$ $\frac{1}{0.4}\left(\frac{1}{2(5)}\right)\ln\left\|\frac{5+v}{5-v}\right\|=t+c$ $\frac{1}{4}\ln\left\|\frac{5+v}{5-v}\right\|=t+c$ $\ln\left\|\frac{5+v}{5-v}\right\|=4t+d, \quad d=4c$ $\frac{5+v}{5-v}=Ae^{4t}, \quad A=\pm e^d$ $\frac{5-v}{5+v}=Be^{-4t}, \quad B=\frac{1}{A}$ when $t=0,\ v=0\ \therefore A=1$ $\frac{5-v}{5+v}=e^{-4t}$ $5-v=(5+v)e^{-4t}$ $\therefore v=\frac{5(1-e^{-4t})}{1+e^{-4t}}$ As $t\to\infty$, $v\to 5\text{ ms}^{-1}$	Those who managed to obtain $\frac{dv}{dt}+kv^2=10$, are able to get $k=0.4$ easily. With $\frac{dv}{dt}+kv^2=10$, majority knew how to separate the variables. However, only a handful include modulus. A common mistake made is $\int \frac{5}{50-2v^2}\,dv=\frac{5}{2(\sqrt{50})}\ln\left\|\frac{\sqrt{50}+\sqrt{2}v}{\sqrt{50}-\sqrt{2}v}\right\|$. Students did not realise that coefficient of v^2 must be 1 if they are applying the same formula from MF26. Students have no idea why $\frac{5(e^{4t}-1)}{e^{4t}+1}=\frac{5(1-e^{-4t})}{1+e^{-4t}}$. Working must be shown explicitly. Students didn't read question. At least a quarter of the cohort missed this part. Those who attempted this part managed to answer correctly.

	$\frac{dx}{dt}=\frac{5(1-e^{-4t})}{1+e^{-4t}}$ $x=\int\frac{5(1-e^{-4t})}{1+e^{-4t}}\,dt$ $=5\int\frac{1}{1+e^{-4t}}-\frac{e^{-4t}}{1+e^{-4t}}\,dt=5\int\frac{e^{4t}}{1+e^{4t}}-\frac{e^{-4t}}{1+e^{-4t}}\,dt$ $=\frac{5}{4}\left[\ln(1+e^{4t})+\ln(1+e^{-4t})\right]+c$ when $t=0$, $x=0$ $\therefore c=-2.5\ln 2$ Hence, $x=\frac{5}{4}\left[\ln(1+e^{4t})+\ln(1+e^{-4t})\right]-\frac{5}{2}\ln 2$. From G.C., when $x=10$, $t=2.3465\approx 2.35$ s Alternatively, $\int_0^t\frac{5(1-e^{-4t})}{1+e^{-4t}}\,dt=10$ From G.C., $t=2.3465\approx 2.35$ s	Badly done. Many students did not attempt this part. Those who attempted, majority have no idea that the question is solving for the particular solution of x. Many students differentiated $\frac{5(1-e^{-4t})}{1+e^{-4t}}$. There is still a handful of students who got $x=\int\frac{5(1-e^{-4t})}{1+e^{-4t}}\,dt$. However, most of them stuck at $\int\frac{1}{1+e^{-4t}}\,dt$, which many mistook as $\ln\left(1+e^{-4t}\right)$ or tangent inverse. Some students interpreted the question wrongly as "when $t=0,\ x=10$". There are some impressive solutions, but a handful of them didn't realise that they need to make use of G.C to solve.
12(i)	F_1: $\mathbf{r}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-4\mathbf{j}+\mathbf{k})$ F_2: $\mathbf{r}=-2\mathbf{i}+\mathbf{j}+3\mathbf{k}+\mu(\mathbf{i}+m\mathbf{j}-7\mathbf{k})$ Since $\begin{pmatrix}2\\-4\\1\end{pmatrix}\neq k\begin{pmatrix}1\\m\\-7\end{pmatrix}$, the paths cannot be parallel. If the paths intersect, $\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\m\\-7\end{pmatrix}$ $2\lambda-\mu=-3$ .... (1) $\lambda+7\mu=0$ .... (2) $4\lambda+m\mu=1$ .... (3) Solving (1) and (2), $\lambda=-\frac{7}{5},\ \mu=\frac{1}{5}$ From (3), $m=33$ Since the paths do not intersect, they are skew lines $\therefore m\neq 33$	Shocking that a significant number of students did not know how or made mistakes/slips when converting from Cartesian to vector form. For non-intersecting lines, many considered parallel lines and concluded that m is not multiples of 4. A lot of students did not even consider case of skew lines.
(ii)	Signal is located at $S(0, 0, 3)$. <u>Method 1:</u> Let $A(1,2,3)$ be a point on F_1. Then $\overrightarrow{AS}=\begin{pmatrix}0\\0\\3\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}$	Many students did not write the position of signal correctly, <u>base</u> of control towel at (0, 0, 0) (info given earlier and found on a different page).

	Perpendicular dist $=\left\|\overrightarrow{AS}\times\hat{\mathbf{b}}\right\|$ $=\left\|\begin{pmatrix}-1\\-2\\0\end{pmatrix}\times\frac{1}{\sqrt{21}}\begin{pmatrix}2\\-4\\1\end{pmatrix}\right\|=\frac{1}{\sqrt{21}}\left\|\begin{pmatrix}-2\\1\\8\end{pmatrix}\right\|$ $=\frac{\sqrt{(-2)^2+1+8^2}}{\sqrt{21}}=\sqrt{\frac{23}{7}}$ <u>Method 2:</u> $\left\|\overrightarrow{AF}\right\|=\left\|\overrightarrow{AS}.\hat{\mathbf{b}}\right\|=\left\|\begin{pmatrix}-1\\-2\\0\end{pmatrix}.\frac{1}{\sqrt{21}}\begin{pmatrix}2\\-4\\1\end{pmatrix}\right\|=\frac{6}{\sqrt{21}}$ $SF=\sqrt{AS^2-AF^2}=\sqrt{(\sqrt{5})^2-(\frac{6}{\sqrt{21}})^2}=\sqrt{\frac{23}{7}}$ <u>Method 3:</u> Let $\overrightarrow{OF}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\1\end{pmatrix}$ for some λ. $\overrightarrow{SF}=\begin{pmatrix}1+2\lambda\\2-4\lambda\\3+\lambda\end{pmatrix}-\begin{pmatrix}0\\0\\3\end{pmatrix}=\begin{pmatrix}1+2\lambda\\2-4\lambda\\\lambda\end{pmatrix}$ $\overrightarrow{SF}.\mathbf{b}=0$ $\begin{pmatrix}1+2\lambda\\2-4\lambda\\\lambda\end{pmatrix}.\begin{pmatrix}2\\-4\\1\end{pmatrix}=0\Rightarrow 2(1+2\lambda)-4(2-4\lambda)+\lambda=0\therefore\lambda=\frac{2}{7}$ $\overrightarrow{SF}=\begin{pmatrix}1+2(2/7)\\2-4(2/7)\\2/7\end{pmatrix}=\frac{1}{7}\begin{pmatrix}11\\6\\2\end{pmatrix}$ $\left\|\overrightarrow{SF}\right\|=\frac{\sqrt{11^2+6^2+2^2}}{7}=\frac{\sqrt{161}}{7}$ <u>Method 4:</u> $\left\|\overrightarrow{SF}\right\|=\left\|\begin{pmatrix}1+2\lambda\\2-4\lambda\\3+\lambda\end{pmatrix}-\begin{pmatrix}0\\0\\3\end{pmatrix}\right\|=\left\|\begin{pmatrix}1+2\lambda\\2-4\lambda\\\lambda\end{pmatrix}\right\|$ $=\sqrt{(1+2\lambda)^2+(2-4\lambda)^2+\lambda^2}$ $=\sqrt{5-12\lambda+21\lambda^2}$ $=\sqrt{\frac{23}{7}+21(\lambda-\frac{6}{21})^2}$ $\therefore$ shortest dist $=\sqrt{\frac{23}{7}}$	

(iii)	$F_2: \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix}$ $F_3: \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$ $\mathbf{a.n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} . \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} = 11$ $\therefore \mathbf{r}. \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} = 11 \Rightarrow 5x + 6y + 5z = 11$	Many students did not know what **cartesian** form "$ax+by+cz=p$" of equation of plane is and left answer in parametric form: $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix}$.
(iv)	When F_2 and F_3 intersect, $\begin{pmatrix} -2+\mu \\ 1+5\mu \\ 3-7\mu \end{pmatrix} = \begin{pmatrix} 1+\alpha \\ 1 \\ -\alpha \end{pmatrix} \quad \therefore \mu = 0, \alpha = -3$ ie. $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ Let B be another point on F_2 ($\mu = 1$). $\overrightarrow{OB} = \begin{pmatrix} -2+1 \\ 1+5(1) \\ 3-7(1) \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix}$ Find p.v. of foot of perpendicular to F_3: <u>Method 1:</u> Let $\overrightarrow{ON} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ for some α. $\overrightarrow{BN} = \begin{pmatrix} 1+\alpha \\ 1 \\ -\alpha \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix} = \begin{pmatrix} 2+\alpha \\ -5 \\ 4-\alpha \end{pmatrix}$ $\overrightarrow{BN}.\mathbf{b} = 0$ $\begin{pmatrix} 2+\alpha \\ -5 \\ 4-\alpha \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \Rightarrow (2+\alpha) - (4-\alpha) = 0 \therefore \alpha = 1$ $\therefore \overrightarrow{ON} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ $\overrightarrow{ON} = \dfrac{\overrightarrow{OB} + \overrightarrow{OB'}}{2} \Rightarrow \overrightarrow{OB'} = 2\overrightarrow{ON} - \overrightarrow{OB}$ $\overrightarrow{OB'} = 2\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$ $\overrightarrow{PB'} = \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ -1 \end{pmatrix} \quad \therefore F_4: \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -5 \\ -1 \end{pmatrix}$ <u>Method 2:</u>	Most students just gave up. Out of those who did not, many used the intersection of F_2 and F_3 (−2, 1, 3) to find foot of perpendicular, which failed.

	$\begin{pmatrix}-1\\6\\-4\end{pmatrix}-\begin{pmatrix}-2\\1\\3\end{pmatrix}=\begin{pmatrix}1\\5\\-7\end{pmatrix}$ $\overrightarrow{PN}=\left(\overrightarrow{PB}.\hat{\mathbf{b}}\right).\hat{\mathbf{b}}=\left(\begin{pmatrix}1\\5\\-7\end{pmatrix}.\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right).\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}=4\begin{pmatrix}1\\0\\-1\end{pmatrix}$ $\overrightarrow{PN}=\frac{\overrightarrow{PB}+\overrightarrow{PB'}}{2}\Rightarrow\overrightarrow{PB'}=2\overrightarrow{PN}-\overrightarrow{PB}$ $\overrightarrow{PB'}=8\begin{pmatrix}1\\0\\-1\end{pmatrix}-\begin{pmatrix}1\\5\\-7\end{pmatrix}=\begin{pmatrix}7\\-5\\-1\end{pmatrix}$ $\therefore F_4:\mathbf{r}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}7\\-5\\-1\end{pmatrix}$	

ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

CANDIDATE NAME

TUTORIAL/ FORM CLASS

INDEX NUMBER

MATHEMATICS 9758/01

Paper 2 **3 September 2019**

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of **5** printed pages.

Anglo-Chinese Junior College

[Turn Over

Section A: Pure Mathematics [40 marks]

1 Referred to the origin O, the points A and B have position vectors **a** and **b,** where **a** and **b** are not parallel, **b** is a unit vector, and $\angle AOB = 45^\circ$. The point R has position vector given by $\mathbf{r} = 3\mathbf{a} + 5\mathbf{b}$.
Find

(i) the position vector of the point where OR meets AB, [3]

(ii) the length of projection of $\overrightarrow{OR}$ on $\overrightarrow{OB}$, leaving your answer in terms of $|\mathbf{a}|$. [3]

2 By considering $\dfrac{1}{r(r-2)} - \dfrac{1}{r(r+2)}$, show that

$$\sum_{r=3}^{n} \frac{1}{(r-2)r(r+2)} = a + \frac{b}{(n-1)(n+1)} + \frac{c}{n(n+2)},$$

where a, b and c are constants to be determined. [4]

(i) State the value of $\displaystyle\sum_{r=3}^{\infty} \frac{1}{(r-2)r(r+2)}$. [1]

(ii) Find $\displaystyle\sum_{r=5}^{n} \frac{1}{r(r+2)(r+4)}$ in terms of n. [3]

3 **(a)** An ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $b > a > 0$.

(i) Show that the area A of the region enclosed by the ellipse is given by

$$A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\ \mathrm{d}x.$$ [1]

By using the substitution $x = a\sin\theta$, find A in terms of a, b and π. [4]

(ii) The region enclosed by the ellipse is rotated about the x-axis through π radians to form an ellipsoid. Find the volume of the ellipsoid formed, in terms of a, b and π. [3]

(b) When a continuous function, $y = \mathrm{f}(x)$, $a \le x \le b$, is rotated completely about the x-axis, the resulting curved surface area is given by

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

By considering the circle with equation $x^2 + y^2 = r^2$ where $-r \le x \le r$, show that the surface area of a sphere with radius r is $4\pi r^2$. [4]

4 The folium of Descartes is an algebraic curve defined by the equation

$$x^3 + y^3 = 3axy,$$

where a is a real constant.
Consider the curve when $a = 1$.
Show that by setting $y = xt$, where t is a parameter such that $t \in \mathbb{R}$,

$$x = \frac{3t}{1+t^3}, \quad t \neq k,$$

where k is a real number to be determined.
Hence write down the expression for y in terms of t. [3]

(i) Show that

$$x + y = \frac{3t}{t^2 - t + 1},$$

and, by considering limits or otherwise, find the equation of the oblique asymptote of the curve. [3]

(ii) Sketch the graph of the folium of Descartes when $a = 1$, showing clearly the coordinates of any intersection with the axes, and the equation of any asymptote(s). [2]

(iii) Find the equation of the tangent to the curve at $t = 2$, and determine if the tangent cuts the curve again, giving the coordinates of the intersection if it does. [6]

Section B: Probability and Statistics [60 marks]

5 A school tennis team comprises of 6 boys and 6 girls.

(i) Given that the members of the tennis team all have different heights, find the number of ways a group of 4 students of increasing heights can be chosen from the team. [1]

(ii) A delegation of at least one student from the team is to be chosen to attend a convention. Find the number of ways this delegation can be chosen. [2]

(iii) Andy, a boy, and Beth, a girl, are members of the tennis team. A group of 8 students from the team is to be chosen to attend a dinner, seated at a round table.
Find the probability that the boys and the girls in the group alternate such that the group includes Andy and Beth who are seated next to each other. [2]

6 A class proposes the following game for their school's fundraising event.
Four discs numbered '1', '3', '5' and '7' respectively are placed in Bag A. Another four discs numbered '2', '4', '6', and '8' respectively are placed in Bag B. A player chooses two discs at random and without replacement from each of the two bags. It is assumed that selections from the two bags are independent of each other.
Let X represent the difference between the numbers on the two discs drawn from Bag A, and Y represent the difference between the numbers on the two discs drawn from Bag B. The player wins \$$X$ if $X = Y$. Otherwise, the player wins nothing.

(i) Show that $\mathrm{P}(X = 2) = \dfrac{1}{2}$ and find the probability distribution of X. [3]

(ii) Find the probability that the player wins a prize. [2]

The game is to be played using a \$$k$ coupon, where k is a positive integer. Find the minimum value of k in order for the class to earn a profit for each game, justifying your answer. [2]

7 A concert promoter claims that the mean price of a ticket to a pop concert is \$200. A media company collected information on ticket price, $\$x$, for 50 randomly chosen people who bought pop concert tickets. The results are summarised as follows.

$$\sum(x-200) = 450 \qquad \sum(x-200)^2 = 55\ 000$$

(i) Test, at the 4% level of significance, the concert promoter's claim that the mean price of a ticket to a pop concert is \$200. You should state your hypotheses and define any symbols you use. [5]

(ii) The media company took another random sample of n tickets, and found that the average ticket price for this sample is \$206. If the standard deviation of ticket price is now known to be \$32.25, find the maximum value of n such that there is insufficient evidence at the 4% level of significance to reject the concert promoter's claim. [2]

8 A company that organizes live concerts believes that the popularity of an artiste affects his/her concert ticket sales. The popularity of an artiste can be measured by an index, x, such that $1 \le x \le 10$, where 10 indicates most popular and 1 indicates least popular. A study was conducted over six months to investigate the relationship between the popularity index of eight artistes and their concert ticket sales. The results are summarised in the following table.

Artiste	A	B	C	D	E	F	G	H
Popularity Index, x	1.2	2.0	2.7	3.8	4.8	5.6	6.9	8.0
Concert Ticket Sales (hundreds of thousands), $\$y$	2.2	4.5	5.8	7.3	7.4	9.0	9.9	10.8

(i) Draw a scatter diagram for these values, labelling the axes. [1]

(ii) It is thought that concert ticket sales y can be modelled by one of the formulae

$$y = ax^2 + b \qquad \text{or} \qquad y = c\ln x + d,$$

where a, b, c and d are positive constants.
Use your diagram in **(i)** to explain which of the two is a more appropriate model, and calculate its product moment correlation coefficient, correct to 4 decimal places. [2]

(iii) The data for a particular artiste appears to be recorded wrongly. Indicate the corresponding point on your diagram by labeling it P. Find the equation of the least squares regression line for the remaining points using the model that you have chosen in **(ii)**. [2]

(iv) The ticket sales for a new artiste is found to be \$800, 000. Estimate the popularity index of this artiste and comment on the reliability of your estimate. Explain why neither the regression line of $\ln x$ on y nor x^2 on y should be used. [3]

9 The teacher in-charge of the Harmonica Ensemble observed that a pair of twins often turned up late for practice. Attendance records show that the older twin, Albert, is late for practice 65% of the time. When the younger twin, Benny, is late for practice, Albert is also late 97.5% of the time. When Benny is not late for practice, Albert is late 56.875% of the time.

(i) Show that the probability that Benny is late for a practice is 0.2. [3]

(ii) Find the probability that only one of the twins will be late for the next practice. [2]

The teacher observed that another student, Carl, is late for practice 50% of the time. The probability that all three will be late for practice is 0.098. Given that the event of either twin being late for practice is independent of the event that Carl is late for practice, find the probability that neither the twins nor Carl is late for a practice. [3]

10 In this question you should state the parameters of any distributions you use.
Mary runs a noodle stall by herself. The noodles are prepared to order, so she prepares the noodles only after each order, and will only take the next customer's order after the previous customer is served his noodles.
The time taken for a customer to place an order at the stall follows a normal distribution with mean 60 s and standard deviation σ s. The time taken for Mary to prepare and serve a customer's order also follows a normal distribution, with mean 300 s and standard deviation 50 s.
The probability that a customer takes not more than 40 s to place an order is 0.16. Show that $\sigma = 20.11$, correct to 2 decimal places. [2]

(i) Let A be the probability that a randomly chosen customer takes between 57 s and 63 s to place an order with Mary, and B be the probability that a randomly chosen customer takes between 49 s and 55 s to place an order with Mary. Without calculating A and B, explain, with the aid of a diagram, how A and B compare with each other. [2]

(ii) There is a 0.1% chance that a randomly chosen customer has to wait for more than k seconds for his noodles to be served after placing his order. Find k. [1]

(iii) A man visits the noodle stall on 10 separate occasions. Find the probability that his average waiting time per visit after placing his order is less than 4.5 minutes. [2]

(iv) In an effort to shorten wait time, Mary improves the ordering and cooking processes such that the time taken for a customer to place an order is reduced by 5%, while the time taken to prepare and serve a customer his noodles is reduced by 10%. Find the largest number of customers she can serve in 1 hour for at least 80% of the time. State one assumption you made in your calculations. [5]

11 A factory manufactures a large number of wine glasses. It is found that on average, a proportion p of the wine glasses are chipped. A distributor purchases batches of wine glasses from the factory, each batch consisting of n wine glasses. A batch of wine glasses is rejected if it has more than 1 chipped wine glass. Let X be the number of wine glasses that are chipped in one batch. State two assumptions for X to be well-modelled by a binomial distribution. [2]
Show that A, the probability that one batch of wine glasses will not be rejected, is given by the formula

$$A = (1-p)^{n-1}\left[1+(n-1)p\right].$$ [2]

(a) Given that $p = 0.02$, and the probability that a batch of wine glasses is not rejected is at least 0.9, find the largest possible value of n. [2]

(b) The distributor also purchases wine glasses from another factory in batches of 40 and it is found that $A = 0.73131$.

(i) Find p correct to 5 decimal places. [1]

(ii) The mean and standard deviation of X are denoted by μ and σ respectively. Find $\mathrm{P}(\mu-\sigma < X < \mu+\sigma)$. [2]

(iii) The distributor has a one-year contract with this factory such that every week, the factory produces 20 batches of wine glasses. According to the contract, the distributor will receive a compensation of \$100 for each batch of wine glasses it rejects every week. Assuming that there are 52 weeks in a year, find the probability that the total compensation in the one-year contract period is more than \$30 000. [4]

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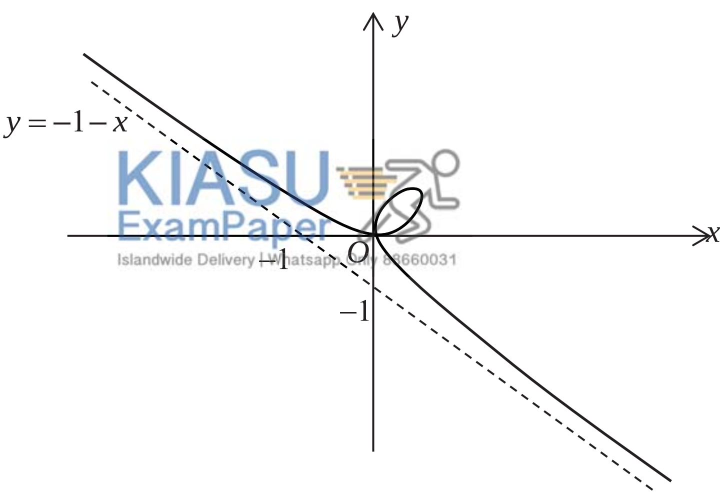
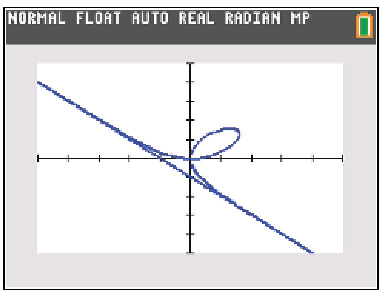
1(i)	$k(3\mathbf{a}+5\mathbf{b})=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$ Comparing coefficient of **a** and **b**, $\left.\begin{array}{l}3k=1-\lambda\\5k=\lambda\end{array}\right\}\therefore\lambda=\frac{5}{8},\ k=\frac{1}{8}$ $\therefore$p.v of the point of intersection is $\frac{3}{8}\mathbf{a}+\frac{5}{8}\mathbf{b}$.	Some made the mistake of using vector AB as the line equation Line AB is $r=a+\lambda(b-a)$ not $r=b-a$ Some assumed that the intersection point is R, making the mistake of equating $(3\mathbf{a}+5\mathbf{b})=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$ when it should have been $k(3\mathbf{a}+5\mathbf{b})=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$
(ii)	$\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 45^o=\frac{\lvert\mathbf{a}\rvert}{\sqrt{2}}$ $\lvert\overrightarrow{OR}\cdot\hat{\mathbf{b}}\rvert=\lvert(3\mathbf{a}+5\mathbf{b})\cdot\mathbf{b}\rvert=\lvert 3\mathbf{a}\cdot\mathbf{b}+5\mathbf{b}\cdot\mathbf{b}\rvert$ $=(3\mathbf{a}\cdot\mathbf{b}+5\lvert\mathbf{b}\rvert^2)=\frac{3\lvert\mathbf{a}\rvert}{\sqrt{2}}+5$	Some wrote the dot product correctly in (i) but didn't put it to use in any way.
2	$\frac{1}{r(r-2)}-\frac{1}{r(r+2)}=\frac{(r+2)-(r-2)}{(r-2)r(r+2)}=\frac{4}{(r-2)r(r+2)}$ $\sum_{r=3}^{n}\frac{1}{(r-2)r(r+2)}=\frac{1}{4}\sum_{r=3}^{n}\left(\frac{1}{r(r-2)}-\frac{1}{r(r+2)}\right)$ $=\frac{1}{4}\left(\frac{1}{3(1)}-\cancel{\frac{1}{3(5)}}\right.$ $+\frac{1}{4(2)}-\cancel{\frac{1}{4(6)}}$ $+\cancel{\frac{1}{5(3)}}-\cancel{\frac{1}{5(7)}}$ $+\cdots$ $+\cancel{\frac{1}{(n-2)(n-4)}}-\cancel{\frac{1}{(n-2)n}}$ $+\cancel{\frac{1}{(n-1)(n-3)}}-\frac{1}{(n-1)(n+1)}$ $\left.+\cancel{\frac{1}{n(n-2)}}-\frac{1}{n(n+2)}\right)$ $=\frac{1}{4}\left(\frac{1}{3}+\frac{1}{8}-\frac{1}{(n-1)(n+1)}-\frac{1}{n(n+2)}\right)$ $=\frac{11}{96}+\frac{-\frac{1}{4}}{(n-1)(n+1)}+\frac{-\frac{1}{4}}{n(n+2)}$ $a=\frac{11}{96},b=c=-\frac{1}{4}$	A minority of students did not consider the given expression. Out of those who did use the result from considering the given expression, ~10-20% of them used the result wrongly (multiplying by 4 instead of by ¼). MOD was done well generally, with minor slips.

(i)	From previous result, $\sum_{r=3}^{n}\frac{1}{(r-2)r(r+2)}=\frac{11}{96}-\frac{1}{4(n-1)(n+1)}-\frac{1}{4n(n+2)}$ As $n\to\infty$, $\frac{1}{(n-1)(n+1)}, \frac{1}{n(n+2)}\to 0$, therefore $\sum_{r=3}^{\infty}\frac{1}{(r-2)r(r+2)}=\frac{11}{96}$.	Done well, but the presentation was off for 20-30%. Some even wrote As $r\to\infty$, $\frac{1}{(r-2)r(r+2)}\to\frac{11}{96}$. Students are reminded to <u>answer the question</u>.
(ii)	$\square_{r=5}^{n}\frac{1}{r(r+2)(r+4)}$ Replace r by $r-2$, $\sum_{r-2=5}^{r-2=n}\frac{1}{(r-2)r(r+2)}=\sum_{r=7}^{n+2}\frac{1}{(r-2)r(r+2)}$ $=\sum_{r=3}^{n+2}\frac{1}{(r-2)r(r+2)}-\sum_{r=3}^{6}\frac{1}{(r-2)r(r+2)}$ $=\left(\frac{11}{96}-\frac{1}{4(n+1)(n+3)}-\frac{1}{4(n+2)(n+4)}\right)$ $-\left(\frac{11}{96}-\frac{1}{4(6-1)(6+1)}-\frac{1}{4(6)(6+2)}\right)$ $=\frac{83}{6720}-\frac{1}{4(n+1)(n+3)}-\frac{1}{4(n+2)(n+4)}$	Poorly done. A variety of similar approaches can be applied, but there were many conceptual errors with regard to how to the general term interacts with the dummy variable in the summation. There was a small number of scripts where the working was filled with errors arriving at the right answer (or close to). Marks may not have been awarded if the terms used in the working were not equal at any juncture.
3(a)(i)	$y^2=b^2\left(1-\frac{x^2}{a^2}\right)=\frac{b^2(a^2-x^2)}{a^2}$ $A=4\int_0^a y\ \mathrm{d}x=4\int_0^a\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}\ \mathrm{d}x$ $=4\int_0^a\sqrt{\frac{b^2(a^2-x^2)}{a^2}}\ \mathrm{d}x=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\ \mathrm{d}x.$ (shown) $A=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\ \mathrm{d}x$ $=\frac{4b}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^2-(a\sin\theta)^2}\,(a\cos\theta)\ \mathrm{d}\theta$ $=\frac{4b}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^2(1-\sin^2\theta)}\,(a\cos\theta)\ \mathrm{d}\theta$ $=\frac{4b}{a}\int_0^{\frac{\pi}{2}}a^2\cos^2\theta\ \mathrm{d}\theta$ $=2ab\int_0^{\frac{\pi}{2}}2\cos^2\theta\ \mathrm{d}x=2ab\int_0^{\frac{\pi}{2}}1+\cos 2\theta\ \mathrm{d}\theta$ $=2ab\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}=2ab(\frac{\pi}{2})=\pi ab$	For the ‘show’ part, it is a simple result that aims to test the students’ conceptual understanding of the application of integration to find area under a curve. Students’ presentation has to demonstrate that understanding. Generally very well done, though the usual mistakes were still present (not changing the limits, making errors substituting in $\frac{\mathrm{d}x}{\mathrm{d}\theta}$). Some students did not know how to proceed with the integrand thereafter (but only minority). Of those who applied the cosine double angle formula, there was a significant minority who integrated $\cos 2\theta$ wrongly.

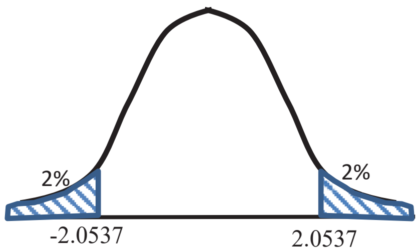
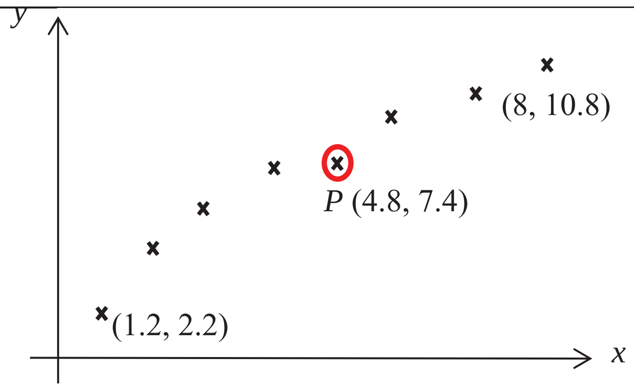
(a)(ii)	$V=2\pi\int_0^a y^2 \text{ d}x \text{ or } \pi\int_{-a}^a y^2 \text{ d}x$ $=2\pi\int_0^a \frac{b^2(a^2-x^2)}{a^2} \text{ d}x=\frac{2\pi b^2}{a^2}\int_0^a a^2-x^2 \text{ d}x$ $=\frac{2\pi b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{2\pi b^2}{a^2}\left(\frac{2a^3}{3}\right)$ $=\frac{4\pi ab^2}{3}$	Poorly done by quite a number of students. These students may have been confused by the integral they showed in (i), not helped by their poor understanding of finding volume of revolution by applying integration. Simplify answers too!
(b)	$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{\text{d}y}{\text{d}x}\right)^2} \text{ d}x$ $x^2+y^2=a^2$ $2x+2y\frac{\text{d}y}{\text{d}x}=0$ $\frac{\text{d}y}{\text{d}x}=-\frac{x}{y}$ $=\int_{-r}^r 2\pi y\sqrt{1+\left(-\frac{x}{y}\right)^2} \text{ d}x$ $=\int_{-r}^r 2\pi\sqrt{x^2+y^2} \text{ d}x$ $=\int_{-r}^r 2\pi\sqrt{x^2+(r^2-x^2)} \text{ d}x$ $=\int_{-r}^r 2\pi r \text{ d}x=2\pi r\ [x]_{-r}^r=4\pi r^2$ Alternative: $y=\sqrt{a^2-x^2}$ $\frac{\text{d}y}{\text{d}x}=\frac{-x}{\sqrt{a^2-x^2}}$ $S=\int_a^b 2\pi y\sqrt{1+\left(\frac{\text{d}y}{\text{d}x}\right)^2} \text{ d}x$ $=\int_{-r}^r 2\pi\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2} \text{ d}x$ $=\int_{-r}^r 2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}} \text{ d}x$ $=\int_{-r}^r 2\pi\sqrt{(r^2-x^2)+x^2} \text{ d}x$ $=\int_{-r}^r 2\pi r \text{ d}x=2\pi r\ [x]_{-r}^r=4\pi r^2$	Students generally made good progress with this part, applying the formula with ease. A common conceptual error was integrating r with respect to x to obtain $\frac{r^2}{2}$.

4	When $a=1$, $x^3+y^3=3xy$. Let $y=xt$, then $x^3+(xt)^3=3x(xt)$ $\Rightarrow\quad x^3+x^3t^3=3x^2t$ $\Rightarrow\quad x^3(1+t^3)=3x^2t$ $\Rightarrow\quad x=\dfrac{3t}{1+t^3}$. (shown) For x to be defined, $1+t^3\neq 0$ Hence $t\neq -1$. Therefore $k=-1$ Since $y=xt$, hence $y=\left(\dfrac{3t}{1+t^3}\right)t=\dfrac{3t^2}{1+t^3}$.	The proof for x was generally well done. Working for find y in terms of t varied from a one-liner to a full page working when the question says "write down the expression…" which students should suspect that the answer is obvious. Some students neglected to write down the value of k.
(i)	$x+y=\dfrac{3t}{1+t^3}+\dfrac{3t^2}{1+t^3}$ $=\dfrac{3t(1+t)}{1+t^3}$ $=\dfrac{3t(1+t)}{(1+t)(t^2-t+1)}$ $=\dfrac{3t}{t^2-t+1}$. (shown) Oblique asymptote is when $x\to\pm\infty$. Since $x=\dfrac{3t}{1+t^3}$, $x\to\pm\infty$ when $t\to -1$. When $t\to -1$, $x+y=\dfrac{3t}{t^2-t+1}\to\dfrac{-3}{(-1)^2-(-1)+1}=\dfrac{-3}{3}=-1$. i.e. $x+y\to -1$ $\Rightarrow\quad y\to -1-x$. Therefore, the oblique asymptote of the curve is $y=-1-x$.	In this show question, students either did not know how to simplify $\dfrac{3t(1+t)}{1+t^3}$, or some just did $\dfrac{3t(1+t)}{1+t^3}=\dfrac{3t}{t^2-t+1}$ without showing the factorization. Students need to realise that nothing in a "show" or "prove" question should be assumed as obvious. Only a few students knew that oblique asymptote occurs when $x\to\pm\infty$, not when $t\to\pm\infty$.
(ii)		Graph was badly drawn. Students need to know how to interpret their G.C. sketch: 

		1. Why is there a "hole" in the graph near the origin? 2. Is that line that looks like a straight line part of the graph? Sketches that included a sharp turn to draw the straight line as part of the graph did not get any credit. Nor did sketches with a gap near the origin.
(iii)	Differentiating the cartesian equation implicitly, $3x^2+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=3x\frac{\mathrm{d}y}{\mathrm{d}x}+3y$ $\Rightarrow\ \left(y^2-x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=y-x^2$ i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y-x^2}{y^2-x}$. When $t=2$, $x=\frac{2}{3}$, $y=\frac{4}{3}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{4}{3}-\left(\frac{2}{3}\right)^2}{\left(\frac{4}{3}\right)^2-\frac{2}{3}}=\frac{4}{5}$. Hence equation of tangent when $t=2$: $\frac{y-\frac{4}{3}}{x-\frac{2}{3}}=\frac{4}{5}\ \Rightarrow\ y=\frac{4}{5}x+\frac{4}{5}$. If the tangent cuts the curve again, using the parametric equations of the curve to substitute into the equation of the tangent, $\frac{3t^2}{1+t^3}=\frac{4}{5}\left(\frac{3t}{1+t^3}\right)+\frac{4}{5}$ $\Rightarrow\ 3t^2=\frac{4}{5}(3t)+\frac{4}{5}\left(1+t^3\right)$ $\Rightarrow\ 15t^2=12t+4+4t^3$ $\Rightarrow\ 4t^3-15t^2+12t+4=0$ $\Rightarrow\ t=-\frac{1}{4}$, or $t=2$ (tangent here) Therefore the tangent cuts the curve again at $t=-\frac{1}{4}$, with coordinates $\left(-\frac{16}{21},\frac{4}{21}\right)$.	This part of the question required students to choose between the given Cartesian equation or the equivalent parametric equations to use to find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Most students chose to differentiate the parametric equations, with many many remembering quotient rule wrongly. Some used product rule instead but also fumbled with the algebra. The most efficient method is to differentiate the Cartesian equation implicitly, then substituting $t=2$ to find *x* and *y* at the point to find gradient of the tangent. The intersection between tangent and original curve should be a familiar question. The parametric equations should be substituted into the equation of the tangent to solve for *t*. Technically, even without having done the preceding parts, this question is a standard one that could have easily been done with just the Cartesian equation and the G.C.
5(i)	Number of ways $={}^{12}C_4$ $=495$	Most common error was to use ${}^{12}P_4$ instead. But order in this case is already

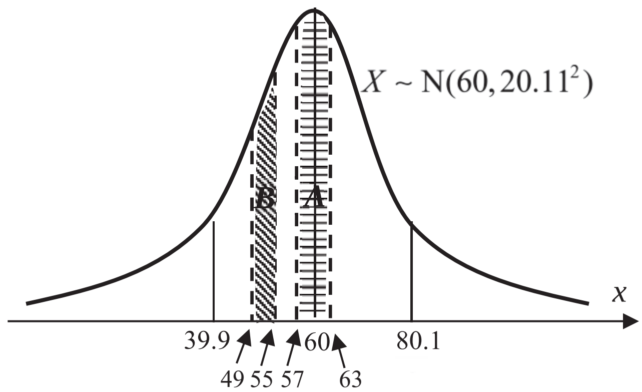
		predetermined by heights, so we just need to find how many ways a group of 4 can be chosen and there is only 1 way to arrange them by height.
5(ii)	Method 1 Number of ways $= 2^{12}-1$ $= 4095$ Method 2 Number of ways $= {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 ... + {}^{12}C_{12}$ $= 4095$	This question, unintentionally, has two interpretations: 1. Total number of ways to choose delegations with at least one student. This was the intended interpretation. 2. If n is assumed constant, then number of ways to choose n students from the team is just ${}^{12}C_n = \frac{12!}{n!(12-n)!}$. For this approach we need to see the formula in order to gain the second mark.
5(iii)	Probability $= \frac{{}^5C_3 3! \times {}^5C_3 3!\ \times 2}{{}^{12}C_8 7!}$ $= \frac{2}{693}$ (or 0.00289)	This question proved a challenge for many with a majority neglecting to choose the number of boys and girls to be seated with Andy and Beth. The team has 12 members but only 8 are chosen to attend the dinner. Another problem is not knowing that the total number of ways is to count how many ways to choose **any** 8 to seat at a round table.
6(i)	$P(X=2) = 2P(\{1,3\}) + 2P(\{3,5\}) + 2P(\{5,7\})$ $= 2\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)$ $= \frac{1}{2}$ (shown) $P(X=4) = 2P(\{1,5\}) + 2P(\{3,7\})$ $= 2\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \frac{1}{3}$ x: 2, 4, 6 $P(X=x)$: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$	(i) Unclear presentation for P(X=2) = 0.5 for many students. Some wrote $6\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \frac{1}{2}$ or $2\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + 4\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)$ without explaining what these numbers meant.
6(ii)	Similarly y: 2, 4, 6 $P(Y=y)$: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$	(ii) some made the mistake $E(W) = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{3}\right) + 6\left(\frac{1}{6}\right)$

	P(player wins a prize) $= P(X=2)P(Y=2)+P(X=4)P(Y=4)+P(X=6)P(Y=6)$ $=\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{6}\right)^2$ $=\frac{7}{18}$	when it should have been $E(W)=2\left(\frac{1}{2}\right)^2+4\left(\frac{1}{3}\right)^2+6\left(\frac{1}{6}\right)^2$
6(ii)	Let W be the winnings per game. <table><tr><td>w</td><td>0</td><td>2</td><td>4</td><td>6</td></tr><tr><td>$P(W=w)$</td><td>$\frac{11}{18}$</td><td>$\left(\frac{1}{2}\right)^2$</td><td>$\left(\frac{1}{3}\right)^2$</td><td>$\left(\frac{1}{6}\right)^2$</td></tr></table>$E(W)=2\left(\frac{1}{2}\right)^2+4\left(\frac{1}{3}\right)^2+6\left(\frac{1}{6}\right)^2=\frac{10}{9}$ Since the expected winnings of player $=\$\frac{10}{9} > \$1$, the game is to be played using \$2 coupon in order to have profit per game. $k=2$	Some didn't realise that to win a prize you need P(X=Y), thus making the mistake of calculating E(X) = 10/3 Some students did not realise that amount won is associated with their different probabilities, they made the mistake of using prob $\frac{7}{18}$ with W (winning per game). Some found E(W) $=\frac{10}{9}$ but made wrong conclusion for k = \$1
7(i)	$\bar{x}=\frac{\sum(x-200)}{50}+200=209$ $s^2=\frac{1}{49}\left[55000-\frac{(450)^2}{50}\right]=\frac{50950}{49}$ or 1039.8 (5 s.f.) Let μ be the population mean price of all concert tickets. To test $H_0:\mu=200$ against $H_1:\mu\neq 200$ at the 4% level of significance. Under H_0, $Z=\frac{\bar{X}-200}{s/\sqrt{50}}\sim N(0,1)$ or $\bar{X}\sim N\left(200,\frac{50950}{49(50)}\right)$ approximately by central limit theorem since sample size, 50, is large. Value of test statistic $z=1.97$ p-value = 0.0484 > 0.04 (do not reject H_0) There is **insufficient** evidence at the **4% level of significance** that the mean concert ticket price is not \$200.	Most students are able to present first step, i.e. $H_0:\mu=200$ $H_1:\mu\neq 200$ at 4% level of sig. Many DID NOT define μ. Some students made the mistake of putting the wrong population mean 209 in place of the value 200 in the distribution, $\bar{X}\sim N\left(200,\frac{50950}{49(50)}\right)$ Some forgot to quote central limit theorem. Phrasing of the conclusion was contradicting for some students, even though they knew it was 'do not reject H_0'.
(ii)	Now given the standard deviation is 32.25 Under H_0, $\bar{X}\sim N\left(200,\frac{32.25^2}{n}\right)$ by CLT since n is large	Some students misread the question.

	 For H_0 to not be rejected, the test statistic, z, must not be in critical region. Hence $-2.0537 < z < 2.0537$ i.e. $-2.0537 < \dfrac{206-200}{\frac{32.25}{\sqrt{n}}} < 2.0537$ i.e. $-2.0537 < \dfrac{6\sqrt{n}}{32.25} < 2.0537$ i.e. $-11.039 < \sqrt{n} < 11.039$ Hence largest value of n is 121.	(ii) Some made the mistake to assuming it is upper tail, writing $\dfrac{206-200}{\frac{32.25}{\sqrt{n}}} < 2.0537$ instead of $-2.0537 < \dfrac{206-200}{\frac{32.25}{\sqrt{n}}} < 2.0537$ Some also made the mistake of confusing it with (i) information, writing it as $-2.0537 < \dfrac{209-206}{\frac{32.25}{\sqrt{n}}} < 2.0537$
8(i)		Students should be using a cross 'x' to make the points instead of a small square box (as seem on the calculator screen).
(ii)	The scatter diagram suggests that as x increases, y increases at a decreasing rate, which is consistent with the model $y = c\ln x + d$ where c and d are positive. The model $y = ax^2 + b$ where a and b are positive suggests instead that as x increases, y increases at an increasing rate, which is not suitable for the data given. $r = 0.9927$ (4dp) 	Quite a number of students said that the points in the scatter diagram were distributed along a line, hence $y = c\ln x + d$ was the appropriate model (incorrectly assuming ln x is a **lin**ear function?) Some students mentioned the turning point in $y = ax^2 + b$. This is not relevant to the current question as we are only looking at a limited range of values of x. Some students also calculated r for both models before deciding $y = c\ln x + d$ was the better model as its value of r was closer to 1. This was

		not accepted as it did not make use of the scatter diagram. Finally, many students did not give r to 4 decimal places as stated in the question.
(iii)	Least squares regression line is $y = 1.3755 + 4.4612\ln x$ (5 s.f.) $= 1.38 + 4.46\ln x$ (3 s.f.)	A large number of students gave $y = 4.46 + 1.38\ln x$, or otherwise chose the wrong model in part (ii) and hence lost marks here. Most students chose P correctly; some students thought it was the bottom left point (1.2, 2.2).
(iv)	When $y = 8$, $8 = 1.3755 + 4.4612\ln x$ Hence $\ln x = \dfrac{8-1.3755}{4.4612}$ $\Rightarrow x = 4.42$ (3 s.f.) This estimate is reliable as $r_{y,\ln x}$ is very close to +1 and $y = 8$ is within the data range of y ($2.2 \le y \le 10.8$) used to obtain the regression line of y on ln$x$. The popularity index (variable x) is the independent variable in this context, therefore it is not appropriate to use the ln x on y or the x^2 on y regression lines to estimate popularity index x.	Some students did not use the regression line in (iii), but recalculated a new regression line based on a linear model. Some students left their answer as $x = \dfrac{8-1.3755}{4.4612} = 1.48$. Many students omitted to mention that r is close to 1. Some students wrote that it was unreliable as the popularity index of a new artiste might not be accurate. This was not accepted. Finally, a large number of students stated the two models were inappropriate for various reasons – such as the fact that y could not be negative, or that the graph did not have a turning point. This reflects a misconception that the regression model can be extrapolated to values of x beyond those of the data.
9(i)	Let A be the event that the older twin Albert is late for practice, and B be the event that the younger twin Benny is late for practice. Given: $\mathrm{P}(A) = 0.65$, $\mathrm{P}(A\|B) = 0.975$, $\mathrm{P}(A\|B') = 0.56875$.	Majority of students were able to do this. Those who could not wrongly interpreted the

	$P(A\|B)=0.975 \Rightarrow \frac{P(A\cap B)}{P(B)}=0.975$ $\Rightarrow P(A\cap B)=0.975\,P(B)$ $P(A\|B')=0.56875 \Rightarrow \frac{P(A\cap B')}{P(B')}=0.56875$ $\Rightarrow P(A\cap B')=0.56875\,P(B')$ Now, $P(A)=P(A\cap B)+P(A\cap B')$ $0.975P(B)+0.56875[1-P(B)]=0.65$ $P(B)=0.2$ (shown)	probability of 0.975 as $P(A\cap B)$ instead of $P(A\|B)$. Some students had difficulty combining $P(A\cap B)$ and $P(A\cap B')$, not realizing that drawing a simple venn diagram would clearly lead to $P(A)=P(A\cap B)+P(A\cap B')$.
9(ii)	$\frac{P(A\cap B)}{0.2}=0.975 \Rightarrow P(A\cap B)=0.195$ Probability $=P(A\cap B')+P(A'\cap B)$ $=[P(A)-P(A\cap B)]+[P(B)-P(A\cap B)]$ $=0.65-0.195+0.2-0.195$ $=0.46$ OR Probability $=P(B')P(A\|B')+P(B)P(A'\|B)$ $=(0.8)(0.56785)+(0.2)(1-0.975)$ $=0.46$ OR $P(A'\cap B')=1-P(A\cup B)$ $=1-[P(A)+P(B)-P(A\cap B)]$ $=1-(0.65+0.2-0.195)$ $=0.345$ Probability $=1-P(A\cap B)-P(A'\cap B')$ $=1-0.195-0.345$ $=0.46$	Very common mistake: $P(A\cap B')+P(A'\cap B)$ $=P(A)P(B')+P(A')P(B)$ $=(0.65)(0.8)+(0.35)(0.2)$. Student need to realise that the question did not state that *A* and *B* are independent, and they must not assume this. In this question, indeed, *A* and *B* are NOT independent.
9(iii)	$P(A'\cap B'\cap C')$ $=1-P(A\cup B\cup C)$ $=1-\begin{bmatrix}P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)\\-P(A\cap C)+P(A\cap B\cap C)\end{bmatrix}$ $=1-[0.65+0.2+0.5-0.195-(0.2)(0.5)-(0.65)(0.5)+0.098]$ $=0.172$	A common mistake: $P(A'\cap B'\cap C')$ $=P(A'\cap B')P(C')$, where students assumed that $A'\cap B'$ is independent of *C'*. Some students wrongly interpreted the phrase 'event of either twin being late for practice is independent of the event that C is late for practice' as $A\cup B$ independent of *C*, and wrote $P((A\cup B)\cap C)$ $=P(A\cup B)\times P(C)$.

10	Let X be the random variable for the time taken to place an order and Y be the random variable for the time taken to prepare and serve an order. $X \sim \mathrm{N}(60, \sigma^2), Y \sim \mathrm{N}(300, 50^2)$ Given: $\mathrm{P}(X < 40) = 0.16$ $\Rightarrow \mathrm{P}\left(Z < \frac{40-60}{\sigma}\right) = 0.16$ $\Rightarrow \frac{-20}{\sigma} = -0.9944578907$ $\Rightarrow \sigma = 20.11146$ $\Rightarrow \sigma = 20.11$(to 2 d.p.) (shown)	This is generally well-done. However, no credit is given if student used the value of 20.11 and compared the probabilities of $\mathrm{P}(X < 40)$ for $\sigma = 20.10,\ 20.11$ and 20.12. For a question that requires the student to show the result, student must not use the value to be shown in the working. There are also some students who wrote the standardized value wrongly as $\frac{60-40}{\sigma}$, instead of $\frac{40-60}{\sigma}$.
10(i)	 Since normal curve is symmetrical about mean 60, $\mathrm{P}(49 < X < 55) < \mathrm{P}(57 < X < 63)$. So $A > B$.	Students need to read the question carefully. Many students wrongly interpreted A and B as events rather than probabilities. Hence the following mistakes: 1. Drawing a venn diagram showing A and B as 2 sets 2. Writing $\mathrm{P}(A) > \mathrm{P}(B)$. Some drew 2 different normal curves, without realizing that it's the same random variable X.
10 (ii)	$Y \sim \mathrm{N}(300, 50^2)$ $\mathrm{P}(Y > k) = 0.001$ From GC, $k = 454.5116154 = 455$ s (to 3 s.f.)	Not well-done despite it being an easy question. Common mistakes: 1. Very careless in interpreting the probability of 0.1%, often writing it as 0.1, or 0.01. 2. Some considered distribution of X, or $X+Y$ instead. Many did additional step of standardizing Y. This is not necessary since the mean and variance of Y are given.

10 (iii)	$\bar{Y}=\frac{Y_1+Y_2+...+Y_{10}}{10}\sim N(300,\frac{50^2}{10})$ $P(\bar{Y}<270)=0.028897188$ $=0.0289$ (to 3 s.f.)	This is generally well-done. Some, however, wrongly quoted the use of CLT. Common mistakes: 1. Considering distribution of $\frac{X_1+...+X_{10}+Y_1+...+Y_{10}}{10}$ 2. Writing variance of $\bar{Y}$ as $10(50^2)$. 3. Not converting 4.5 mins to 270s
10 (iv)	Let T be random variable for time taken to take the order and prepare the order of 1 customer. $T=0.95X+0.9Y$ $T\sim N(0.95(60)+0.9(300),0.95^2(20.11146^2)+0.9^2(50^2))$ $T\sim N(327,2390.01822)$ Time taken to serve n customers is $T_1+T_2+...+T_n$ $T_1+T_2+...+T_n\sim N(327n,2390.01822n)$ Given: $P(T_1+...+T_n<3600)\geq 0.8$ $P(Z<\frac{3600-327n}{\sqrt{2390.01822n}})\geq 0.8$ $\frac{3600-327n}{\sqrt{2390.01822n}}\geq 0.84162$ From GC, let $Y_1=\frac{3600-327n}{\sqrt{2390.01822n}}$ n | Y_1 9 | $4.4796 > 0.84162$ 10 | $2.1346 > 0.84162$ 11 | $0.0185 < 0.84162$ Largest n = 10 Assume that the time taken to place an order is independent of the time taken to prepare and serve an order for a customer OR Assume that the time taken to place an order for a customer is independent of the time taken to place an order for another customer.	Common mistakes: 1. Writing Var(T) as 20.11146^2+50^2 or $(0.95)(20.11146^2)+(0.9)(50^2)$ 2. Interpreting the time taken for n customers as nT instead of $T_1+T_2+...+T_n$, and hence writing Var(nT) as $2390.01822n^2$ Students should realise that for normal distributions, independence of random variables is a necessary assumption. There are students who simply assume that the new improved time for serving customers is normally distributed, and even quote use of CLT.
11	The probability of a randomly chosen wine glass being chipped is constant. Selections of chipped wine glasses are independent of each other.	Keywords for **<u>assumptions</u>**: • **probability … constant** • **Selections/events … independent** probability … independent is WRONG! Note the difference between conditions & assumptions.

<table>
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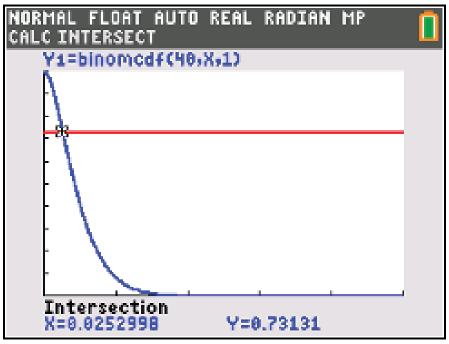
<td>Let X be the number of chipped wine glasses in a batch of n glasses.
$X \sim \mathrm{B}(n, p)$
$A = \mathrm{P}(X \le 1)$
$= \mathrm{P}(X=0) + \mathrm{P}(X=1)$
$= \binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1}$
$= (1-p)^n + np(1-p)^{n-1}$
$= (1-p)^{n-1}(1-p+np)$
$= (1-p)^{n-1}\left[1+(n-1)p\right]$ (shown)</td>
<td>Generally well done except –
• Some students mistook A to be $\mathrm{P}(X > 1)$.
• Some only had one of $\mathrm{P}(X = 0)$ or $\mathrm{P}(X = 1)$.
• Some had missing 2nd last step of factoring out $(1 - p)^{n-1}(1 - p + np)$</td>
</tr>
<tr>
<td>11
(a)</td>
<td>Given $p = 0.02$, $X \sim \mathrm{B}(n, 0.02)$
$A = \mathrm{P}(\text{batch of glasses will not be rejected}) = \mathrm{P}(X \le 1) \ge 0.9$
From GC,
<table>
<tr><td>n</td><td>$A = P(X \le 1)$</td></tr>
<tr><td>25</td><td>0.9114 > 0.9</td></tr>
<tr><td>26</td><td>0.9052 > 0.9</td></tr>
<tr><td>27</td><td>0.8989 < 0.9</td></tr>
</table>
Thus largest $n = 26$</td>
<td>Generally well done except that some students wasted time in writing out a few steps of working before using GC, instead of comparing with 0.9 directly. Another common mistake is $n = 27$ because of wrong inequality or using equation instead.</td>
</tr>
<tr>
<td>11
(b)</td>
<td>Given that $n = 40$, $A = \mathrm{P}(X \le 1) = 0.73131$ where $X \sim \mathrm{B}(40, p)$


From GC, $p = 0.02530$ (5 d.p.)</td>
<td>Common mistake – ignoring ‘5 decimal places’ in the question. Answer such as 0.025300 was common.</td>
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<tr>
<td>11
(b)
(i)</td>
<td>$X \sim \mathrm{B}(40, 0.02530)$
$\mu = \mathrm{E}(X) = 40 \times 0.02530 = 1.012$
$\sigma^2 = \mathrm{Var}(X) = 40 \times 0.02530 \times (1 - 0.02530) = 0.9863964$
$\mathrm{P}(\mu - \sigma < X < \mu + \sigma)$
$= \mathrm{P}(0.018805 < X < 2.005175)$
$= \mathrm{P}(X=1) + \mathrm{P}(X=2)$
$= 0.56107$
$= 0.561$ (to 3 s.f.)</td>
<td>Confusion in the distribution of X is common:
$X \sim \mathrm{B}(40, 0.02530)$ at the start becomes
$X \sim \mathrm{N}(1.1012, 0.9863964)$ a few steps later!

A few students took the values of $n = 26$ and $p = 0.02$ from part (a) instead of using the current values stated in part (b) with $n = 40$ and p found in (b) (i)</td>
</tr>
<tr>
<td>11
(b)
(ii)</td>
<td>Let R be number of rejected batches out of 20×52 batches in a year.
$R \sim \mathrm{B}(20 \times 52, 1 - 0.73131)$</td>
<td>There were a few scripts with no mention of any distributions!</td>
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<td>
$R \sim \mathrm{B}(1040, 0.26869)$
$\mathrm{P}(\text{total compensation} > \$30000) = \mathrm{P}(R > 300)$
$= 1 - \mathrm{P}(R \le 300) = 0.0711$

<u>OR</u>

Let C be number of rejected batches out of 20 batches in a week.
$C \sim \mathrm{B}(20, 1-0.73131)$
$C \sim \mathrm{B}(20, 0.26869)$
$\mathrm{E}(C) = 20 \times 0.26869 = 5.3738$
$\mathrm{Var}(C) = 20 \times 0.26869 \times 0.73131 = 3.929913678$

Since n = 52 is large, by Central Limit Theorem,
$C_1 + C_2 + \ldots + C_{52} \sim \mathrm{N}(279.4376, 204.3555113)$
approximately

Let total compensation be W.
$W = 100(C_1 + \ldots + C_{52})$
$\mathrm{E}(W) = 100(279.4376) = 27943.76$
$\mathrm{Var}(W) = 100^2(204.3555113) = 2043555.113$
$W \sim \mathrm{N}(27943.76, 2043555.113)$

$\mathrm{P}(W > 30000) = 0.075160$
$= 0.0752$ (to 3 s.f.)
</td>
<td>
Central Limit Theorem was wrongly used for 20 batches instead of 52 weeks.

Students should note the random variables involved with their respective n and p:
• $X \sim \mathrm{B}(40, 0.02530)$ for the number of chipped wine glass
• $C \sim \mathrm{B}(20, 1 - 0.73131)$ for the number of rejected batches
</td>
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ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Paper 1 (100 marks)

30 August 2019

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
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11	
Total	

This document consists of 23 printed pages and 1 blank page.

[Turn Over

1 **(a)** Show that $\sqrt{\dfrac{1-2x}{1+2x}}$ can be written in the form $\dfrac{\mathrm{f}(x)}{\sqrt{1-4x^2}}$ where f(x) is a polynomial to be determined. Hence, find $\displaystyle\int \sqrt{\frac{1-2x}{1+2x}}\,\mathrm{d}x$ [3]

(b) Show that $\displaystyle\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{1}{x\ln x}\,\mathrm{d}x = \ln 2$. [3]

2 It is given that $\tan\frac{1}{2}y = \sqrt{2}x$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

(i) Show that $\left(1+2x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{2}$. [2]

(ii) Use the result from part **(i)** to find the first two non-zero terms in the Maclaurin series for y, giving the coefficients in exact form. [3]

(iii) Hence, using standards series from the List of Formulae (MF26), find the expansion of $\dfrac{2\tan^{-1}\sqrt{2}x}{\cos 2x}$ in ascending powers of x, up to and including the term in x^3, giving the coefficients in exact form. [3]

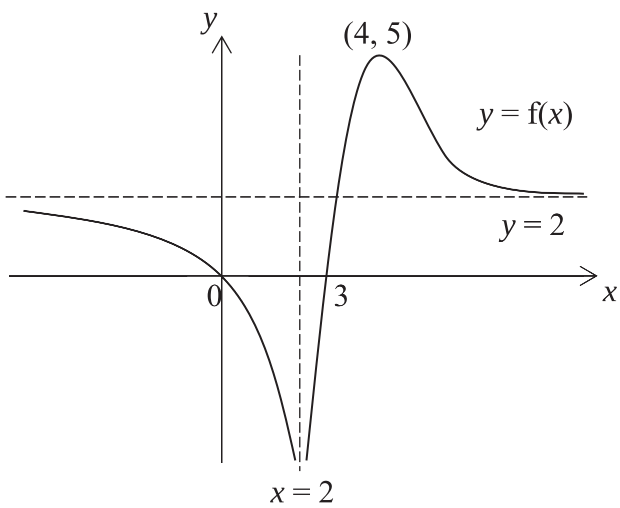
3 The position vectors of A, B and C referred to a point O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The point N is on AB such that AN:NB = 2:1.

(i) If O is the midpoint of CN, prove that $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = \mathbf{0}$. [2]

(ii) Show that A, O and M are collinear, and find the ratio AO:OM [3]

(iii) If the point P is such that $\overrightarrow{NP} = \overrightarrow{AM}$, show that the ratio of the area of $PNAM$: area of $PNOM$ = 12:7 [3]

4 **(a)**



The diagram shows the graph of $y = \mathrm{f}(x)$. The curve crosses the x-axis at $x = 0$ and $x = 3$. It has a turning point at (4, 5) and asymptotes with equations $y = 2$ and $x = 2$.

Showing clearly the coordinates of turning points, axial intercepts and equations of asymptotes where possible, sketch the graphs of

(i) $y = \mathrm{f}\left(|x|\right)$; [2]

(ii) $y = \mathrm{f}'(x)$ [3]

(b) The curve with equation $y = \mathrm{f}(x)$ is transformed by a stretch with scale factor 2 parallel to the x–axis, followed by a translation of 2 units in the negative x–direction, followed by a translation of 3 units in the positive y–direction. The equation of the resulting curve is $y = \ln\left(\mathrm{e}^3 x\right)$.

Find the equation of the curve $y = \mathrm{f}(x)$. [3]

5 A curve C has parametric equations

$$x = a\sin^2 t, \qquad y = a\cos t,$$

where $0 \le t \le \dfrac{\pi}{2}$ and $a > 0$.

(i) Find the cartesian equation of C, stating clearly any restrictions on the values of x and y. [2]

(ii) Sketch C, showing clearly the axial intercepts. [1]

(iii) The region bounded by C, the line $y + x = \dfrac{5}{4}a$ and the y-axis is rotated through 2π radians about the y-axis. Show that the exact volume of the solid obtained is $k\pi a^3$ where k is a constant to be determined. [5]

6 A function f is defined by

$$\mathrm{f}(x) = \begin{cases} \dfrac{6}{x-4} & \text{for } x < -2, \\ -|1+x| & \text{for } -2 \le x \le 1, \\ x^2 - x - 2 & \text{for } x > 1. \end{cases}$$

(i) Sketch the graph of f. [3]

(ii) Evaluate exactly $\displaystyle\int_{-3}^{4} |\mathrm{f}(x)|\,\mathrm{d}x$. [4]

7 **Do not use a calculator in answering this question.**

The roots of the equation $z^2 + (2 - 2\mathrm{i})z = -3 - 2\mathrm{i}$ are z_1 and z_2.

(i) Find z_1 and z_2 in cartesian form $x + \mathrm{i}y$, showing clearly your working. [5]

(ii) The complex numbers z_1 and z_2 are also roots of the equation

$$z^4 + 4z^3 + 14z^2 + 4z + 13 = 0.$$

Find the other roots of the equation, explaining clearly how the answers are obtained. [2]

(iii) Using your answer in part **(i)**, solve $z^2 + (2 + 2\mathrm{i})z = 3 + 2\mathrm{i}$. [2]

8 **(i)** Show that $\dfrac{3r+4}{(r+2)(r+1)r}=\dfrac{A}{r+2}+\dfrac{B}{r+1}+\dfrac{C}{r}$ where A, B and C are constants to be determined. [1]

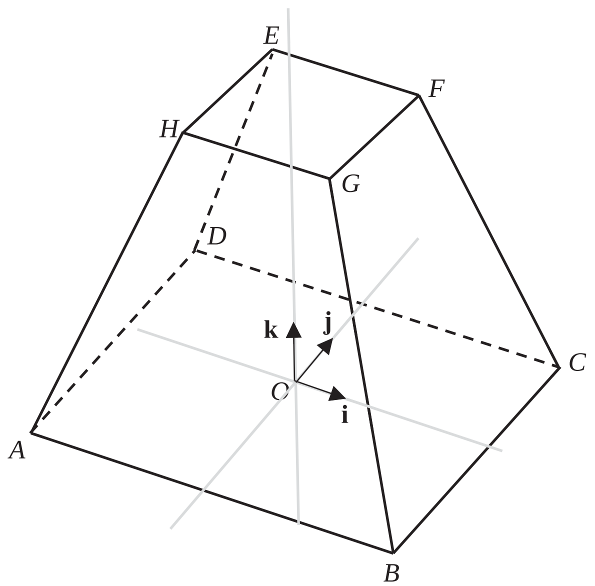
(ii) Find the sum to n terms of

$$\frac{7}{3\times2\times1}+\frac{10}{4\times3\times2}+\frac{13}{5\times4\times3}+\ldots$$

(There is no need to express your answer as a single algebraic fraction). [4]

(iii) Hence show that $\displaystyle\sum_{r=5}^{n+3}\frac{3r-5}{(r-1)(r-2)(r-3)}<\frac{4}{3}$. [3]

9



The diagram above shows an object with O at the centre of its rectangular base $ABCD$ where AB = 8 cm and BC = 4 cm. The top side of the object, $EFGH$ is a square with side 2 cm long and is parallel to the base. The centre of the top side is vertically above O at a height of h cm.

(i) Show that the equation of the line BG may be expressed as $\mathbf{r}=\begin{pmatrix}4\\-2\\0\end{pmatrix}+t\begin{pmatrix}-3\\1\\h\end{pmatrix}$, where t is a parameter. [1]

(ii) Find the sine of the angle between the line BG and the rectangular base $ABCD$ in terms of h. [2]

It is given that $h = 6$.

(iii) Find the cartesian equation of the plane $BCFG$. [3]

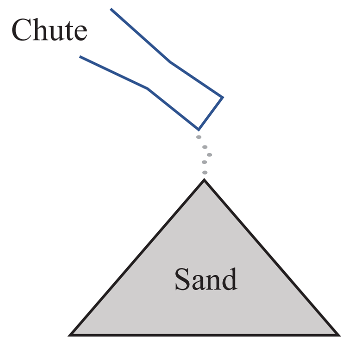
(iv) Find the shortest distance from the point A to the plane $BCFG$. [2]

(v) The line l, which passes through the point A, is parallel to the normal of plane $BCFG$. Given that, the line l intersects the plane $BCFG$ at a point M, use your answer in part (iv) to find the shortest distance from point M to the rectangular base $ABCD$. [2]

10 Commonly used in building materials, sand is the second largest world resource used by humans after water. To reduce the environmental impacts of sand mining, an alternative approach is to make "sand" by crushing rock.

A machine designed for this purpose produces sand in large quantities. Sand falling from the output chute of the machine forms a pile in the shape of a right circular cone such that the height of the cone is always equal to $\frac{4}{3}$ of the radius of its base.

[Volume of a cone = $\frac{1}{3}\pi r^2 h$ and curved surface area of a cone = πrl where r is the radius of the base area, h is the height of the cone and l is the slant length of the cone]



A machine operator starts the machine.

(a) **(i)** Given that V and A denote the volume and the curved surface area of the conical pile respectively, write down V and A in terms of r, the radius of its base. [2]

(ii) Hence show that the rate of change of A with respect to V is inversely proportional to the radius of the conical pile. [3]

An architect is tasked to design sand-lined walking paths in a large park. He decides to base his design of the paths on the shape of astroids, which are shapes with equations

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}} \ (k > 0).$$

On a piece of graph paper, he sketches an astroid with the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}}$.

(b) The tangent at a point $P(x_1, y_1)$ on the curve meets the x-axis at Q and the y-axis at R. Show that the length of QR is independent of where P lies on the curve. [7]

11 **(a)** Find the sum of all integers between 200 and 1000 (both inclusive) that are not divisible by 7. [4]

(b)

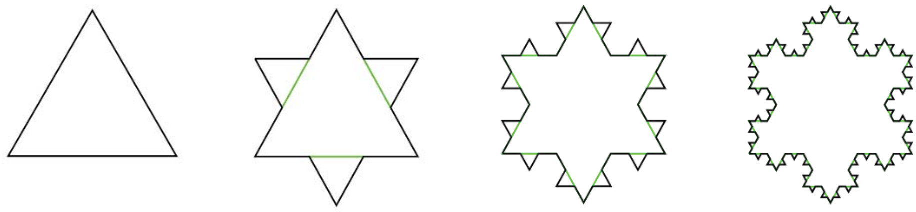


Fig. 1 **Fig. 2** **Fig. 3** **Fig. 4**

Snowflakes can be constructed by starting with an equilateral triangle (Fig. 1), then repeatedly altering each line segment of the resulting polygon as follows:

1. Divide each outer line segments into three segments of equal length.
2. Add an equilateral triangle that has the middle segment from step 1 as its base.
3. Remove the line segment that is the base of the triangle from step 2.
4. Repeat the above steps for a number of iterations, n.

The 1st, 2nd and 3rd iteration produces the snowflakes in Fig. 2, Fig. 3 and Fig. 4 respectively.

(i) If a_0 denotes the area of the original triangle and the area of each new triangle added in the n^{th} iteration is denoted by a_n, show that

$$a_n = \frac{1}{9}a_{n-1} \text{ for all positive integers } n.$$ [2]

(ii) Write down the number of sides in the polygons in Fig.1, Fig. 2 and Fig. 3, and deduce, with clear explanations, that the number of new triangles added in the n^{th} iteration is $T_n = k\left(4^n\right)$ where k is a constant to be determined. [2]

(iii) Find the total area of triangles added in the n^{th} iteration, A_n in terms of a_0 and n. [2]

(iv) Show that the total area of the snowflake produced after the n^{th} iteration is $a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]$ units^2. [3]

The snowflake formed when the above steps are followed indefinitely is called the Koch snowflake.

(v) Determine the least number of iterations needed for the area of the snowflake to exceed 99% of the area of a Koch snowflake. [3]

ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Paper 1 (100 marks)

30 August 2019

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of 23 printed pages and 1 blank page.

[Turn Over

1 **(a)** Show that $\sqrt{\dfrac{1-2x}{1+2x}}$ can be written in the form $\dfrac{\mathrm{f}(x)}{\sqrt{1-4x^2}}$ where f(x) is a polynomial to be determined. Hence, find $\displaystyle\int \sqrt{\frac{1-2x}{1+2x}}\,\mathrm{d}x$ [3]

<u>Solution</u>

$$\int \sqrt{\frac{1-2x}{1+2x}}\,\mathrm{d}x = \int \frac{\sqrt{1-2x}}{\sqrt{1+2x}} \times \frac{\sqrt{1-2x}}{\sqrt{1-2x}}\,\mathrm{d}x$$

$$= \int \frac{1-2x}{\sqrt{1-4x^2}}\,\mathrm{d}x$$

$$= \int \frac{1}{\sqrt{1-4x^2}} - \frac{2x}{\sqrt{1-4x^2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{2}{\sqrt{1-(2x)^2}}\,\mathrm{d}x + \frac{1}{2}\int \left(\frac{-8x}{2}\right)\left(1-4x^2\right)^{\frac{-1}{2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2} + c$$, where c is an arbitrary constant.

(b) Show that $\displaystyle\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{1}{x\ln x}\,\mathrm{d}x = \ln 2$. [3]

<u>Solution</u>

$$\int_{\mathrm{e}}^{\mathrm{e}^2} \frac{1}{x\ln x}\,\mathrm{d}x = \int_{\mathrm{e}}^{\mathrm{e}^2} \frac{\frac{1}{x}}{\ln x}\,\mathrm{d}x$$

$$= \Big[\ln\left|\ln x\right|\Big]_{\mathrm{e}}^{\mathrm{e}^2}$$

$= \ln|\ln \mathrm{e}^2| - \ln|\ln \mathrm{e}|$

$= \ln 2 - \ln 1$

$= \ln 2$ (Shown)

2 It is given that $\tan\frac{1}{2}y=\sqrt{2}x$, where $-\frac{\pi}{2}<y<\frac{\pi}{2}$.

(i) Show that $\left(1+2x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2\sqrt{2}$. [2]

Solution

$\tan\frac{1}{2}y=\sqrt{2}x$ ---(1)

Differentiate w.r.t x, $\frac{1}{2}\sec^2\left(\frac{1}{2}y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{2}$

$$\left[1+\tan^2\left(\frac{1}{2}y\right)\right]\frac{\mathrm{d}y}{\mathrm{d}x}=2\sqrt{2}$$

$\left(1+2x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2\sqrt{2}$ ---(2)

(ii) Use the result from part **(i)** to find the first two non-zero terms in the Maclaurin series for y, giving the coefficients in exact form. [3]

Solution

Differentiate Eq(2) w.r.t x, $\left(1+2x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ---(3)

$\left(1+2x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ ---(4)

When $x=0$, from (1), (2), (3) & (4),

$y=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=2\sqrt{2}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-8\sqrt{2}$

$$\therefore\quad y=0+2\sqrt{2}x+0x^2+\frac{-8\sqrt{2}}{3!}x^3+\ldots$$

$$=2\sqrt{2}x-\frac{4\sqrt{2}}{3}x^3+\ldots$$

(iii) Hence, using standards series from the List of Formulae (MF26), find the expansion of $\dfrac{2\tan^{-1}\sqrt{2}x}{\cos 2x}$ in ascending powers of x, up to and including the term in x^3, giving the coefficients in exact form. [3]

Solution

$$\frac{2\tan^{-1}\sqrt{2}x}{\cos 2x} = \frac{y}{\cos 2x}$$

$$= \left(2\sqrt{2}x - \frac{4\sqrt{2}}{3}x^3 + \ldots\right)\left[1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 + \ldots\right]^{-1}$$

$$= \left(2\sqrt{2}x - \frac{4\sqrt{2}}{3}x^3 + \ldots\right)\left[1 - 2x^2 + \frac{2}{3}x^4 + \ldots\right]^{-1}$$

$$= \left(2\sqrt{2}x - \frac{4\sqrt{2}}{3}x^3 + \ldots\right)\left[1 - \left(-2x^2 + \frac{2}{3}x^4 + \ldots\right)\right]$$

$$= \left(2\sqrt{2}x - \frac{4\sqrt{2}}{3}x^3 + \ldots\right)\left(1 + 2x^2 + \ldots\right)$$

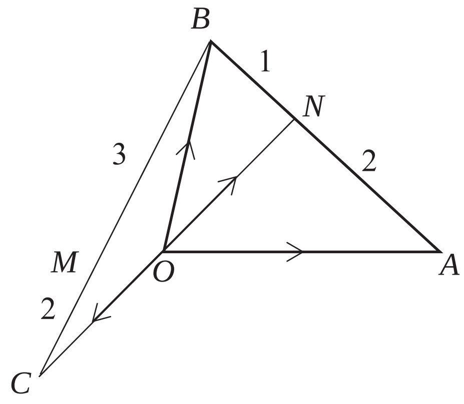
$$= 2\sqrt{2}x + 4\sqrt{2}x^3 - \frac{4\sqrt{2}}{3}x^3 + \ldots$$

$$= 2\sqrt{2}x + \frac{8\sqrt{2}}{3}x^3 \text{ (up to the } x^3 \text{ term)}$$

3 The position vectors of A, B and C referred to a point O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The point N is on AB such that AN:NB = 2:1.

(i) If O is the midpoint of CN, prove that $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = \mathbf{0}$. [2]

<u>Solution</u>



By ratio theorem, $\overrightarrow{ON} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b})$

Since O is the midpoint of CN, $\overrightarrow{ON} = -\mathbf{c}$

$\therefore\ \overrightarrow{ON} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b}) = -\mathbf{c}$

$\Rightarrow \mathbf{a} + 2\mathbf{b} = -3\mathbf{c}$

$\Rightarrow \mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = \mathbf{0}$

The point M is on CB such that CM:MB = 2:3.

(ii) Show that A, O and M are collinear, and find the ratio AO:OM [3]

<u>Solution</u>

By the ratio theorem, $\overrightarrow{OM} = \dfrac{2\mathbf{b} + 3\mathbf{c}}{2+3}$

$$= \frac{-\mathbf{a}}{5} = -\frac{1}{5}\overrightarrow{OA}$$

Hence, A, O and M are collinear (Shown)

Since $\left|\overrightarrow{OM}\right| = \frac{1}{5}\left|\overrightarrow{OA}\right|$, the ratio AO:OM = 5:1

6

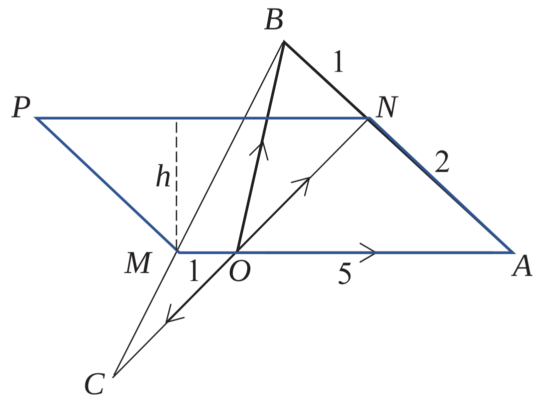
(iii) If the point P is such that $\overrightarrow{NP} = \overrightarrow{AM}$, show that the ratio of the area of $PNAM$: area of $PNOM$ = 12:7 [3]

<u>Solution</u>

<u>Method 1</u>

Area of //gram $PNAM$

$= h\left|\overrightarrow{AM}\right|$ where h is the height of the //gram



Area of trapezium $PNOM$

$= \frac{1}{2}\left(\left|\overrightarrow{PN}\right| + \left|\overrightarrow{OM}\right|\right)h$

$= \frac{1}{2}\left(\left|\overrightarrow{AM}\right| + \frac{1}{6}\left|\overrightarrow{AM}\right|\right)h$

$= \frac{7}{12}h\left|\overrightarrow{AM}\right| = \frac{7}{12} \times$ Area of //gram $PNAM$

$\therefore$ the ratio of the area of $PNAM$: area of $PNOM$ = 12:7

<u>Method 2</u>

Area of //gram $PNAM$

$= \left|\overrightarrow{PN} \times \overrightarrow{PM}\right|$

$= \left|\frac{6}{5}\underset{\sim}{a} \times \left(\frac{2}{3}\underset{\sim}{a} - \frac{2}{3}\underset{\sim}{b}\right)\right| = \frac{4}{5}\left|\underset{\sim}{a} \times \underset{\sim}{b}\right|$

Area of trapezium $PNOM$

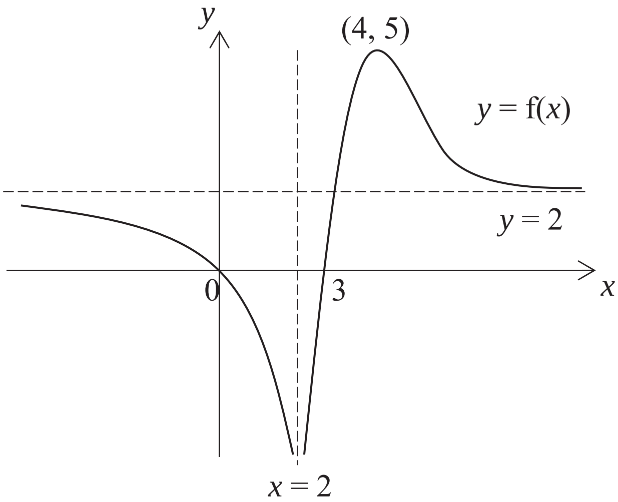
$= \left|\overrightarrow{PN} \times \overrightarrow{PM}\right| - \frac{1}{2}\left|\overrightarrow{ON} \times \overrightarrow{OA}\right|$

$= \frac{4}{5}\left|\underset{\sim}{a} \times \underset{\sim}{b}\right| - \frac{1}{2}\left|\left(\frac{1}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b}\right) \times \underset{\sim}{a}\right|$

$= \frac{4}{5}\left|\underset{\sim}{a} \times \underset{\sim}{b}\right| - \frac{1}{2}\left|-\frac{2}{3}\underset{\sim}{a} \times \underset{\sim}{b}\right| = \frac{7}{15}\left|\underset{\sim}{a} \times \underset{\sim}{b}\right|$

$\therefore$ the ratio of the area of $PNAM$: area of $PNOM$ = 12:7

4 **(a)**

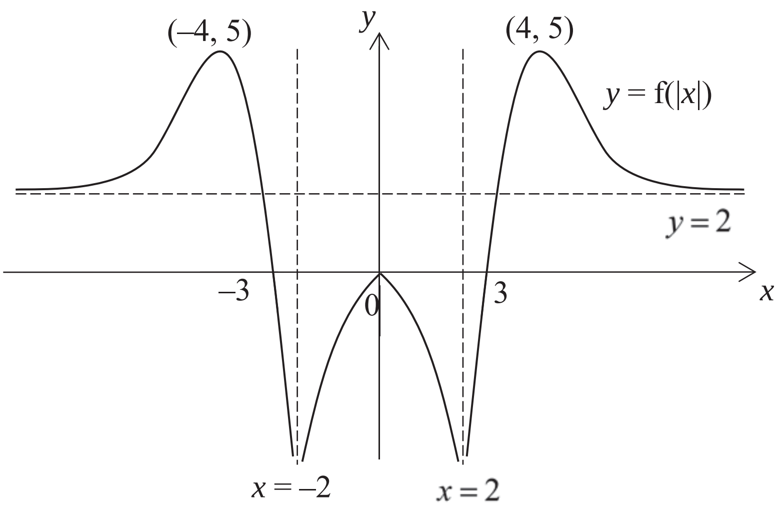


The diagram shows the graph of $y = \mathrm{f}(x)$. The curve crosses the x-axis at $x = 0$ and $x = 3$. It has a turning point at (4, 5) and asymptotes with equations $y = 2$ and $x = 2$.

Showing clearly the coordinates of turning points, axial intercepts and equations of asymptotes where possible, sketch the graphs of

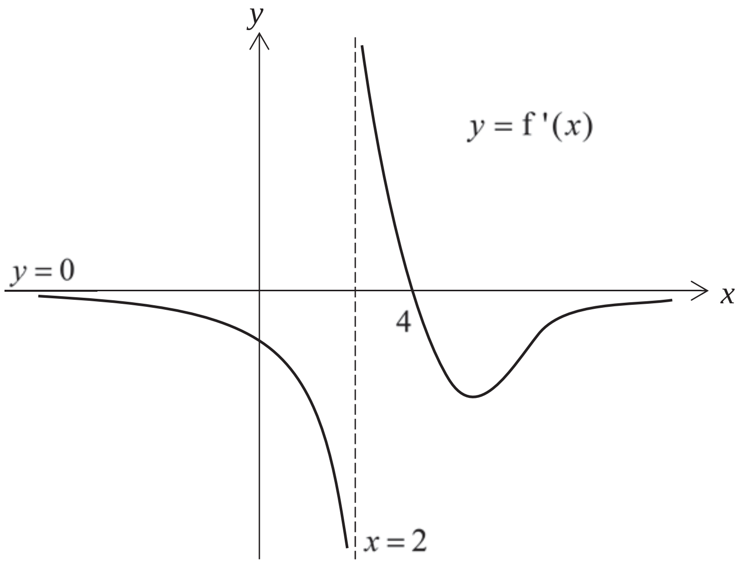
(i) $y = \mathrm{f}(|x|)$; [2]

Solution



(ii) $y = \mathrm{f}'(x)$ [3]

<u>Solution</u>



(b) The curve with equation $y = \mathrm{f}(x)$ is transformed by a stretch with scale factor 2 parallel to the x–axis, followed by a translation of 2 units in the negative x–direction, followed by a translation of 3 units in the positive y–direction. The equation of the resulting curve is $y = \ln\left(\mathrm{e}^3 x\right)$.

Find the equation of the curve $y = \mathrm{f}(x)$. [3]

<u>Solution</u>

Translation of 3 units in the negative y-direction

$$y = \ln\left(\mathrm{e}^3 x\right) \xrightarrow{\text{replace } y \text{ with } y+3} y = \ln\left(\mathrm{e}^3 x\right) - 3 = 3 + \ln x - 3 = \ln x$$

Translation of 2 units in the positive x-direction

$$y = \ln x \xrightarrow{\text{replace } x \text{ with } x-2} y = \ln(x-2)$$

Stretch with scale factor $\frac{1}{2}$ parallel to the x – axis

$$y = \ln(x-2) \xrightarrow{\text{replace } x \text{ with } 2x} y = \ln(2x-2)$$

5 A curve C has parametric equations

$$x = a\sin^2 t, \qquad y = a\cos t,$$

where $0 \le t \le \dfrac{\pi}{2}$ and $a > 0$.

(i) Find the cartesian equation of C, stating clearly any restrictions on the values of x and y. [2]

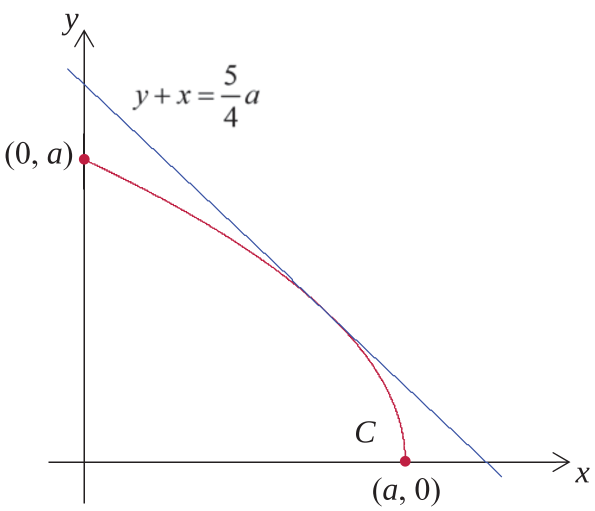
<u>Solution</u>

$\sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad \dfrac{x}{a} + \left(\dfrac{y}{a}\right)^2 = 1$

$ax + y^2 = a^2$ where $0 \le x \le a$ and $0 \le y \le a$

(ii) Sketch C, showing clearly the axial intercepts. [1]

<u>Solution</u>



(iii) The region bounded by C, the line $y+x=\frac{5}{4}a$ and the y-axis is rotated through 2π radians about the y-axis. Show that the exact volume of the solid obtained is $k\pi a^3$ where k is a constant to be determined. [5]

Solution

$$y+x=\frac{5}{4}a \Rightarrow x=\frac{5}{4}a-y$$

$$\therefore a\left(\frac{5}{4}a-y\right)+y^2=a^2$$

$$y^2-ay+\frac{1}{4}a^2=0$$

$$\left(y-\frac{1}{2}a\right)^2=0$$

$$\therefore y=\frac{1}{2}a$$

$$\Rightarrow \text{Volume required} =\frac{1}{3}\pi\left(\frac{5}{4}a-\frac{1}{2}a\right)^2\left(\frac{5}{4}a-\frac{1}{2}a\right)-\pi\int_{\frac{1}{2}a}^{a}\left(\frac{a^2-y^2}{a}\right)^2 \mathrm{d}y$$

$$=\frac{1}{3}\pi\left(\frac{3}{4}a\right)^3-\frac{\pi}{a^2}\int_{\frac{1}{2}a}^{a} a^4-2a^2y^2+y^4 \ \mathrm{d}y$$

$$=\frac{9}{64}\pi a^3-\frac{\pi}{a^2}\left[a^4y-\frac{2}{3}a^2y^3+\frac{1}{5}y^5\right]_{\frac{1}{2}a}^{a}$$

$$=\frac{9}{64}\pi a^3-\frac{\pi}{a^2}\left[a^5-\frac{2}{3}a^5+\frac{1}{5}a^5-\frac{1}{2}a^5+\frac{1}{12}a^5-\frac{1}{160}a^5\right]$$

$$=\frac{29}{960}\pi a^3 \text{ units}^3 \ ; k=\frac{29}{960} \qquad \text{(Shown)}$$

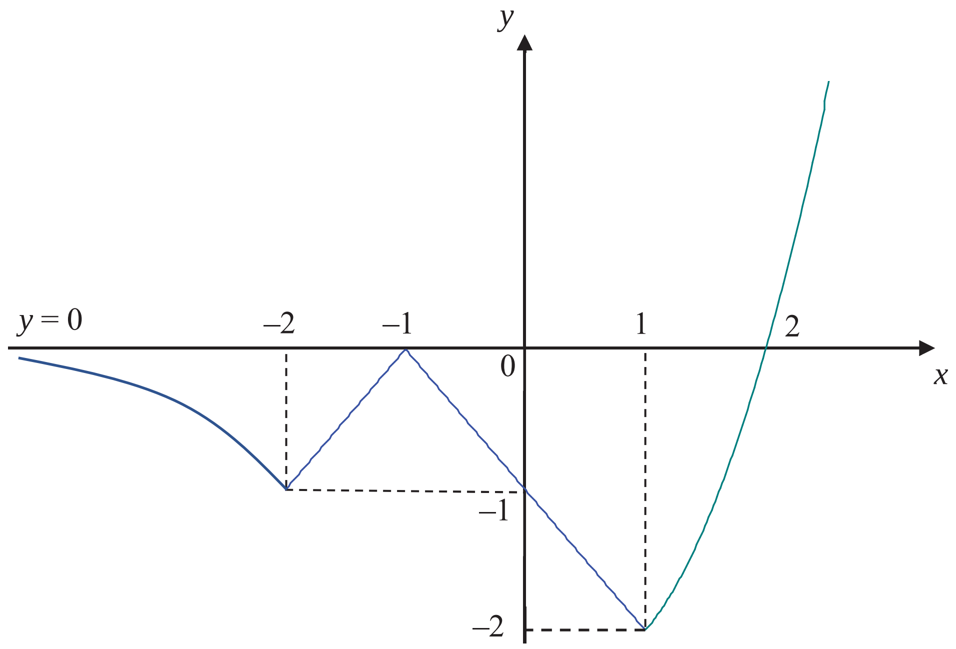
6 A function f is defined by

$$\mathrm{f}(x)=\begin{cases}\dfrac{6}{x-4} & \text{for } x<-2,\\ -|1+x| & \text{for } -2\le x\le 1,\\ x^2-x-2 & \text{for } x>1.\end{cases}$$

(i) Sketch the graph of f. [3]

Solution

(i)



(ii) Evaluate exactly $\int_{-3}^{4}|\mathrm{f}(x)|\,\mathrm{d}x$. [4]

$$\int_{-3}^{4}|\mathrm{f}(x)|\,\mathrm{d}x=\int_{-3}^{-2}\frac{-6}{x-4}\,\mathrm{d}x+\frac{1}{2}(1)(1)+\frac{1}{2}(2)(2)-\int_{1}^{2}x^2-x-2\ \mathrm{d}x+\int_{2}^{4}x^2-x-2\ \mathrm{d}x$$

$$=-6\left[\ln|x-4|\right]_{-3}^{-2}+\frac{5}{2}-\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{1}^{2}+\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{2}^{4}$$

$$=-6\ln\frac{6}{7}+\frac{5}{2}-\left[\frac{8}{3}-\frac{4}{2}-4-\frac{1}{3}+\frac{1}{2}+2\right]+\left[\frac{64}{3}-\frac{16}{2}-8-\frac{8}{3}+\frac{4}{2}+4\right]$$

$$=-6\ln\frac{6}{7}+\frac{37}{3}$$

7 **Do not use a calculator in answering this question.**

The roots of the equation $z^2 + (2-2i)z = -3 - 2i$ are z_1 and z_2.

(i) Find z_1 and z_2 in cartesian form $x + iy$, showing clearly your working. [5]

Solution

(i) Let $z = x + iy$

$\therefore (x+iy)^2 + (2-2i)(x+iy) = -3-2i$

$x^2 + 2xyi - y^2 + 2x + 2yi - 2xi + 2y = -3 - 2i$

$(x^2 + 2x - y^2 + 2y) + i(2xy + 2y - 2x) = -3 - 2i$

By comparing real and imaginary parts,

$x^2 + 2x - y^2 + 2y = -3 \qquad \ldots(1)$

$2xy + 2y - 2x = -2$

$y(x+1) = x - 1$

$$y = \frac{x-1}{x+1} \qquad \ldots(2)$$

Subt (2) into (1):

$$x^2 + 2x - \left(\frac{x-1}{x+1}\right)^2 + 2\left(\frac{x-1}{x+1}\right) = -3$$

$$x^2(x+1)^2 + 2x(x+1)^2 - (x-1)^2 + 2(x-1)(x+1) = -3(x+1)^2$$

$$x^2(x^2+2x+1) + 2x(x^2+2x+1) - (x^2-2x+1) + 2(x^2-1) = -3(x^2+2x+1)$$

$x^4 + 4x^3 + 9x^2 + 10x = 0$

Let $f(x) = x(x^3 + 4x^2 + 9x + 10)$

$f(0) = 0$ and $f(-2) = -8 + 16 - 18 + 10 = 0$

$\Rightarrow x = 0, -2$ are roots of $x^4 + 4x^3 + 9x^2 + 10x = 0$

When $x = 0$, $y = -1$ and when $x = -2$, $y = 3$

$\therefore z_1 = -i$ and $z_2 = -2 + 3i$

(ii) The complex numbers z_1 and z_2 are also roots of the equation

$$z^4 + 4z^3 + 14z^2 + 4z + 13 = 0.$$

Find the other roots of the equation, explaining clearly how the answers are obtained. [2]

<u>Solution</u>

Since all the coefficients of $z^4 + 4z^3 + 14z^2 + 4z + 13 = 0$ are real, the other roots of the equation must be complex conjugates of $-\mathrm{i}$ and $-2 + 3\mathrm{i}$.

Hence the other two roots are i and $-2 - 3\mathrm{i}$.

(iii) Using your answer in part **(i)**, solve $z^2 + (2 + 2\mathrm{i})z = 3 + 2\mathrm{i}$. [2]

<u>Solution</u>

$z^2 + (2 - 2\mathrm{i})z = -3 - 2\mathrm{i}$

$-z^2 + (-2 + 2\mathrm{i})z = 3 + 2\mathrm{i}$

$-z^2 + (2\mathrm{i}^2 + 2\mathrm{i})z = 3 + 2\mathrm{i}$

$(\mathrm{i}z)^2 + (2\mathrm{i} + 2)\mathrm{i}z = 3 + 2\mathrm{i}$

Hence from part (i), $-\mathrm{i}z = -\mathrm{i}$ or $-2 + 3\mathrm{i}$

$$\Rightarrow\ z = 1 \text{ or } \frac{-2+3\mathrm{i}}{-\mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}} = -3 - 2\mathrm{i}$$

8 **(i)** Show that $\dfrac{3r+4}{(r+2)(r+1)r}=\dfrac{A}{r+2}+\dfrac{B}{r+1}+\dfrac{C}{r}$ where A, B and C are constants to be determined. [1]

<u>Solution</u>

$$\frac{3r+4}{(r+2)(r+1)r}=\frac{A}{r+2}+\frac{B}{r+1}+\frac{C}{r}$$

By cover-up rule, $A=\dfrac{-2}{(-1)(-2)}=-1, B=\dfrac{1}{(1)(-1)}=-1, C=\dfrac{4}{(2)(1)}=2$

(ii) Find the sum to n terms of

$$\frac{7}{3\times2\times1}+\frac{10}{4\times3\times2}+\frac{13}{5\times4\times3}+\ldots$$

(There is no need to express your answer as a single algebraic fraction). [4]

<u>Solution</u>

$$\text{The sum to } n\text{th terms } = \sum_{r=1}^{n}\frac{3r+4}{(r+2)(r+1)r}$$

$$=\sum_{r=1}^{n}\left[-\frac{1}{r+2}-\frac{1}{r+1}+\frac{2}{r}\right]$$

$$=\left[\begin{array}{lll}
-\cancel{\frac{1}{3}} & -\frac{1}{2} & +2 \\
-\cancel{\frac{1}{4}} & -\cancel{\frac{1}{3}} & +1 \\
-\frac{1}{5} & -\cancel{\frac{1}{4}} & +\cancel{\frac{2}{3}} \\
+\ldots & & \\
-\frac{1}{n+1} & -\cancel{\frac{1}{n}} & +\cancel{\frac{2}{n-1}} \\
-\frac{1}{n+2} & -\frac{1}{n+1} & +\cancel{\frac{2}{n}}
\end{array}\right]$$

$$=\frac{5}{2}-\frac{1}{n+2}-\frac{2}{n+1}$$

(iii) Hence show that $\displaystyle\sum_{r=5}^{n+3} \frac{3r-5}{(r-1)(r-2)(r-3)} < \frac{4}{3}$. [3]

Solution

$$\sum_{r=5}^{n+3} \frac{3r-5}{(r-1)(r-2)(r-3)}$$

$$= \sum_{r+3=5}^{r+3=n+3} \frac{3(r+3)-5}{(r+3-1)(r+3-2)(r+3-3)}$$

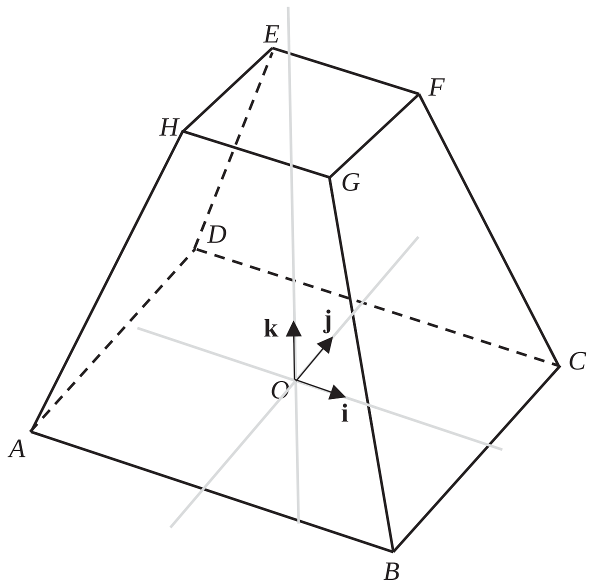
$$= \sum_{r=2}^{r=n} \frac{3r+4}{(r+2)(r+1)(r)}$$

$$= \sum_{r=1}^{n} \frac{3r+4}{(r+2)(r+1)(r)} - \frac{3(1)+4}{(3)(2)(1)}$$

$$= \left[\frac{5}{2} - \frac{1}{n+2} - \frac{2}{n+1}\right] - \frac{7}{6}$$

$$= \frac{4}{3} - \left[\frac{1}{n+2} + \frac{2}{n+1}\right] < \frac{4}{3} \text{ since } \frac{1}{n+2} + \frac{2}{n+1} > 0 \text{ for all } n.$$

9



The diagram above shows an object with O at the centre of its rectangular base $ABCD$ where AB = 8 cm and BC = 4 cm. The top side of the object, $EFGH$ is a square with side 2 cm long and is parallel to the base. The centre of the top side is vertically above O at a height of h cm.

(i) Show that the equation of the line BG may be expressed as $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ h \end{pmatrix}$,

where t is a parameter. [1]

<u>Solution</u>

$B(4,-2,0)$ and $G(1,-1,h)$

$$\overrightarrow{BG} = \begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ h \end{pmatrix}$$

$$l_{BG} : \underset{\sim}{r} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ h \end{pmatrix}, t \in \mathbb{R}$$

(ii) Find the sine of the angle between the line *BG* and the rectangular base *ABCD* in terms of *h*. [2]

Solution

Let θ be the angle between the line *BG* and the rectangular base *ABCD*.

$$\sin\theta = \frac{1}{\sqrt{10+h^2}}\begin{pmatrix}-3\\1\\h\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\sin\theta = \frac{h}{\sqrt{10+h^2}}$$

It is given that $h = 6$.

(iii) Find the cartesian equation of the plane $BCFG$. [3]

Solution

A normal perpendicular to plane *BCFG* is

$$\underset{\sim}{n} = \overrightarrow{BG}\times\overrightarrow{GF} = \begin{pmatrix}-3\\1\\6\end{pmatrix}\times\begin{pmatrix}0\\2\\0\end{pmatrix} = \begin{pmatrix}-12\\0\\-6\end{pmatrix} = -6\begin{pmatrix}2\\0\\1\end{pmatrix}$$

Equation of plane *BCFG* is $\underset{\sim}{r}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\0\end{pmatrix}$

$$\underset{\sim}{r}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix} = 8$$

Cartesian equation of plane is $2x + z - 8 = 0$

(iv) Find the shortest distance from the point *A* to the plane *BCFG*. [2]

Solution

Shortest distance from point *A* to plane *BCFG* $= \dfrac{\left|\underset{\sim}{a}\cdot\underset{\sim}{n} - d\right|}{\left|\underset{\sim}{n}\right|}$

$$= \frac{1}{\sqrt{5}}\left|\begin{pmatrix}-4\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix} - 8\right| = \frac{16}{\sqrt{5}} = \frac{16\sqrt{5}}{5} \text{ units}$$

(v) The line l, which passes through the point A, is parallel to the normal of plane $BCFG$. Given that, the line l intersects the plane $BCFG$ at a point M, use your answer in part (iv) to find the shortest distance from point M to the rectangular base $ABCD$. [2]

<u>Solution</u>

$$\overrightarrow{AM} = \frac{16}{\sqrt{5}}\hat{\underset{\sim}{n}} = \frac{16}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix}\right) = \frac{16}{5}\begin{pmatrix}2\\0\\1\end{pmatrix}$$

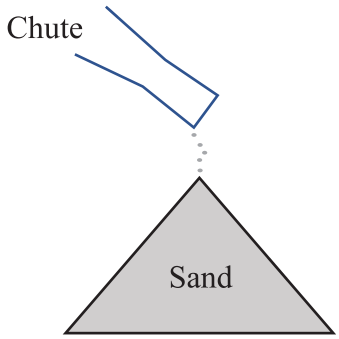
Shortest distance from M to the base = z-coordinate of $\overrightarrow{AM}$

$$= \frac{16}{5}$$

10 Commonly used in building materials, sand is the second largest world resource used by humans after water. To reduce the environmental impacts of sand mining, an alternative approach is to make "sand" by crushing rock.

A machine designed for this purpose produces sand in large quantities. Sand falling from the output chute of the machine forms a pile in the shape of a right circular cone such that the height of the cone is always equal to $\frac{4}{3}$ of the radius of its base.

[Volume of a cone = $\frac{1}{3}\pi r^2 h$ and curved surface area of a cone = $\pi r l$ where r is the radius of the base area, h is the height of the cone and l is the slant length of the cone]



A machine operator starts the machine.

(a) **(i)** Given that V and A denote the volume and the curved surface area of the conical pile respectively, write down V and A in terms of r, the radius of its base. [2]

Solution

$$V = \frac{1}{3}\pi r^2\left(\frac{4}{3}r\right) = \frac{4}{9}\pi r^3$$

$$A = \pi r l$$

$$= \pi r\sqrt{r^2 + h^2}$$

$$= \pi r\sqrt{r^2 + \frac{16}{9}r^2} = \frac{5}{3}\pi r^2$$

(ii) Hence show that the rate of change of A with respect to V is inversely proportional to the radius of the conical pile. [3]

Solution

$$\frac{dV}{dr} = \frac{4}{9}\pi\left(3r^2\right) = \frac{4}{3}\pi r^2 \quad \text{and} \quad \frac{dA}{dr} = \frac{5}{3}\pi\left(2r\right) = \frac{10}{3}\pi r$$

$$\frac{\mathrm{d}A}{\mathrm{d}V} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}V}$$

$$= \frac{10}{3}\pi r \times \frac{1}{\frac{4}{3}\pi r^2}$$

$$= \frac{5}{2r} \qquad \text{(Shown)}$$

An architect is tasked to design sand-lined walking paths in a large park. He decides to base his design of the paths on the shape of astroids, which are shapes with equations

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}} \ (k > 0).$$

On a piece of graph paper, he sketches an astroid with the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}}$.

(b) The tangent at a point $P(x_1, y_1)$ on the curve meets the x-axis at Q and the y-axis at R. Show that the length of QR is independent of where P lies on the curve. [7]

Solution

(ii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}}$

Differentiating wrt x, $\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$\therefore$ the equation of the tangent at $P(x_1, y_1)$ is $y - y_1 = \left(-\frac{y_1}{x_1}\right)^{\frac{1}{3}}(x - x_1)$

When $x = 0$, $y - y_1 = \left(-\frac{y_1}{x_1}\right)^{\frac{1}{3}}(-x_1)$

$$y = y_1^{\frac{1}{3}} x_1^{\frac{2}{3}} + y_1$$

When $y = 0$, $\quad -y_1 = \left(-\frac{y_1}{x_1}\right)^{\frac{1}{3}}(x - x_1)$

$$x = y_1^{\frac{2}{3}} x_1^{\frac{1}{3}} + x_1$$

The length of QR $= \sqrt{x^2 + y^2}$

$$= \sqrt{\left(y_1^{\frac{2}{3}} x_1^{\frac{1}{3}} + x_1\right)^2 + \left(y_1^{\frac{1}{3}} x_1^{\frac{2}{3}} + y_1\right)^2}$$

$$= \sqrt{x_1^{\frac{2}{3}}\left(y_1^{\frac{2}{3}} + x_1^{\frac{2}{3}}\right)^2 + y_1^{\frac{2}{3}}\left(x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}}\right)^2}$$

$$= \sqrt{x_1^{\frac{2}{3}}\left(k^{\frac{2}{3}}\right)^2 + y_1^{\frac{2}{3}}\left(k^{\frac{2}{3}}\right)^2}$$

$$= k^{\frac{2}{3}}\sqrt{x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}}}$$

$$= k^{\frac{2}{3}}\sqrt{k^{\frac{2}{3}}}$$

$= k$ which is a constant.

Hence, the length of QR is independent of where P lies on the curve. (Shown)

11 **(a)** Find the sum of all integers between 200 and 1000 (both inclusive) that are not divisible by 7. [4]

Solution

Sum required $= (200 + 201 + 202 + \ldots + 1000) - (203 + 210 + 217 + \ldots + 994)$

$= \dfrac{1000-200+1}{2}(200+1000) - 7(29 + 30 + 31 + \ldots + 142)$

$= 480600 - 7\left(\dfrac{142-29+1}{2}\right)(29+142)$

$= 412371$

(b)

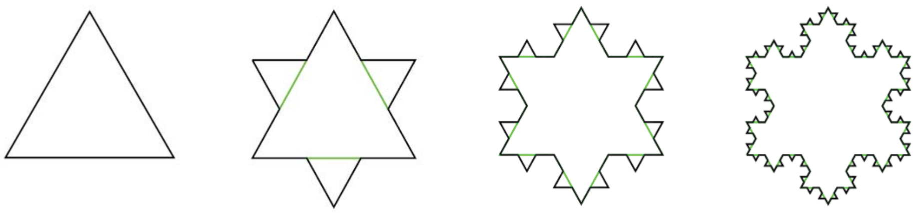


Fig. 1 **Fig. 2** **Fig. 3** **Fig. 4**

Snowflakes can be constructed by starting with an equilateral triangle (Fig. 1), then repeatedly altering each line segment of the resulting polygon as follows:

1. Divide each outer line segments into three segments of equal length.
2. Add an equilateral triangle that has the middle segment from step 1 as its base.
3. Remove the line segment that is the base of the triangle from step 2.
4. Repeat the above steps for a number of iterations, *n*.

The 1st, 2nd and 3rd iteration produces the snowflakes in Fig. 2, Fig. 3 and Fig. 4 respectively.

(i) If a_0 denotes the area of the original triangle and the area of each new triangle added in the n^{th} iteration is denoted by a_n, show that

$$a_n = \frac{1}{9}a_{n-1} \text{ for all positive integers } n. \qquad [2]$$

Solution

Each new triangle added in the n^{th} iteration is similar to the triangle added in the previous iteration (AAA).

Since the length of a side of the new triangle is $\frac{1}{3}$ of the length of a side of the triangle in the previous iteration, $a_n = \left(\frac{1}{3}\right)^2 a_{n-1} = \frac{1}{9}a_{n-1}$ (Shown)

(ii) Write down the number of sides in the polygons in Fig.1, Fig. 2 and Fig. 3, and deduce, with clear explanations, that the number of new triangles added in the n^{th} iteration is $T_n = k\left(4^n\right)$ where k is a constant to be determined. [2]

Solution

No of sides in the polygon (Fig 1) = 3, No of sides in the polygon (Fig 2) = 12

No of sides in the polygon (Fig 3) = 48

GP with first term 3 and common ratio 4

Since there is 1 new triangle for every side in the next iteration, the number of new triangles added in the n^{th} iteration = the number of sides in the polygon after the $(n-1)^{\text{th}}$ iteration

i.e. $T_1 = 3$, $T_2 = 12$, $T_3 = 48$, …

$\therefore T_n = 3(4)^{n-1} = \frac{3}{4}(4)^n$ and $k = \frac{3}{4}$

(iii) Find the total area of triangles added in the n^{th} iteration, A_n in terms of a_0 and n. [2]

Solution

$$A_n = T_n \times a_n = \frac{3}{4}\left(4^n\right)\times\left(\frac{1}{9}a_{n-1}\right)$$

$$= \frac{3}{4}\left(4^n\right)\times\left(\underbrace{\frac{1}{9}\times\frac{1}{9}\times\frac{1}{9}\times\ldots\times\frac{1}{9}}_{n\text{ terms}}a_0\right) = \frac{3}{4}\left(\frac{4}{9}\right)^n a_0 \text{ units}^2$$

(iv) Show that the total area of the snowflake produced after the n^{th} iteration is $a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]$ units^2. [3]

Solution

Total area required $= a_0+\sum_{r=1}^{n}\frac{3}{4}\left(\frac{4}{9}\right)^r a_0$

$$= a_0+\frac{3}{4}a_0\frac{\frac{4}{9}\left(1-\frac{4}{9}^n\right)}{1-\frac{4}{9}}$$

$$= a_0\left[1+\frac{3}{5}\left(1-\frac{4}{9}^n\right)\right]$$

$$= a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right] \qquad \text{(Shown)}$$

The snowflake formed when the above steps are followed indefinitely is called the Koch snowflake.

(v) Determine the least number of iterations needed for the area of the snowflake to exceed 99% of the area of a Koch snowflake. [3]

Solution

Area of a Koch snowflake $= \lim_{n\to\infty} a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]=\frac{8}{5}a_0$

Consider $a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]>0.99\times\frac{8}{5}a_0$

$$0.016-\frac{3}{5}\left(\frac{4}{9}\right)^n>0$$

From GC,

n	$0.016-\frac{3}{5}\left(\frac{4}{9}\right)^n$
4	$-0.007<0$
5	$0.0056>0$
6	$0.0114>0$

Hence the least value of n is 5.

ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Paper 2 (100 marks)

18 September 2019

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

This document consists of 20 printed pages and 4 blank pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1 Given that the curve $y = \dfrac{ax^2 + bx + c}{x-1} = \mathrm{f}(x)$ passes through the points $(-1, -3)$, $\left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$ and $\left(5, \dfrac{33}{2}\right)$, find the values of a, b and c. [3]

Hence find the exact range of values of x for which the gradient of $y = \mathrm{f}(x)$ is positive. [3]

2 The functions f and g are defined by:

$$\mathrm{f} : x \mapsto \left|ax - x^2\right|, \qquad x \in \mathbb{R}$$

$$\mathrm{g} : x \mapsto a\mathrm{e}^{-x} + 1, \qquad x \geq 0$$

where a is a positive constant and $a \geq 5$.

(i) Sketch the graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{f}(x)$ on separate diagrams.

Hence find the range of the composite function fg, leaving your answer in exact form in terms of a. [5]

(ii) Given that the domain of f is restricted to the subset of $\mathbb{R}$ for which $x \geq k$, find the smallest value of k, in terms of a, for which f^{-1} exists. Hence, without finding the expression for $\mathrm{f}^{-1}(x)$, sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on a single diagram, stating the exact coordinates of any points of intersection with the axes. [3]

(iii) With the restricted domain of f in part (ii), find with clear working, the set of values of x such that $\mathrm{f}\,\mathrm{f}^{-1}(x) = \mathrm{f}^{-1}\,\mathrm{f}(x)$. [2]

[Turn Over

3 A student is investigating a chemical reaction between two chemicals A and B. In his experiments, he found that the reaction produces a new product C which does not react with A and B after being formed.

His experiments also suggest that the rate at which the product, C is formed is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=(a-x)\sqrt{b-x},$$

where a and b are the initial concentrations (gram/litre) of A and B just before the reaction started and x is the concentration (gram/litre) of C at time t (mins).

(i) If $a = b = 9$, solve the differential equation to show that

$$x = 9-\frac{36}{(3t+2)^2}.$$ [4]

(ii) Sketch the graph of x and describe the behaviour of x as t increases. Find the amount of time needed for the concentration of C to reach 4.5 gram/litre. [3]

It is now given that $a > b$ instead.

(iii) Solve the differential equation to find t in terms of x, a and b by using the substitution $u = \sqrt{b-x}$ for $0 \le x < b$. [6]

4 The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

A small object, P, is placed on the straight line passing through two light sources A and B that are eight metres apart. It is known that the light sources are similar except that S, the strength of light source A, is a positive constant and is three times that of light source B. It can be assumed that I, the total illumination of the object P by the two light source is the sum of the illumination due to each light source.

The distance between the object P and light source A is denoted by x.

(i) Show that when P is between A and B,

$$I = kS\left[\frac{1}{x^2}+\frac{1}{\alpha(\beta-x)^2}\right]$$ where k is a positive constant and,

α and β are constants to be determined. [1]

(ii) Show that there is only one value of x that will give a stationary value of I. Determine this value and the nature of the stationary value without the use of a graphic calculator. [5]

It is now given that $I=\dfrac{4x^2-48x+192}{x^2\left(x^2-16x+64\right)}$ for $0 < x < 8$.

(iii) Sketch the graph of I, indicating clearly any equations of asymptote and coordinates of the stationary point. [3]

(iv) It is desired to place P such that $I = \sqrt{\lambda - (x - x_1)^2}$ where x_1 is the value of x found in part **(ii)** and λ is a positive constant. Find the value of λ if there is only one possible position to place P. [2]

Section B: Statistics [60 marks]

5 Greenhouse gases generated by Singapore comes mainly from the burning of fossil fuels to generate energy for industries, buildings, households and transportation.

The following table* shows the total greenhouse gas emissions, X (kilotonnes of CO_2), and the total deaths in Singapore, Y from 2007 to 2016.

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Total Greenhouse Gas Emissions (X)	42613	41606	43100	47063	49930	48712	50299	50908	51896	51519
Total Deaths (Y)	17140	17222	17101	17610	18027	18481	18938	19393	19862	20017

**Extracted from www.singstat.gov.sg*

(i) Draw a scatter diagram for these values, labelling the axes and state the product moment correlation between X and Y. Use your diagram to explain why the best model for the relationship between X and Y may not be given by $Y = aX + b$, where a and b are constants. [3]

(ii) Comment on the correctness of the following statement.

"Since there exists a high positive linear correlation between total greenhouse gas emission and total deaths, the deaths are a result of the greenhouse gas emissions." [1]

(iii) Explain clearly which of the following equations, where a and b are constants, provides the more accurate model of the relationship between X and Y.

(A) $Y = aX^4 + b$

(B) $\lg Y = aX + b$ [3]

The Government has pledged that Singapore's greenhouse gas emissions will peak around 2030 at the equivalent of 65 million tonnes of carbon dioxide.

(iv) Using the model you have chosen in part **(iii)**, write down the equation for the relationship, giving the regression coefficients correct to 5 significant figures. Hence find the estimated total deaths in the year 2030 if Singapore's pledge is fulfilled and comment on the reliability of the estimate. [3]

[Turn Over

6 Birth weight can be used to predict short and long-term health complications for babies. Studies show that the birth weight of babies born to mothers who do not smoke in a certain hospital can be assumed to follow a normal distribution with mean 3.05 kg and variance σ^2 kg^2.

The hospital classifies babies based on their birth weight as shown in the table below.

Birth weight	Classification
Less than 1.5 kg	Very low birth weight
1.5 kg to 2.5 kg	Low birth weight
2.5 kg to 4.0 kg	Normal birth weight
More than 4.0 kg	High birth weight

(i) A sample showed that 20.2% of the babies born to mothers who do not smoke have low birth weight. If this is true for the entire population, find two possible values of σ, corrected to 2 decimal places. Explain clearly why one of the values of σ found should be rejected. [4]

Studies also show that babies born to mothers who smoke have a lower mean birth weight of 2.80 kg.

For the remaining of the question, you may assume the birth weight of babies born to mothers who smoke is also normally distributed and that the standard deviations for the birth weight of babies born to mothers who do not smoke and mothers who smoke are both 0.86 kg.

(ii) Three pregnant mothers-to-be who smoke are randomly chosen. Find the probability that all their babies will not be of normal birth weight. State an assumption that is needed for your working. [3]

(iii) Find the probability that the average birth weight of two babies from mothers who do not smoke differs from twice the birth weight of a baby from a mother who smokes by less than 2 kg. [3]

Babies whose birth weight not classified as normal will have to remain in hospital for further observation until their condition stabilises. Depending on the treatment received and length of stay, the mean hospitalisation cost per baby is \$2800 and standard deviation is \$500.

(iv) Find the probability that the average hospitalisation cost from a random sample of 50 babies, whose birth weight not classified as normal, exceeds \$3000. [2]

7 Three cards are to be drawn at random without replacement from a pack of six cards numbered 0, 0, 1, 1, 1, 1. The random variables X_1, X_2 and X_3 denotes the numbers on the first, second and third card respectively.

(i) If $Y = X_1 - X_2 + X_3$, show that $\text{P}(Y = 1) = \dfrac{1}{3}$ and find the probability distribution of Y. [3]

(ii) Find the values of $\text{E}(Y)$ and $\text{Var}(Y)$. [2]

(iii) Find the probability that the difference between Y and $\text{E}(Y)$ is less than one standard deviation of Y. [2]

8 A factory makes a certain type of muesli bar which are packed in boxes of 20. The factory claims that each muesli bar weigh at least 200 grams. Muesli bars that weigh less than 200 grams are sub-standard. It is known that the probability that a muesli bar is sub-standard is 0.02 and the weight of a muesli bar is independent of other muesli bars.

(i) Find the probability that, in a randomly chosen box of muesli bars, there are at least one but no more than five sub-standard muesli bars. [2]

The boxes of muesli bars are sold in cartons of 12 boxes each.

(ii) Find the probability that, in a randomly chosen carton, at least 75% of the boxes will have at most one sub-standard muesli bar. [2]

As part of quality control for each batch of muesli bars produced daily, one box of muesli bars is randomly selected to be tested. If all the muesli bars in the box weigh at least 200 grams, the batch will be released for sales. If there are more than one sub-standard muesli bar, the batch will not be released. If there is exactly one sub-standard muesli bar, a second box of muesli bars will be tested. If all the muesli bars in the second box weigh at least 200 grams, the batch will be released. Otherwise, the batch will not be released.

(iii) Find the probability that the batch is released on a randomly selected day. [2]

(iv) If the factory operates 365 days in a year, state the expected number of days in the year when the batch will not be released. [1]

To entice more people to buy their muesli bars, a scratch card is inserted into each box. The probability that a scratch card will win a prize is p. It may be assumed that whether a scratch card will win a prize or not, is independent of the outcomes of the other scratch cards.

Jane buys a carton of muesli bars.

(v) Write in terms of p, the probability that Jane win 2 prizes. [1]

The probability that Jane win 2 prizes is more than twice the probability that she win 3 prizes.

(vi) Find in exact terms, the range of values that p can take. [2]

[Turn Over

9 In preparation for an upcoming event, a student management team is considering having meetings on this Friday, Saturday and Sunday. The probability that the meeting is conducted on Friday is $\frac{1}{6}$. On each of the other days, the probability that a meeting is conducted when a meeting has already been conducted on the previous day, is $\frac{2}{5}$ and the probability that a meeting is conducted, when a meeting has not been conducted on the previous day, is $\frac{1}{3}$.

Find the probability that

(i) a meeting is conducted on Sunday, [3]

(ii) a meeting is not conducted on Friday given that there is a meeting on Sunday. [3]

The student management team consists of 7 girls and 3 boys. At one of the meetings, they sit at a round table with 10 chairs.

(iii) Find the probability that the girls are seated together. [2]

Two particular boys are absent from the meeting.

(iv) Find the probability that two particular girls do not sit next to each other. [3]

10 Along a 3km stretch of a road, the speed in km/h of a vehicle is a normally distributed random variable T. Over a long period of time, it is known that the mean speed of vehicles traveling along that stretch of the road is 90.0 km/h. To deter speeding, the traffic governing body introduced a speed monitoring camera. Subsequently, the speeds of a random sample of 60 vehicles are recorded. The results are summarised as follows.

$$\sum t = 5325, \qquad \sum \left(t - \bar{t}\right)^2 = 2000.$$

(i) Find unbiased estimates of the population mean and variance, giving your answers to 2 decimal places. [2]

(ii) Test, at the 5% significance level, whether the speed-monitoring camera is effective in deterring the speeding of vehicles on the stretch of road. [4]

(iii) In another sample of size n ($n > 30$) that was collected independently, it is given that $\bar{t} = 89.0$. The result of the subsequent test using this information and the unbiased estimate of the population variance in **part (i)** is that the null hypothesis is not rejected. Obtain an inequality involving n, and hence find the largest possible value n can take. [4]

ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Paper 2 (100 marks)

18 September 2019

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

This document consists of 20 printed pages and 4 blank pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1 Given that the curve $y = \dfrac{ax^2+bx+c}{x-1} = \mathrm{f}(x)$ passes through the points $(-1, -3)$, $\left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$ and $\left(5, \dfrac{33}{2}\right)$, find the values of a, b and c. [3]

Solution

$$\frac{a(-1)^2-b+c}{-1-1} = -3 \quad \Rightarrow \quad a-b+c=6 \quad \text{---(1)}$$

$$\frac{a\left(\frac{1}{2}\right)^2+\frac{1}{2}b+c}{\frac{1}{2}-1} = -\frac{3}{2} \quad \Rightarrow \quad \frac{1}{4}a+\frac{1}{2}b+c=\frac{3}{4} \quad \text{---(2)}$$

$$\frac{a(5)^2+5b+c}{5-1} = \frac{33}{2} \quad \Rightarrow \quad 25a+5b+c=66 \quad \text{---(3)}$$

Using GC, $a=3$, $b=-2$, $c=1$.

Hence find the exact range of values of x for which the gradient of $y = \mathrm{f}(x)$ is positive. [3]

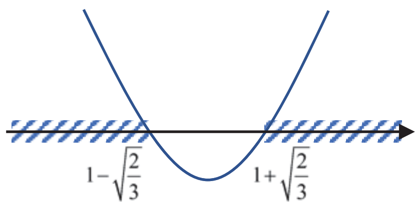
Solution

$$\mathrm{f}'(x) > 0 \quad \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3x^2-2x+1}{x-1}\right) > 0$$

$$\Rightarrow \frac{(x-1)(6x-2)-(3x^2-2x+1)(1)}{(x-1)^2} > 0$$

$$\Rightarrow \frac{3x^2-6x+1}{(x-1)^2} > 0$$

Since $(x-1)^2 > 0$, $3x^2-6x+1 > 0$



$$3(x-1)^2-2>0$$

$$\left(\sqrt{3}x-\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}x-\sqrt{3}+\sqrt{2}\right) > 0$$

$$x > 1+\sqrt{\frac{2}{3}} \quad \text{or} \quad x < 1-\sqrt{\frac{2}{3}}$$

2 The functions f and g are defined by:

$$\mathrm{f}: x \mapsto \left|ax - x^2\right|, \qquad x \in \mathbb{R}$$

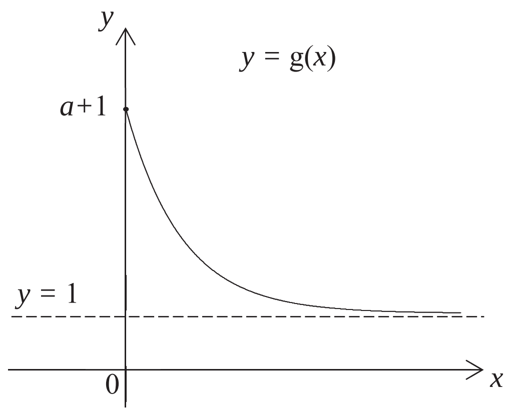
$$\mathrm{g}: x \mapsto a\mathrm{e}^{-x} + 1, \qquad x \geq 0$$

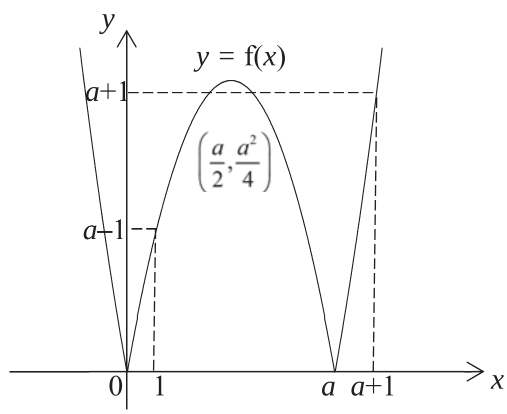
where a is a positive constant and $a \geq 5$.

(i) Sketch the graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{f}(x)$ on separate diagrams.

Hence find the range of the composite function fg, leaving your answer in exact form in terms of a. [5]

Solution





From the graphs,
$R_g = (1, a+1]$

Since $a \geq 5$, $\dfrac{a}{2} \geq \dfrac{5}{2} > 1 \Rightarrow 0 < 1 < \dfrac{a}{2} \Rightarrow \dfrac{a^2}{4} \geq a - 1$

and $a^2 \geq 5a \Rightarrow \dfrac{a^2}{4} \geq \dfrac{5a}{4} = a + \dfrac{1}{4}a \geq a + 1$

$[0, \infty) \xrightarrow{\mathrm{g}} (1, a+1] \xrightarrow{\mathrm{f}} R_{fg}$

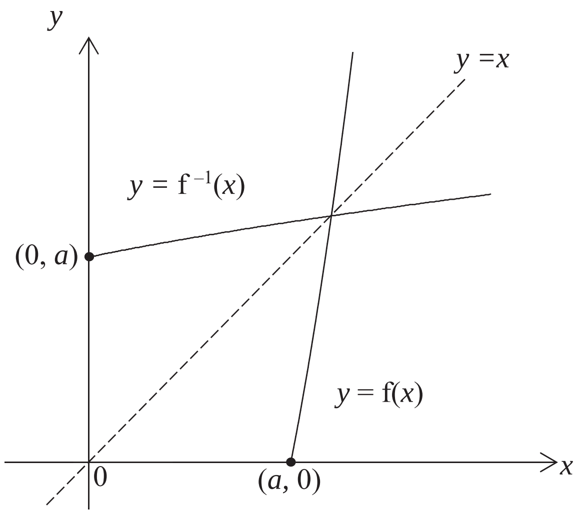
From the graphs, $R_{fg} = \left[0, \dfrac{a^2}{4}\right]$

[Turn Over

(ii) Given that the domain of f is restricted to the subset of $\mathbb{R}$ for which $x \geq k$, find the smallest value of k, in terms of a, for which f^{-1} exists. Hence, without finding the expression for $\mathrm{f}^{-1}(x)$, sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on a single diagram, stating the exact coordinates of any points of intersection with the axes. [3]

<u>Solution</u>

The smallest value of k for which f^{-1} exists = a.



(iii) With the restricted domain of f in part (ii), find with clear working, the set of values of x such that $\mathrm{f}\,\mathrm{f}^{-1}(x) = \mathrm{f}^{-1}\,\mathrm{f}(x)$. [2]

<u>Solution</u>

$\mathrm{f}^{-1}\mathrm{f}(x) = x$ for $x \in D_{\mathrm{f}}$, i.e $x \in [a, \infty)$

$\mathrm{f}\mathrm{f}^{-1}(x) = x$ for $x \in D_{\mathrm{f}^{-1}}$, i.e $x \in [0, \infty)$

For $\mathrm{f}\,\mathrm{f}^{-1}(x) = \mathrm{f}^{-1}\,\mathrm{f}(x)$, $[0,\infty) \cap [a,\infty) = [a,\infty)$

Set of values of x required = $[a, \infty)$

3 A student is investigating a chemical reaction between two chemicals A and B. In his experiments, he found that the reaction produces a new product C which does not react with A and B after being formed.

His experiments also suggest that the rate at which the product, C is formed is given by the differential equation

$$\frac{dx}{dt} = (a-x)\sqrt{b-x},$$

where a and b are the initial concentrations (gram/litre) of A and B just before the reaction started and x is the concentration (gram/litre) of C at time t (mins).

(i) If $a = b = 9$, solve the differential equation to show that

$$x = 9 - \frac{36}{(3t+2)^2}.$$ [4]

Solution

(i) $\frac{dx}{dt} = (9-x)(9-x)^{\frac{1}{2}}$

$\int (9-x)^{-\frac{3}{2}}\, dx = \int 1\, dt$

$-2(-1)(9-x)^{-\frac{1}{2}} = t + c$

When $t = 0$, $x = 0$, $2(9-0)^{-\frac{1}{2}} = 0 + c$

$\therefore\ c = \frac{2}{3}$

$\Rightarrow \frac{2}{\sqrt{9-x}} = t + \frac{2}{3} = \frac{3t+2}{3}$

$\frac{4}{9-x} = \frac{(3t+2)^2}{9}$

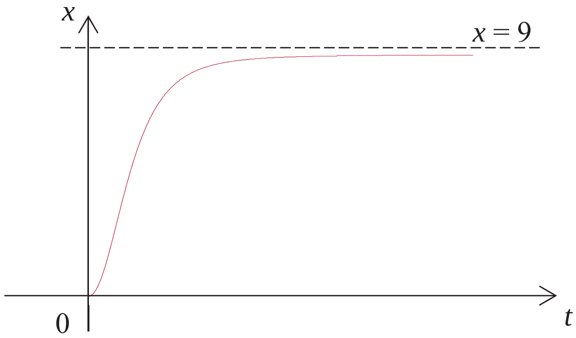
$9 - x = \frac{36}{(3t+2)^2}$

$\therefore\ x = 9 - \frac{36}{(3t+2)^2}$

[Turn Over

(ii) Sketch the graph of x and describe the behaviour of x as t increases. Find the amount of time needed for the concentration of C to reach 4.5 gram/litre. [3]

Solution



[G1 – shape with asymptote (no need for correct value of limit)]

As t increase, x increases and approaches the value of 9 gram/litre.

From GC or otherwise, $t = 0.276$ mins or 16.6 s (3sf) or $\dfrac{2\left(\sqrt{2}-1\right)}{3}$

It is now given that $a > b$ instead.

(iii) Solve the differential equation to find t in terms of x, a and b by using the substitution $u = \sqrt{b-x}$ for $0 \le x < b$. [6]

Solution

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-1}{2\sqrt{b-x}} = -\frac{1}{2u}$$

$$u = \sqrt{b-x} \Rightarrow x = b-u^2, \therefore \frac{\mathrm{d}u}{\mathrm{d}t} \times \frac{\mathrm{d}x}{\mathrm{d}u} = \left(a-b+u^2\right)u$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} \times (-2u) = \left(a-b+u^2\right)u$$

$$-2\int \frac{1}{(a-b)+u^2}\,\mathrm{d}u = \int 1\,\mathrm{d}t$$

$$-\frac{2}{\sqrt{a-b}}\tan^{-1}\frac{u}{\sqrt{a-b}} + C = t$$

When $t = 0$, $x = 0$, $u = \sqrt{b}$, $\therefore 0 = C - \dfrac{2}{\sqrt{a-b}}\tan^{-1}\sqrt{\dfrac{b}{a-b}}$

$$\therefore C = \frac{2}{\sqrt{a-b}}\tan^{-1}\sqrt{\frac{b}{a-b}}$$

$$\Rightarrow t = \frac{2}{\sqrt{a-b}}\tan^{-1}\sqrt{\frac{b}{a-b}} - \frac{2}{\sqrt{a-b}}\tan^{-1}\sqrt{\frac{b-x}{a-b}}$$

4 The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

A small object, P, is placed on the straight line passing through two light sources A and B that are eight metres apart. It is known that the light sources are similar except that S, the strength of light source A, is a positive constant and is three times that of light source B. It can be assumed that I, the total illumination of the object P by the two light source is the sum of the illumination due to each light source.

The distance between the object P and light source A is denoted by x.

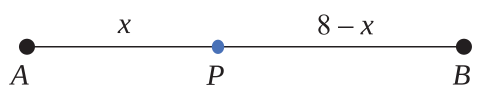
(i) Show that when P is between A and B,

$$I = kS\left[\frac{1}{x^2}+\frac{1}{\alpha\left(\beta-x\right)^2}\right]$$ where k is a positive constant and,

α and β are constants to be determined. [1]

<u>Solution</u>

$$I = \frac{kS}{x^2}+\frac{k\left(\frac{1}{3}S\right)}{\left(8-x\right)^2} \quad \text{where } k>0$$



$$= kS\left[\frac{1}{x^2}+\frac{1}{3\left(8-x\right)^2}\right] \quad \text{so } \alpha = 3 \text{ and } \beta = 8$$

(ii) Show that there is only one value of x that will give a stationary value of I. Determine this value and the nature of the stationary value without the use of a graphic calculator. [5]

<u>Solution</u>

$$\frac{dI}{dx} = kS\left[-\frac{2}{x^3}-2\left(-1\right)\frac{1}{3\left(8-x\right)^3}\right] = 2kS\left[-\frac{1}{x^3}+\frac{1}{3\left(8-x\right)^3}\right]$$

Let $\frac{dI}{dx}=0 \quad \Rightarrow \quad -\frac{1}{x^3}+\frac{1}{3\left(8-x\right)^3}=0$

$$x^3 = 3(8-x)^3$$

$$x = \sqrt[3]{3}\left(8-x\right)$$

$$x = \frac{8\sqrt[3]{3}}{1+\sqrt[3]{3}}$$

Hence I will have a stationary value only when $x = \frac{8\sqrt[3]{3}}{1+\sqrt[3]{3}}$. (Shown)

[Turn Over

$$\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = 2kS\left[\frac{3}{x^4} + \frac{(-3)(-1)}{3(8-x)^4}\right]$$

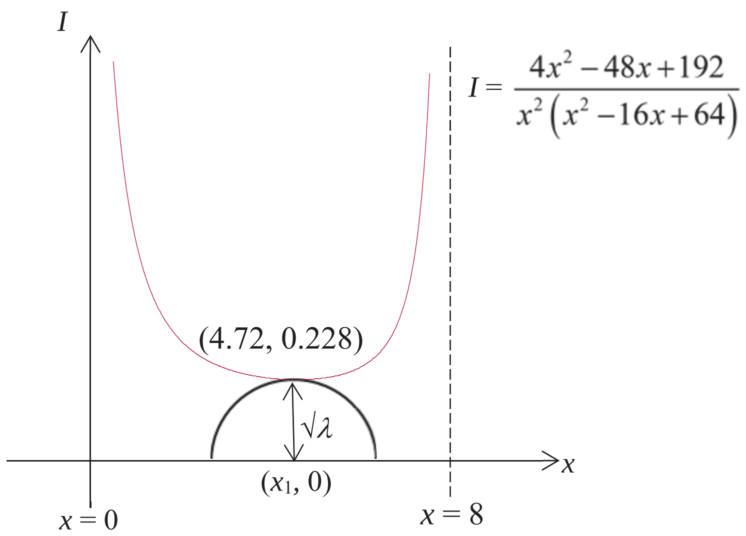
$$= 2kS\left[\frac{3}{x^4} + \frac{1}{(8-x)^4}\right] > 0 \text{ for all } x.$$

$\Rightarrow I$ is minimum when $x = \dfrac{8\sqrt[3]{3}}{1+\sqrt[3]{3}}$.

It is now given that $I = \dfrac{4x^2 - 48x + 192}{x^2\left(x^2 - 16x + 64\right)}$ for $0 < x < 8$.

(iii) Sketch the graph of I, indicating clearly any equations of asymptote and coordinates of the stationary point. [3]

Solution



(iv) It is desired to place P such that $I = \sqrt{\lambda - (x - x_1)^2}$ where x_1 is the value of x found in part **(ii)** and λ is a positive constant. Find the value of λ if there is only one possible position to place P. [2]

Solution

$I = \sqrt{\lambda - (x - x_1)^2}$ is the equation of a semi-circle centred at $(x_1, 0)$ and radius $\sqrt{\lambda}$.

Hence when there is only 1 position to place P, it must happen when $\sqrt{\lambda} = 0.2276094$

i.e. $\lambda = 0.0518$ (3sf)

Section B: Statistics [60 marks]

5 Greenhouse gases generated by Singapore comes mainly from the burning of fossil fuels to generate energy for industries, buildings, households and transportation.

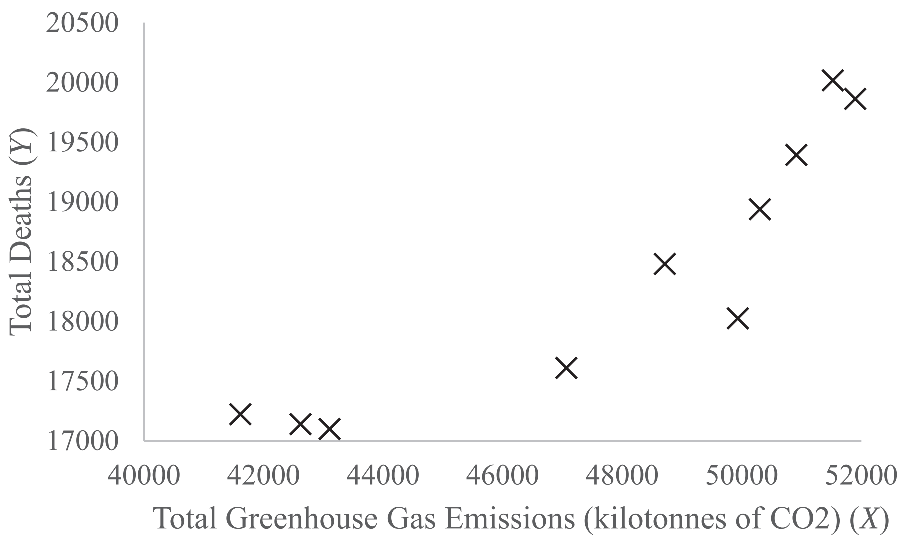
The following table* shows the total greenhouse gas emissions, X (kilotonnes of CO_2), and the total deaths in Singapore, Y from 2007 to 2016.

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Total Greenhouse Gas Emissions (X)	42613	41606	43100	47063	49930	48712	50299	50908	51896	51519
Total Deaths (Y)	17140	17222	17101	17610	18027	18481	18938	19393	19862	20017

**Extracted from www.singstat.gov.sg*

(i) Draw a scatter diagram for these values, labelling the axes and state the product moment correlation between X and Y. Use your diagram to explain why the best model for the relationship between X and Y may not be given by $Y = aX + b$, where a and b are constants. [3]

<u>Solution</u>



From the GC, r = 0.90527 = 0.905 (3 sf)

From the scatter diagram, the points follow a curvilinear trend rather than a linear trend, hence $Y = aX + b$ may not be the best model.

(ii) Comment on the correctness of the following statement.

"Since there exists a high positive linear correlation between total greenhouse gas emission and total deaths, the deaths are a result of the greenhouse gas emissions." [1]

Solution

High positive linear correlation does not indicate causation, hence it is incorrect to state that the deaths are a result of the greenhouse gas emission. Moreover, there are other factors affecting the total deaths in Singapore.

(iii) Explain clearly which of the following equations, where a and b are constants, provides the more accurate model of the relationship between X and Y.

(A) $Y = aX^4 + b$

(B) $\lg Y = aX + b$ [3]

Solution

From GC, r = 0.928 for $Y = aX^4 + b$

r = 0.912 for $\lg Y = aX + b$

Since the product moment correlation coefficient for $Y = aX^4 + b$ is closer to 1 as compared to the one for $\lg Y = aX + b$, $Y = aX^4 + b$ is the better model.

The Government has pledged that Singapore's greenhouse gas emissions will peak around 2030 at the equivalent of 65 million tonnes of carbon dioxide.

(iv) Using the model you have chosen in part **(iii)**, write down the equation for the relationship, giving the regression coefficients correct to 5 significant figures. Hence find the estimated total deaths in the year 2030 if Singapore's pledge is fulfilled and comment on the reliability of the estimate. [3]

Solution

The required equation is $Y = 6.4326 \times 10^{-16} X^4 + 14911$

When $X = 65000$, $Y = 6.4326 \times 10^{-16} (65000)^4 + 14911$

$= 26393.6$

$= 26400$ (3 sf)

The estimate is not reliable as extrapolation is carried out where the linear correlation outside the range of X, i.e. [41606, 51896] may not be valid.

6 Birth weight can be used to predict short and long-term health complications for babies. Studies show that the birth weight of babies born to mothers who do not smoke in a certain hospital can be assumed to follow a normal distribution with mean 3.05 kg and variance σ^2 kg^2.

The hospital classifies babies based on their birth weight as shown in the table below.

Birth weight	Classification
Less than 1.5 kg	Very low birth weight
1.5 kg to 2.5 kg	Low birth weight
2.5 kg to 4.0 kg	Normal birth weight
More than 4.0 kg	High birth weight

(i) A sample showed that 20.2% of the babies born to mothers who do not smoke have low birth weight. If this is true for the entire population, find two possible values of σ, corrected to 2 decimal places. Explain clearly why one of the values of σ found should be rejected. [4]

Solution

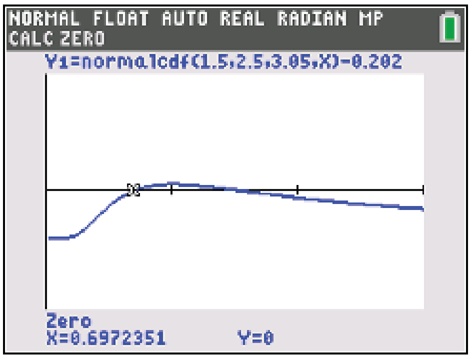
Let X be random variable representing the birth weight of a baby in kg.

$\therefore X \sim \mathrm{N}(3.05,\ \sigma^2)$

$\mathrm{P}(1.5 < X < 2.5) = 0.202$

By plotting the graph of $y = \mathrm{P}(1.5 < X < 2.5) - 0.202$ using GC,

we have $\sigma = 0.70$ (2 dp) or $\sigma = 1.56$ (2 dp)



If $\sigma = 0.70$, $\mathrm{P}(2.5 < X < 4) = 0.70$

If $\sigma = 1.56$, $\mathrm{P}(2.5 < X < 4) = 0.37$

Reject $\sigma = 1.56$ as this would mean that it is very much more likely that a baby will be born with "abnormal" birth weight than normal birth weight.

(Or when $\sigma = 1.56$, $\mathrm{P}(X < 0) = 0.03$ compared with $\mathrm{P}(X < 0) = 6.6\times10^{-6}$ for $\sigma = 0.7$)

Studies also show that babies born to mothers who smoke have a lower mean birth weight of 2.80 kg.

For the remaining of the question, you may assume the birth weight of babies born to mothers who smoke is also normally distributed and that the standard deviations for the birth weight of babies born to mothers who do not smoke and mothers who smoke are both 0.86 kg.

(ii) Three pregnant mothers-to-be who smoke are randomly chosen. Find the probability that all their babies will not be of normal birth weight. State an assumption that is needed for your working. [3]

[Turn Over

Solution

Let Y be r.v. denoting the birth weight of a baby in kg from a mother who smokes.
$Y \sim \mathrm{N}(2.8,\ 0.86^2)$

$$\begin{aligned}\text{Required probability} &= [1-\mathrm{P}(2.5<Y<4)]^3\\ &=(1-0.55493)^3\\ &= 0.088157\\ &= 0.0882 \text{ (3 sf)}\end{aligned}$$

Assumption: The birth weights of babies are independent of one another.

(iii) Find the probability that the average birth weight of two babies from mothers who do not smoke differs from twice the birth weight of a baby from a mother who smokes by less than 2 kg. [3]

Solution

Let $A=\dfrac{X_1+X_2}{2}-2Y$

$\mathrm{E}(A)=3.05-2(2.80)=-2.55$

$\mathrm{Var}(A)=0.5^2\left(0.86^2\times 2\right)+2^2(0.86^2)=3.3282$

$\therefore\ A\sim \mathrm{N}(-2.55,\ 3.3282)$

Required probability $=\ \mathrm{P}(|A|<2)\ =\ \mathrm{P}(-2<A<2)=0.37521$

$=0.375$ (3 s.f.)

Babies whose birth weight not classified as normal will have to remain in hospital for further observation until their condition stabilises. Depending on the treatment received and length of stay, the mean hospitalisation cost per baby is \$2800 and standard deviation is \$500.

(iv) Find the probability that the average hospitalisation cost from a random sample of 50 babies, whose birth weight not classified as normal, exceeds \$3000. [2]

Solution

Let W be random variable representing the hospitalisation charge of a baby.

$\mathrm{E}(W)=2800$ and $\mathrm{Var}(W)=500^2=250000$

Since $n=50$ is large, by Central Limit Theorem,

$\overline{W}=\dfrac{W_1+W_2+\ldots+W_{50}}{50}\sim \mathrm{N}\left(2800,\ \dfrac{500^2}{50}\right)$ approximately

i.e. $\overline{W}\sim \mathrm{N}(2800,\ 5000)$ approximately [M1]

$\Rightarrow\ \mathrm{P}(\overline{W}>3000)=0.00234$ (3.s.f) [A1]

7 Three cards are to be drawn at random without replacement from a pack of six cards numbered 0, 0, 1, 1, 1, 1.

The random variables X_1, X_2 and X_3 denotes the numbers on the first, second and third card respectively.

(i) If $Y = X_1 - X_2 + X_3$, show that $\mathrm{P}(Y = 1) = \frac{1}{3}$ and find the probability distribution of Y. [3]

Solution

$\mathrm{P}(Y = 1) = \mathrm{P}(0, 0, 1) + \mathrm{P}(1, 0, 0) + \mathrm{P}(1, 1, 1)$

$$= \frac{2}{6}\times\frac{1}{5}\times\frac{4}{4}+\frac{4}{6}\times\frac{2}{5}\times\frac{1}{4}+\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}$$

$$= \frac{1}{3} \quad \text{(Shown)}$$

y	-1	0	1	2
$\mathrm{P}(Y = y)$	$\mathrm{P}(0, 1, 0) = \frac{2}{6}\times\frac{4}{5}\times\frac{1}{4}$ $= \frac{1}{15}$	$\mathrm{P}(1, 1, 0) + \mathrm{P}(0, 1, 1)$ $= \frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}+\frac{2}{6}\times\frac{4}{5}\times\frac{3}{4}$ $= \frac{2}{5}$	$\frac{1}{3}$	$\mathrm{P}(1, 0, 1) = \frac{4}{6}\times\frac{2}{5}\times\frac{3}{4}$ $= \frac{1}{5}$

(ii) Find the values of $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. [2]

Solution

$$\mathrm{E}(Y) = -\frac{1}{15}+0+\frac{1}{3}+\frac{2}{5} = \frac{2}{3} = 0.667 \text{ (3sf)}$$

$$\mathrm{Var}(Y) = \frac{1}{15}+0+\frac{1}{3}+\frac{4}{5}-\left(\frac{2}{3}\right)^2 = \frac{34}{45} = 0.756 \text{ (3sf)}$$

(iii) Find the probability that the difference between Y and $\mathrm{E}(Y)$ is less than one standard deviation of Y. [2]

Solution

$$\mathrm{P}\left(\left|Y-\frac{2}{3}\right| < \sqrt{\frac{34}{45}}\right) = \mathrm{P}(Y=0) + \mathrm{P}(Y=1)$$

$$= \frac{2}{5} + \frac{1}{3} = \frac{11}{15} = 0.733 \text{ (3 sf)}$$

8 A factory makes a certain type of muesli bar which are packed in boxes of 20. The factory claims that each muesli bar weigh at least 200 grams. Muesli bars that weigh less than 200 grams are sub-standard. It is known that the probability that a muesli bar is sub-standard is 0.02 and the weight of a muesli bar is independent of other muesli bars.

(i) Find the probability that, in a randomly chosen box of muesli bars, there are at least one but no more than five sub-standard muesli bars. [2]

<u>Solution</u>

Let X be the random variable denoting the number of sub-standard muesli bars, out of 20. $\therefore$ $X \sim \mathrm{B}(20, 0.02)$

$\Rightarrow$ Probability required $= \mathrm{P}(1 \le X \le 5)$

$= \mathrm{P}(X \le 5) - \mathrm{P}(X = 0)$

$= 0.33239$

$= 0.332$ (3 sf)

The boxes of muesli bars are sold in cartons of 12 boxes each.

(ii) Find the probability that, in a randomly chosen carton, at least 75% of the boxes will have at most one sub-standard muesli bar. [2]

<u>Solution</u>

Let Y be the random variable denoting the number of boxes of muesli bars that have at most one sub-standard muesli bars, out of 12 boxes.

$\mathrm{P}(X \le 1) = 0.94010$

$\therefore$ $Y \sim \mathrm{B}(12, 0.94010)$

Probability required $= \mathrm{P}(Y \ge 0.75 \times 12)$

$= \mathrm{P}(Y \ge 9)$

$= 1 - \mathrm{P}(Y \le 8)$

$= 0.99568$

$= 0.996$

As part of quality control for each batch of muesli bars produced daily, one box of muesli bars is randomly selected to be tested. If all the muesli bars in the box weigh at least 200 grams, the batch will be released for sales. If there are more than one sub-standard muesli bar, the batch will not be released. If there is exactly one sub-standard muesli bar, a second box of muesli bars will be tested. If all the muesli bars in the second box weigh at least 200 grams, the batch will be released. Otherwise, the batch will not be released.

(iii) Find the probability that the batch is released on a randomly selected day. [2]

Solution

Probability required $= \mathrm{P}(X_1 = 0) + \mathrm{P}(X_1 = 1)\,\mathrm{P}(X_2 = 0)$

$= 0.84953$

$= 0.850$ (3 sf)

(iv) If the factory operates 365 days in a year, state the expected number of days in the year when the batch will not be released. [1]

Solution

Expected number of days when the batch will not be released

$= 365 - 365 \times 0.84953$

$= 54.9$ (3 sf)

To entice more people to buy their muesli bars, a scratch card is inserted into each box. The probability that a scratch card will win a prize is p. It may be assumed that whether a scratch card will win a prize or not, is independent of the outcomes of the other scratch cards.

Jane buys a carton of muesli bars.

(v) Write in terms of p, the probability that Jane win 2 prizes. [1]

Solution

Let W be the random variable denoting the number of prizes Jane wins, out of 12.

$\therefore W \sim \mathrm{B}(12, p)$

$\mathrm{P}(W = 2) = \binom{12}{2} p^2 (1-p)^{10}$

$= 66 p^2 (1-p)^{10}$

The probability that Jane win 2 prizes is more than twice the probability that she win 3 prizes.

(vi) Find in exact terms, the range of values that p can take. [2]

Solution

$\text{P}(W=2) > 2\,\text{P}(W=3)$

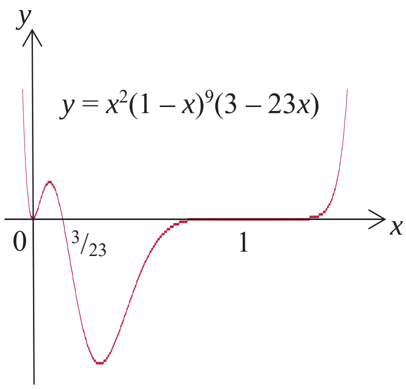
$66p^2(1-p)^{10} > 2\binom{12}{3}p^3(1-p)^9$

$66p^2(1-p)^{10} - 440p^3(1-p)^9 > 0$

$p^2(1-p)^9\left[3(1-p)-20p\right] > 0$

$p^2(1-p)^9\left[3-23p\right] > 0$

Since $0 \le p \le 1$, $0 < p < \dfrac{3}{23}$



9 In preparation for an upcoming event, a student management team is considering having meetings on this Friday, Saturday and Sunday. The probability that the meeting is conducted on Friday is $\frac{1}{6}$. On each of the other days, the probability that a meeting is conducted when a meeting has already been conducted on the previous day, is $\frac{2}{5}$ and the probability that a meeting is conducted, when a meeting has not been conducted on the previous day, is $\frac{1}{3}$.

Find the probability that

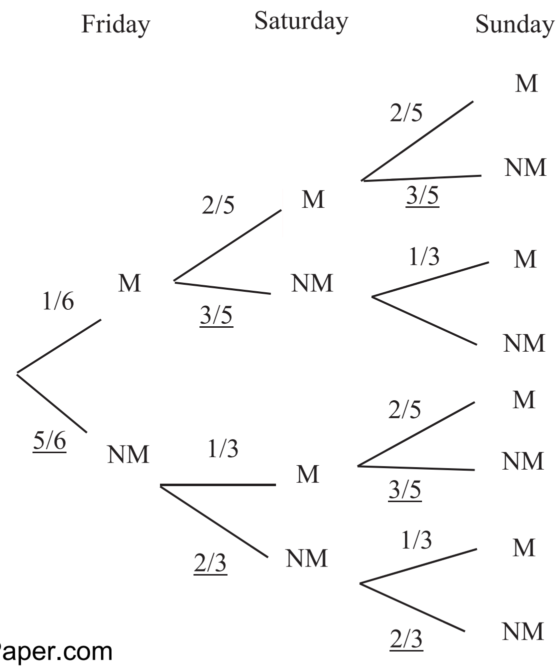
(i) a meeting is conducted on Sunday, [3]

Solution

P(meeting is conducted on Sunday)

$= \frac{1}{6}\left[\frac{2}{5}\cdot\frac{2}{5}+\frac{3}{5}\cdot\frac{1}{3}\right] + \frac{5}{6}\left[\frac{1}{3}\cdot\frac{2}{5}+\frac{2}{3}\cdot\frac{1}{3}\right]$

$= \frac{481}{1350}$ (or 0.356)



(ii) a meeting is not conducted on Friday given that there is a meeting on Sunday. [3]

<u>Solution</u>

P(no meeting on Friday | meeting on Sunday)

$$= \frac{\mathrm{P}\left(\text{no meeting on Friday and meeting on Sunday}\right)}{\mathrm{P}\left(\text{no meeting on Friday}\right)}$$

$$= \frac{\frac{5}{6}\left[\frac{1}{3}\cdot\frac{2}{5}+\frac{2}{3}\cdot\frac{1}{3}\right]}{\left(\frac{481}{1350}\right)}$$

$$= \frac{400}{481}$$

$$= 0.83160$$

$$= 0.832 \text{ (correct to 3 s.f.)}$$

The student management team consists of 7 girls and 3 boys. At one of the meetings, they sit at a round table with 10 chairs.

(iii) Find the probability that the girls are seated together. [2]

<u>Solution</u>

$$\mathrm{P}\left(\text{girls are seated together}\right) = \left[\frac{(4-1)!7!}{(10-1)!}\right]$$

$$= \frac{1}{12}$$

Two particular boys are absent from the meeting.

(iv) Find the probability that two particular girls do not sit next to each other. [3]

<u>Solution</u>

$$\text{Probability required} = \frac{(8-1)!\,{}^{8}\mathrm{P}_{2}\frac{1}{2!}}{(10-1)!\frac{1}{2!}}$$

$$= \frac{7}{9}$$

[Turn Over

10 Along a 3km stretch of a road, the speed in km/h of a vehicle is a normally distributed random variable T. Over a long period of time, it is known that the mean speed of vehicles traveling along that stretch of the road is 90.0 km/h. To deter speeding, the traffic governing body introduced a speed monitoring camera. Subsequently, the speeds of a random sample of 60 vehicles are recorded. The results are summarised as follows.

$$\sum t = 5325, \qquad \sum \left(t - \bar{t}\right)^2 = 2000.$$

(i) Find unbiased estimates of the population mean and variance, giving your answers to 2 decimal places. [2]

Solution

Unbiased estimate of the population mean = $\bar{t} = \dfrac{5325}{60} = 88.75$

Unbiased estimate of the population variance = $s^2 = \dfrac{\sum\left(t-\bar{t}\right)^2}{59} = \dfrac{2000}{59}$

$= 33.89830508$

$= 33.90$

(ii) Test, at the 5% significance level, whether the speed-monitoring camera is effective in deterring the speeding of vehicles on the stretch of road. [4]

Solution

Let μ denote the population mean speed of the vehicles traveling along the stretch of the road.

To test H_0: $\mu = 90.0$

Against H_1: $\mu < 90.0$

Conduct a one-tail test at 5% level of significance, i.e., $\alpha = 0.05$

Under H_0, since $n = 60$ and is sufficiently large, by Central Limit Theorem,

$\bar{T} \sim N\left(90.0, \dfrac{33.89830508}{60}\right)$ approximately.

Using GC, p-value = 0.0481545117

Since p-value < 0.05, we reject H_0. There is sufficient evidence at 5% level of significance to conclude that that the mean speed along the 3km stretch of road has been reduced.

(iii) In another sample of size n ($n > 30$) that was collected independently, it is given that $\bar{t} = 89.0$. The result of the subsequent test using this information and the unbiased estimate of the population variance in **part (i)** is that the null hypothesis is not rejected. Obtain an inequality involving n, and hence find the largest possible value n can take. [4]

Solution

If the null hypothesis is not rejected, z_{calc} must lie outside the critical region.

Critical Region: $z \le -1.644853626$

Test Statistics, $Z = \dfrac{\overline{T} - 90.0}{\sqrt{\dfrac{33.89830508}{n}}} \sim \mathrm{N}(0, 1)$

$$\therefore z_{\text{calc}} = \frac{89 - 90.0}{\sqrt{\dfrac{33.89830508}{n}}} > -1.644853626$$

$$\frac{-\sqrt{n}}{\sqrt{33.89830508}} > -1.644853626$$

$$\sqrt{n} < 9.576708062$$

$$n < 91.713$$

Since n is an integer, the largest possible value n can take is 91.

CJC 2019 H2 Mathematics

Prelim Paper 1

1 Differentiate e^{-x^2} with respect to x. Hence find $\int x^3\mathrm{e}^{-x^2}\,\mathrm{d}x$. [3]

2 Without using a calculator, solve the inequality $x^2+4x+3<\dfrac{x+3}{2x+1}$. [4]

3 The curve C has the equation $\dfrac{(x-5)^2}{16}+\dfrac{(y-3)^2}{9}=1$.

(i) Sketch C, showing clearly the coordinates of the stationary points and the vertices. [2]

The region R is bounded by curve C, the line $x = 9$ and the x axis.

(ii) Find the volume of the solid generated when region R is rotated completely about the x axis. [3]

(iii) Describe fully a sequence of two transformations which would transform the curve C to the curve $\dfrac{(2x-5)^2}{16}+\dfrac{y^2}{9}=1$. [2]

4 (i) Show that $\dfrac{3r(r+1)+1}{r^3(r+1)^3}=\dfrac{A}{r^3}+\dfrac{B}{(r+1)^3}$, where A and B are to be determined. [2]

(ii) Hence find $\dfrac{7}{(1)^3(2)^3}+\dfrac{19}{(2)^3(3)^3}+\ldots+\dfrac{6n(2n+1)+1}{(2n)^3(2n+1)^3}$. [3]

(iii) Use your answer in part (ii) to find $\displaystyle\sum_{r=1}^{\infty}\left[\frac{3r(r+1)+1}{r^3(r+1)^3}+\left(\frac{1}{2}\right)^r\right]$. [3]

5 A developer has won the tender to build a stadium. Sunrise Singapore, who would finance the construction, wants to have a capacity of at least 50 000 seats. There is a limited land area to the stadium, and in order for the stadium to have a full sized track and football pitch, the first row of seats can seat 300 people, and every subsequent row has an additional capacity of 20 seats.

(i) What is the least number of rows the stadium must have to meet Sunrise Singapore's requirement? [3]

Assume now that the stadium has been built with 60 rows, with row 1 nearest to the football pitch.

(ii) For the upcoming international football match between Riverloop FC and Gunners FC, tickets are priced starting with \$60 for Category 1, with each subsequent category cheaper by 10%. How much would a seat in the 45th row cost? [1]

Category 1	Rows 1 – 20
Category 2	Rows 21 – 40
Category 3	Rows 41 – 60

There is a total crowd of 51 000 for the football match.

(iii) Assuming that all the tickets from the cheapest category are sold out first before people purchase tickets from the next category, calculate the total revenue collected for the football match. [5]

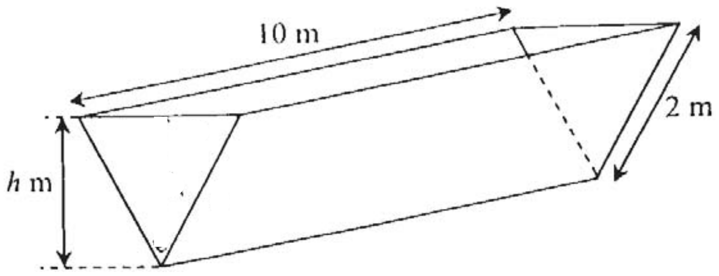
6 (i) Given that $y=\sqrt{e^x \cos x}$, show that $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2y\frac{\mathrm{d}y}{\mathrm{d}x}-y^2$. [4]

(ii) Find the Maclaurin series for $y=\sqrt{e^x \cos x}$, up to and including the term in x^3.

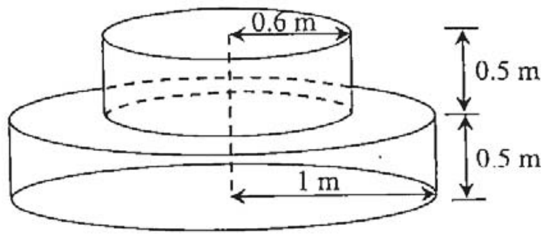
[4]

(iii) Verify the correctness of the expansion found in part (ii) using standard series found in the List of Formulae (MF26). [3]

7 (i) A water trough, shown in the diagram below, in the shape of a triangular prism is used to collect rainwater. The trough consists of two rectangular zinc sheets of negligible thickness, each with fixed dimensions 10 m by 2 m, and two triangular zinc sheets with height h m, as shown in the diagram below. Use differentiation to find the maximum volume of the trough, proving that it is a maximum. [5]



(ii) Water collected in the trough is drained at a rate of 0.001m3 /s into a container consisting of two cylinders, as shown below. The larger cylinder has radius 1 m and height 0.5 m. The smaller cylinder has radius 0.6 m and height 0.5 m. Find the rate at which the depth of water is increasing after 29 minutes. [4]



(iii) Let H be the height of the water level in the container at time t.

Sketch the graph of $\dfrac{\mathrm{d}H}{\mathrm{d}t}$ against t. [3]

8 **Do not use a calculator in answering this question.**

(a) Showing your working clearly, find the complex number z and w which satisfy the simultaneous equations

$$3z - \mathrm{i}w = 12 + 9\mathrm{i},$$
$$2z^* + 3w = 16 - 23\mathrm{i}.$$

[5]

(b) It is given that z_1 is a root of the equation $z^4+3z^3+4z^2-8=0$, where $z_1=-1+\sqrt{3}\mathrm{i}$.

(i) Express $z^4+3z^3+4z^2-8=0$ as a product of two quadratic factors with real coefficients. [4]

(ii) Given that $\mathrm{e}^{p+\mathrm{i}q}=z_1^5$, determine the exact values of p and q, where $-\pi<q<\pi$. [4]

9 Line l_1 has equation $r=\begin{pmatrix}-1\\8\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\5\end{pmatrix}$ where λ is a real parameter, and plane p has equation $\mathbf{r}\cdot\begin{pmatrix}a\\b\\0\end{pmatrix}=17$. It is given that l_1 lies completely on p and the point Q has coordinates $(-1,-2,3)$.

(i) Show that $a=-1$ and $b=2$. [3]

(ii) Find the foot of perpendicular from point Q to point P. Hence find the shortest distance between point Q and p in exact form. [4]

Given that the shortest distance between l_1 and the foot of perpendicular of Q on p is $\frac{5}{3}\sqrt{6}$.

(iii) Using the result obtained in part (ii) or otherwise, find the shortest distance between Q and l_1. [2]

The line l_2 is parallel to p, perpendicular to l_1 and passes through P and Q.

(iv) Show that the Cartesian equation of the line l_2 is $\frac{x+1}{2}=y+2=3-z$. [3]

(v) Find the vector equation of line l_3, which is the reflection of l_2 about p. [3]

10 Tumours develop when cells in the body divide and grow at an excessive rate. If the balance of cell growth and death is disturbed, a tumour may form. A medical scientist investigates the change of the tumour size, L mm at time t days of a particular patient using models A and B. For both of the models, it is given that the initial rate of the tumour size is 1 mm per day when the tumour size is 1 mm.

(i) Under Model A, the scientist observes that the patient's turmour is growing at a rate proportional to the square root of its size. At the same time, the tumour is reduced by radiation at a rate proportional to its size. It is further observed that the patient's tumour is decreasing at 2 mm per day when the tumour is 4 mm.

Show that L and t are related by the differential equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = 3\sqrt{L} - 2L\,.$$ [2]

(ii) Using the substitution $L = y^2$ where $y > 0$, show that the differential equation in part (i) can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3-2y}{2}\,.$$

Find y in terms of t and hence find L in terms of t only. [7]

(iii) Under Model B, the scientist suggests that L and t are related by the differential equation

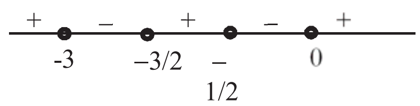
$$\frac{\mathrm{d}^2L}{\mathrm{d}t^2} = \frac{-2t}{(1+t^2)^2}\,.$$

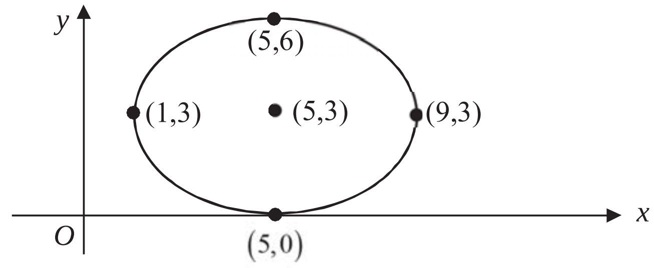
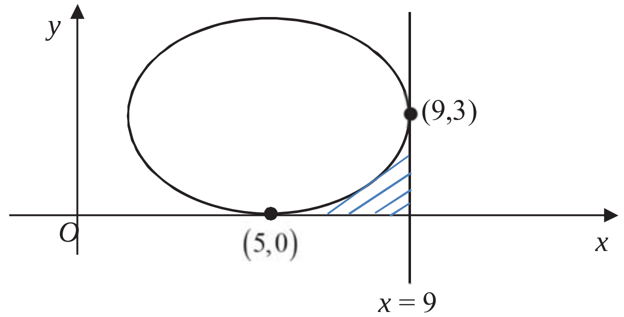
Find the particular solution of this differential equation. [4]

(iv) Find tumour sizes predicted by Models A and B in the long run. [2]

Q1. Techniques of Integration		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Integration by parts	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x^2}=-2x\mathrm{e}^{-x^2}$ $\int x^3\mathrm{e}^{-x^2}\,\mathrm{d}x$ $=\int -\frac{1}{2}x^2\cdot\left(-2x\mathrm{e}^{-x^2}\right)\mathrm{d}x$ $=-\frac{1}{2}x^2\mathrm{e}^{-x^2}-\int -x\mathrm{e}^{-x^2}\,\mathrm{d}x$ $=-\frac{1}{2}x^2\mathrm{e}^{-x^2}-\frac{1}{2}\int -2x\mathrm{e}^{-x^2}\,\mathrm{d}x$ $=-\frac{1}{2}x^2\mathrm{e}^{-x^2}-\frac{1}{2}\mathrm{e}^{-x^2}+C$ $u=-\frac{1}{2}x^2$ $\frac{\mathrm{d}v}{\mathrm{d}x}=-2x\mathrm{e}^{-x^2}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=-x$ $v=\mathrm{e}^{-x^2}$	Differentiation was well done except for a few students who clearly do not know how to differentiate exponential form. Many did not observe the word 'HENCE' in the question. They went on to do by parts, splitting e^{-x^2} and x^3 and was unable to continue.

Q2. Inequalities

Assessment Objectives	Solution	Examiner's Feedback
Use of algebraic method to solve inequality.	$x^2+4x+3-\frac{x+3}{2x+1}<0$ $(x+3)(x+1)-\frac{x+3}{2x+1}<0$ $(x+3)\frac{(x+1)(2x+1)-1}{2x+1}<0$ $\frac{(x+3)\left[2x^2+3x+1-1\right]}{2x+1}<0$ $\frac{(x+3)(x)(2x+3)}{2x+1}<0$  $\therefore -3<x<-\frac{3}{2}$ or $-\frac{1}{2}<x<0$	Despite reminders, many students still went to cross-multiply for inequality. Since this questions says "without the use of a calculator", it is expected that students factorize completely and obtain the roots. Incomplete factorization will result in loss of marks. A number of students wrote $-3\le x\le -\frac{3}{2}$ or $-\frac{1}{2}<x\le 0$ as the final answer. This should not be as the question is a strict inequality to begin with. Many students also wrote 'and' instead of 'or'.
Use of replacement	$-3\le x^2\le -\frac{3}{2}$ or $-\frac{1}{2}<x^2\le 0$ $\therefore \quad x=0$	Many students could identify the correct replacement but could not go on to obtain the final answer mark for $x=0$ as the answer in the earlier part was wrong.

<table>
<tr><th colspan="3">Q3. Definite Integrals + Conics + Transformations</th></tr>
<tr><th>Assessment Objectives</th><th>Solution</th><th>Examiner's Feedback</th></tr>
<tr><td>Standard graph - ellipse</td><td>(i)

</td><td>This part was generally well attempted with some students losing marks for not indicating the coordinates of points specifically requested by the question. There is a significant number of students who sketched a hyperbola instead and given no credit at all.</td></tr>
<tr><td>Volume about x-axis</td><td>(ii)


$\frac{(x-5)^2}{16}+\frac{(y-3)^2}{9}=1 \Rightarrow y=3\pm\sqrt{9\left(1-\frac{(x-5)^2}{16}\right)}$
$\Rightarrow y=3-\sqrt{9\left(1-\frac{(x-5)^2}{16}\right)} \quad (\because y<3)$
Volume (about x-axis)
$=\pi\int_5^9 y^2\,dx=\pi\int_5^9\left(3-\sqrt{9\left(1-\frac{(x-5)^2}{16}\right)}\right)^2 dx$
$=10.84267894=10.8$ (to 3 s.f.)</td><td>This part was very badly attempted or not attempted at all by students. Most students who attempted the question did not realize which area was being referred to and did not managed to get the correct expression for y, but instead assumed the negative square root to be rejected since the $y>0$. A large number of students are also confused about the limits to be used in the definite integral or did not realize in their algebraic manipulation that $\sqrt{a^2+b^2}\neq a+b$ and they proceed to obtain an overly simplified expression. A small but significant number of students set up the integral with respect to y instead when the axis of rotation was clearly defined.</td></tr>
</table>

Translation and Scaling	**(iii)**	Scaling parallel to x axis with scale factor of ½ followed by translation of 3 units in the negative y axis. **OR** Translation of 3 units in the negative y axis followed by a scaling parallel to x axis with scale factor of ½.	This part was very badly attempted. Many students used their own colloquial terms such as "Stretch", "Multiple", "Shift", "Move" which were not accepted. Some used the generic term "Transform" in place of "scale" and "translate" when they were required to describe the transformation. Many students also failed to include the key word "scale factor" in their description of the scaling step. Other common mistakes includes the confusion of ½ vs 2 for scale factor and + vs – 3 for the magnitude of the translation. Quite a number of students also omitted the axis in which the transformation was applied or stated the wrong axis.

Q4. Sigma Notation		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Simplifying expression by partial fractions	**(i)** **Method 1:** $\frac{3r(r+1)+1}{r^3(r+1)^3} = \frac{3r^2+3r+1}{r^3(r+1)^3}$ $= \frac{(r^3+3r^2+3r+1)-r^3}{r^3(r+1)^3}$ $= \frac{(r+1)^3-r^3}{r^3(r+1)^3}$ $= \frac{1}{r^3} - \frac{1}{(r+1)^3}$ where $A=1$ and $B=-1$. **Method 2:** $\frac{3r(r+1)+1}{r^3(r+1)^3} = \frac{A}{r^3} + \frac{B}{(r+1)^3}$ $3r(r+1)+1 = A(r+1)^3 + Br^3$ By comparing coefficient of r^0: $A=1$ By comparing coefficient of r^3: $A+B=0 \Rightarrow B=-1$ $\frac{3r(r+1)+1}{r^3(r+1)^3} = \frac{1}{r^3} - \frac{1}{(r+1)^3}$	This part was generally well attempted. Students should compare the coefficients on both sides to find the unknowns.
Summation of series by the method of differences	**(ii)** $\frac{7}{(1)^3(2)^3} + \frac{19}{(2)^3(3)^3} + \dots + \frac{6n(2n+1)+1}{(2n)^3(2n+1)^3}$ $= \sum_{r=1}^{2n} \frac{3r(r+1)+1}{r^3(r+1)^3}$	This part was not well attempted. The result in (i) can be used to find the expression for the series without using sigma notation.

	$=\sum_{r=1}^{2n}\left[\frac{1}{r^3}-\frac{1}{(r+1)^3}\right]$ $=\frac{1}{1^3}-\cancel{\frac{1}{2^3}}$ $+\cancel{\frac{1}{2^3}}-\cancel{\frac{1}{3^3}}$ $+\ldots$ $+\cancel{\frac{1}{(2n-1)^3}}-\cancel{\frac{1}{(2n)^3}}$ $+\cancel{\frac{1}{(2n)^3}}-\frac{1}{(2n+1)^3}$ $=1-\frac{1}{(2n+1)^3}$	Students were generally not familiar with the sigma notation to represent the series. The sigma notation should be used with integral limits. A significant percentage of students wrote the following sigma notation: $\sum_{r=0.5}^{n}\frac{3r(r+1)+1}{r^3(r+1)^3}$ where the lower limit was 0.5 and the running index was assumed to increase by 0.5 each time. Some students were also confused by the last term in the series. Though they recognized that r has been replaced by $2n$, they did not think of $2n$ as the upper limit. Instead they represented the series wrongly by the following sigma notation $\sum_{r=1}^{n}\frac{6r(2r+1)+1}{(2r)^3(2r+1)^3}$ which only included the even terms of the given series (i.e. 2^{nd}, 4^{th}, $6^{\text{th}},\ldots,(2n)^{\text{th}}$). Some students found a positive value of B in (i) and blindly performed method of difference even though it was not possible. Credit was not given in such cases.
Limits, Sum to infinity and use of standard series expansion	**(iii)** As $2n\to\infty$, $\frac{1}{(2n+1)^3}\to 0$, $1-\frac{1}{(2n+1)^3}\to 1$	This part was not well attempted. Students were able to recognize that the question asked for the long-term average. However, they were confused by the running index r and the unknown constant n . A significant percentage of students considered the case where $r\to\infty$ instead of $2n\to\infty$.

	$\sum_{r=1}^{\infty}\left[\frac{3r(r+1)+1}{r^3(r+1)^3}+\left(\frac{1}{2}\right)^r\right]$ $=\sum_{r=1}^{\infty}\frac{3r(r+1)+1}{r^3(r+1)^3}+\sum_{r=1}^{\infty}\left(\frac{1}{2}\right)^r$ $=1+\left[\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots\right]$ $=1+\frac{1/2}{1-1/2}$ $=2$	Some students did not know how to present their analysis and solutions mathematically for the first summation. They wrongly wrote $\sum_{r=1}^{\infty}\frac{3r(r+1)+1}{r^3(r+1)^3}=1-\frac{1}{(2(\infty)+1)^3}$. As the symbol ∞ is not a number, the presentation should be $\sum_{r=1}^{\infty}\frac{3r(r+1)+1}{r^3(r+1)^3}=\lim_{n\to\infty}\left(1-\frac{1}{(2n+1)^3}\right)=1$. A significant percentage of students had the misconception that if $\left(\frac{1}{2}\right)^r\to 0$ as $r\to\infty$, $\sum_{r=1}^{\infty}\left(\frac{1}{2}\right)^r=0$. For students who recognized that they should use the formula for sum to infinity, • some were not able to identify the first term correctly where they assumed $a=1$, • some were not able to identify the value of common ratio r as they were confused with the running index r.

<table>
<tr><th colspan="3">Q5. A.P. & G.P.</th></tr>
<tr><th>Assessment Objectives</th><th>Solution</th><th>Examiner’s Feedback</th></tr>
<tr><td>Finding sum of a finite arithmetic series</td><td>(i) $a = 300$, $d = 20$
Sum of seats in the first n rows ≥ 50000
$\frac{n}{2}\left(2(300)+(n-1)20\right) \geq 50000$
<table><tr><th>n</th><th>$\frac{n}{2}\left(2(300)+(n-1)20\right)$</th></tr><tr><td>57</td><td>49020</td></tr><tr><td>58</td><td>50460</td></tr><tr><td>59</td><td>51920</td></tr></table>
$n \geq 58$
Minimum no. of rows is 58.</td><td>Some students wrote
$T_n = 300+(n-1)20 \geq 50000$
instead of
$\frac{n}{2}\left(2(300)+(n-1)20\right) \geq 50000$.
Some students put "> 50000" instead of "≥ 50000".</td></tr>
<tr><td>Finding nth term of a finite geometric series</td><td>(ii) $a = 60$, $r = 0.9$
Price of seats in 45th row $= 60(0.9)^{3-1}$ $= \$48.60$</td><td>Part (ii) is generally well done.</td></tr>
<tr><td>Finding nth term of a finite arithmetic series</td><td>(iii) Number of seats in 60th row $= 300+(60-1)20$
$= 1480$
Creating new AP starting from last row, $a = 1480$, $d = -20$
$\frac{n}{2}\left(2(1480)+(n-1)(-20)\right) = 51000$
$n = 53.28$
Number of seats in Category 3
$= S_{20} = \frac{20}{2}\left[2(1480)+(20-1)(-20)\right] = 25800$</td><td>There were confusion in the number of seats in each category as students tend to mixed up the calculations for the number of seats in the first and third category. A number of students tried to use Geometric Progression to calculate the number of seats for each category.</td></tr>
</table>

Number of seats in Category 2

$= S_{40} - S_{20} = \frac{40}{2}\left[2(1480)+(40-1)(-20)\right]-25800 = 43600-25800$

$= 17800$

Number of seats in Category 1

$= 51000 - S_{40} = 51000 - 43600 = 7400$

Total revenue $= 7400(60)+17800(60)(0.9)+25800(60)(0.9)^2$

$= \$2659080$

<u>Alternative</u>

$a = 300$, $d = 20$

Number of seats in Category 3

$= S_{60} - S_{40}$

$= \frac{60}{2}\left[2(300)+(60-1)(20)\right]-\frac{40}{2}\left[2(300)+(40-1)(20)\right]$

$= 53400 - 27600$

$= 25800$

Number of seats in Category 2

$= S_{40} - S_{20}$

$= 27600 - \frac{20}{2}\left[2(300)+(20-1)(20)\right]$

$= 27600 - 9800$

$= 17800$

Number of seats in Category 1

$= 51000 - 25800 - 17800 = 7400$

	Total revenue $= 7400(60)+17800(60)(0.9)+25800(60)(0.9)^2$ $= \$2659080$	

<table>
<tr><th colspan="3">Q6. Maclaurin's Series</th></tr>
<tr><th>Assessment Objectives</th><th>Solution</th><th>Examiner's Feedback</th></tr>
<tr><td>Implicit differentiation and use of product rule</td><td>

$y=\sqrt{e^x \cos x}$

$y^2 = e^x \cos x$

Differentiating w.r.t. x ,

$2y\frac{dy}{dx} = e^x \cos x - e^x \sin x$

$2y\frac{dy}{dx} = y^2 - e^x \sin x$

Differentiating w.r.t. x ,

$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} - \left(e^x \sin x + e^x \cos x\right)$

$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} + \left(2y\frac{dy}{dx} - y^2\right) - y^2$

$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 4y\frac{dy}{dx} - 2y^2$

$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} - y^2$ (shown)

</td><td>

A lot of candidates did not square the y and then implicitly differentiate, but instead tried to bulldoze their way through the repeated differentiation, to various outcomes. Much time could have been saved if the correct preparation for implicit differentiation was used.

Many students could not get to the final required form of the equation because they could not see the appropriate substitution for $e^x \sin x$ or $e^x \cos x$

Some notations which are not correct are used, eg: $\frac{dy}{dx}(y^2) = \frac{dy}{dx}(e^x \cos x)$

Or silly errors like $e^x \cos x$ becoming $ex \cos x$ and $\cos^2 x$ becoming $\cos 2x$

</td></tr>
</table>

	Differentiating w.r.t. x , $2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2y\frac{\mathrm{d}y}{\mathrm{d}x}$ When $x=0$, $\mathrm{f}(0)=1$, $\mathrm{f}'(0)=\frac{1}{2}$, $\mathrm{f}''(0)=-\frac{1}{4}$, $\mathrm{f}'''(0)=-\frac{5}{8}$ $y=1+\frac{x}{2}+\frac{x^2}{2}\left(-\frac{1}{4}\right)+\frac{x^3}{3!}\left(-\frac{5}{8}\right)$ $y=1+\frac{x}{2}-\frac{x^2}{8}-\frac{5x^3}{48}$	
Verification	Using MF26, $y=\sqrt{\mathrm{e}^x\cos x}$ $y=\mathrm{e}^{\frac{x}{2}}(\cos x)^{\frac{1}{2}}$ $y\approx\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}\right)\left(1-\frac{x^2}{2}\right)^{\frac{1}{2}}$ $y\approx\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}\right)\left(1+\frac{1}{2}\left(-\frac{x^2}{2}\right)\right)$ $y\approx1-\frac{x^2}{4}+\frac{x}{2}-\frac{x^3}{8}+\frac{x^2}{8}+\frac{x^3}{48}$ $y=1+\frac{x}{2}-\frac{x^2}{8}-\frac{5x^3}{48}$ (verified)	Some students misunderstand the question. They are supposed to verify that the expression in (ii) is correct. They interpreted it as checking IF the expression is correct and some concluded that it is not. A worrying situation is that many students commit basic algebraic mistakes, like saying $\left(1+x-\frac{x^3}{3}\right)^{\frac{1}{2}}=1+x^{\frac{1}{2}}-\frac{1}{3}x^{\frac{3}{2}}$

<table>
<tr>
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OR

Using MF26,
$y=\sqrt{\mathrm{e}^x \cos x}$
$y \approx \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)^{\frac{1}{2}}\left(1-\frac{x^2}{2}\right)^{\frac{1}{2}}$
$y \approx \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^2}{2}-\frac{x^3}{2}\right)^{\frac{1}{2}}$
$y=\left(1+x-\frac{x^3}{3}\right)^{\frac{1}{2}}$
$y=1+\frac{1}{2}\left(x-\frac{x^3}{3}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(x-\frac{x^3}{3}\right)^2$
$+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(x-\frac{x^3}{3}\right)^3+\ldots$
$y=1+\frac{x}{2}-\frac{x^3}{6}-\frac{1}{8}\left(x^2\right)+\frac{1}{16}x^3+\ldots$
$y \approx 1+\frac{x}{2}-\frac{x^2}{8}-\frac{5x^3}{48}$ (verified)
</td>
<td>
Some students gave too many terms and had a hard time manipulating. They have to realize that they can truncate the series for terms beyond x^3. Some students truncate too much too early, ending up with the incorrect answer too.

A few students square their expression in (ii) to compare with the standard series expansion of y^2 and concluded it is correct. Credit is not given because $A^2=B^2$ does not necessarily imply $A=B$
</td>
</tr>
</table>

Q7. Applications of Differentiation

Assessment Objectives	Solution	Examiner's Feedback
Maxima/Minima problems	**(i)** Volume of trough $V=\frac{1}{2}(2)(10)h\sqrt{4-h^2}$ $V=10h\sqrt{4-h^2}$ $\frac{\mathrm{d}V}{\mathrm{d}h}=10\sqrt{4-h^2}+\frac{1}{2}(10h)\left(4-h^2\right)^{-\frac{1}{2}}(-2h)$ $\frac{\mathrm{d}V}{\mathrm{d}h}=10\sqrt{4-h^2}-10h^2\left(4-h^2\right)^{-\frac{1}{2}}$ To maximize volume of trough, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$ $10h^2\left(4-h^2\right)^{-\frac{1}{2}}=10\sqrt{4-h^2}$ $10h^2=10\left(4-h^2\right)$ $4=2h^2$ $h=\sqrt{2}\ \left(\text{reject } -\sqrt{2} \text{ since } h>0\right)$ **OR** $V^2=100h^2\left(4-h^2\right)$ $V^2=400h^2-100h^4$ $2V\frac{\mathrm{d}V}{\mathrm{d}h}=800h-400h^3$ To maximize volume of trough, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$ $400h(2-h^2)=0$ $h=\sqrt{2}\ \left(\text{reject } h=-\sqrt{2} \text{ and } h=0 \text{ since } h>0\right)$	Many students were not able to form the expression of the volume correctly. This is the most commonly seen mistake: $V=\frac{1}{2}(10)h\sqrt{4-h^2}$. When it comes to differentiation, some students made mistake in the chain rule while performing the product rule. Example: $-2h$ was often missed out.

<table>
<tr><td></td><td>1st derivative test:
<table><tr><td>h</td><td>$\sqrt{2}^-$</td><td>$\sqrt{2}$</td><td>$\sqrt{2}^+$</td></tr><tr><td>$\frac{dV}{d\theta}$</td><td>+</td><td>0</td><td>-</td></tr></table>
OR
2nd derivative test:
$\frac{d^2V}{dh^2}=10h\left(4-h^2\right)^{-\frac{3}{2}}\left(2h^2-12\right)$
When $h=\sqrt{2}$, $\frac{d^2V}{dh^2}<0$

It is a maximum value.

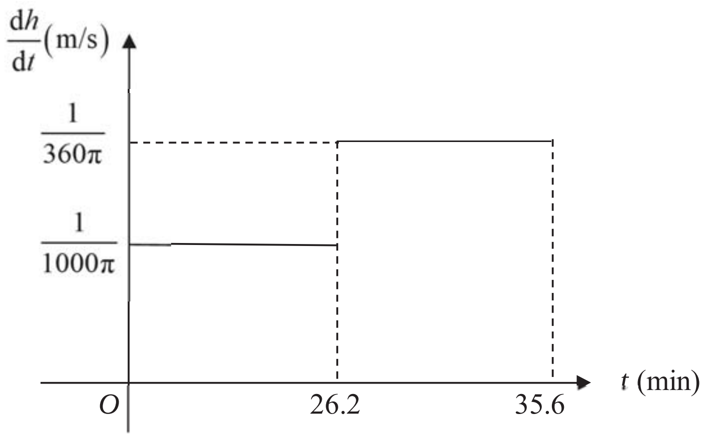
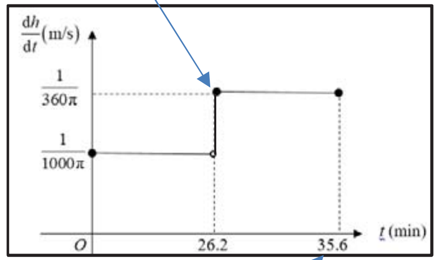
$V=10\sqrt{2}\sqrt{4-2}=20\text{m}^3$</td><td>Most students were able to verify the maximum volume but the presentation was quite bad for some scripts.

Some students lost one mark as they forgot to find the max volume.</td></tr>
<tr><td>Rate of change</td><td>(ii) $\frac{dV}{dt}=0.001\text{ cm}^3/\text{s}$
After 29 min, $V=0.001(60)(29)=1.74\text{m}^3$
Volume of larger cylinder, $V_L=\pi r^2h=\pi(1)^2(0.5)=1.570796\text{m}^3$
Thus the water level is at the smaller cylinder after 29 minutes.

$V_S=\pi(0.6)^2h$
$\frac{dV_S}{dh}=\pi(0.6)^2$
$\frac{dV_S}{dt}=\frac{dV_S}{dh_S}\cdot\frac{dh_S}{dt}$</td><td>Generally quite OK. Most students attempted this part using a variety of methods to obtain the answer.

Common mistake: students forgot to check the water level at 29 minutes and just made their own assumption while majority attempted to form the expression of the volume by considering both large and small containers.</td></tr>
</table>

	$\dfrac{\mathrm{d}h_s}{\mathrm{d}t}=\dfrac{1}{\pi(0.6)^2}\cdot 0.001$ for $r=0.6$ $=\dfrac{1}{360\pi}$ m/s	
Sketch graph	**(iii)** For the larger cylinder: $V_L=\pi(1)^2 h$ $\dfrac{\mathrm{d}V_L}{\mathrm{d}h}=\pi(1)^2$ $\dfrac{\mathrm{d}V_L}{\mathrm{d}t}=\dfrac{\mathrm{d}V_L}{\mathrm{d}h_L}\cdot\dfrac{\mathrm{d}h_L}{\mathrm{d}t}$ $\dfrac{\mathrm{d}h_L}{\mathrm{d}t}=\dfrac{1}{\pi(1)^2}\cdot 0.001$ for $r=1$ $=\dfrac{1}{1000\pi}$ m/s For the smaller cylinder: $\dfrac{\mathrm{d}h_s}{\mathrm{d}t}=\dfrac{1}{360\pi}$ Time taken to fill the larger cylinder $=\dfrac{\pi(1)^2(0.5)}{0.001}=1570.796\text{ s}=26.179938\text{ min}=26.2\text{ min}$ Time taken to fill up the smaller cylinder $=\dfrac{\pi(0.6)^2(0.5)}{0.001}=565.4867\text{ s}=9.42478\text{ min}$ Total time taken to fill up the container $=26.179938+9.42478$ min $=35.6$ min	Many students attempted this part and also able to obtained at least 1 or 2 out of 3 marks.

		Common mistake: (extra vertical line)  Many lost the 1 mark as they didn't manage to find this.

Q8. Complex Numbers

Assessment Objectives	Solution	Examiner's Feedback
Solving of simultaneous equations	**(a)** $3z-\mathrm{i}w=12+19\mathrm{i} \quad \cdots\cdots(1)$ $2z^*+3w=16-23\mathrm{i} \quad \cdots\cdots(2)$ $(1)\times 3$: $9z-3\mathrm{i}w=36+57\mathrm{i}\cdots\cdots(3)$ $(2)\times \mathrm{i}$: $2\mathrm{i}z^*+3\mathrm{i}w=16\mathrm{i}+23\cdots\cdots(4)$ $(3)+(4): 9z+2\mathrm{i}z^*=73\mathrm{i}+59$ Let $z=x+\mathrm{i}y$, so $z^*=x-\mathrm{i}y$ $9(x+\mathrm{i}y)+2\mathrm{i}(x-\mathrm{i}y)=73\mathrm{i}+59$ **Comparing real and imaginary parts,** $9x+2y=59$ and $9y+2x=73$ **Solving,** $x=5$ and $y=7$ $\therefore \quad z=5+7\mathrm{i}$ **From (2),** $2z^*+3w=16-23\mathrm{i}$ $w=\frac{1}{3}(16-23\mathrm{i}-2z^*)=\frac{1}{3}(16-23\mathrm{i}-2(5-7\mathrm{i}))$ $w=2-3\mathrm{i}$	Majority of the candidates lost their final mark because they simply cannot copy the question properly. Many wrote $9\mathrm{i}$ instead of $19\mathrm{i}$, this is unacceptable! Some candidates miss out on the instruction "Do not use calculator in answering the question", they still write "using GC" while solving the simultaneous equation after comparing real and imaginary parts. Some candidates took a longer way by letting $z=a+b\mathrm{i}$ and $w=c+d\mathrm{i}$ and solved for 4 unknowns. It is a tedious process and they ended up using GC to solve. This is not recommended. Some candidates made a conceptual error. After making z the subject from equation (1): $z=\frac{1}{3}(12+(19+w)\mathrm{i})$, they went on to concude that $z^*=\frac{1}{3}(12-(19+w)\mathrm{i})$ **which is incorrect!** They cannot do this step here as w is a complex number.
Complex conjugate roots	**(b)(i)** Since all coefficients are real, the other root is $(-1-\sqrt{3}\mathrm{i})$. $\left[z-(-1+\sqrt{3}\mathrm{i})\right]\left[z-(-1-\sqrt{3}\mathrm{i})\right]=z^2+2z+4$ $z^4+3z^3+4z^2-8=(z^2+2z+4)(z^2+az+b)$	Many candidates are unsure about factors and roots. Some candidates gave the conjugate as $1+\sqrt{3}\mathrm{i}$ (negate the real part) which is wrong.

	Comparing coefficient of z^3: $3=a+2 \quad \Rightarrow a=1$ Comparing coefficient of z^2: $4=b+2a+4 \quad \Rightarrow b=-2$ Comparing coefficient of z: $0=2b+4a \quad \Rightarrow b=-2$ $z^4+3z^3+4z^2-8=\left(z^2+2z+4\right)\left(z^2+z-2\right)$	Candidates who did not get full credit struggled at algebraic manipulations when comparing coefficients/ long division. There is no need to solve for the roots.
Complex numbers in exponential form	**(b)(ii)** $\mathrm{e}^{p+\mathrm{i}q}=z_1^{\,5}$ $z_1=-1+\sqrt{3}\mathrm{i}=2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$ $\mathrm{e}^p\mathrm{e}^{\mathrm{i}q}=\left(2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}\right)^5$ $\mathrm{e}^p\mathrm{e}^{\mathrm{i}q}=32\mathrm{e}^{\mathrm{i}\left(\frac{10\pi}{3}\right)}$ $\mathrm{e}^p=32 \quad \Rightarrow p=\ln 32$ $q=\frac{10\pi}{3}-\frac{12\pi}{3}=-\frac{2\pi}{3}$	Many left this blank. Candidates who expanded this using Cartesian form often struggled, they are recommended to convert to modulus-argument form. Some forgot about the power 5. Candidates need to practice more on finding arguments of complex number. $z_1=-1+\sqrt{3}\mathrm{i}$ lies in the 2nd quadrant and hence, $\arg\left(z_1\right)=\pi-\tan^{-1}\left\lvert\frac{\sqrt{3}}{-1}\right\rvert$ (when finding basic angle, always $\tan^{-1}$ the positive real/ im parts) Some did not know how to convert q into the principal range. They need to subtract multiples of 2π

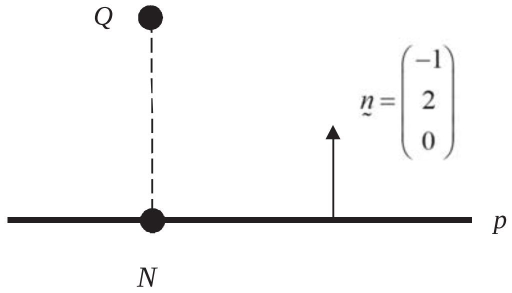
Q9. Vectors (Lines & Planes)		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Line lies on plane	**(i)** $l_1 : \mathbf{r} = \begin{pmatrix} -1 \\ 8 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1+2\lambda \\ 8+\lambda \\ 3+5\lambda \end{pmatrix}$ $p : \mathbf{r} \cdot \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 17$ Since l_1 lies completely on p, $\begin{pmatrix} -1 \\ 8 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 17$ $-a+8b = 17 \cdots\cdots (1)$ Since l_1 lies in p, $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ are perpendicular, $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 0$ $2a+b=0$ $b=-2a$ Substitute $b=-2a$ into (1), $-a-16a=17$ $-17a=17$ $\therefore a=-1$ and $b=-2a=2$ (Shown)	Most students were able to use the information provided to formulate 2 equations in a and b, and proceeded to solve simultaneously. Alternative method of obtaining the two equations was to substitute different values of λ in order to obtain different points on l_1.

Foot of perpendicular

(ii)

$$\overrightarrow{OQ}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}$$

Let N be the foot of perpendicular from point Q to p.



Equation of the line passing though N and Q is

$$\underset{\sim}{r}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\0\end{pmatrix},\mu\in\mathbb{R}$$

$$=\begin{pmatrix}-1-\mu\\-2+2\mu\\3\end{pmatrix}$$

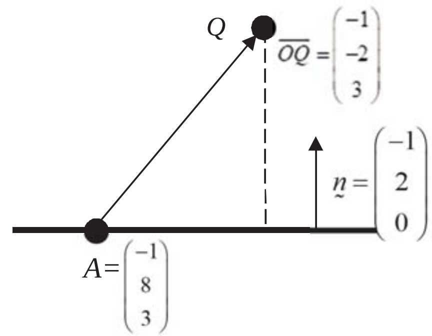
When the line passes through N on p

$$\begin{pmatrix}-1-\mu\\-2+2\mu\\3\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\0\end{pmatrix}=17$$

$$(1+\mu)+(-4+4\mu)=17$$
$$5\mu-3=17$$
$$5\mu=20$$
$$\mu=4$$

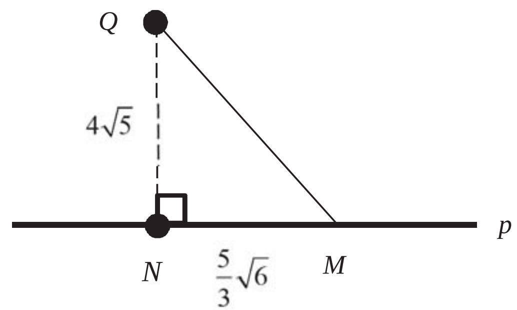
Common mistake was for students to erroneously assume that point Q lies on the plane. A quick check would have showed otherwise.

A handful of students were able to visualize the relationship between l_1, p and Q to discover that the perpendicular distance can be found efficiently using the length of projection formula.



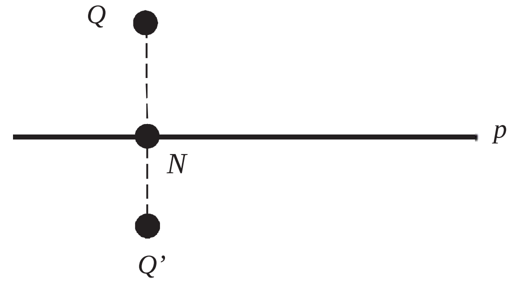
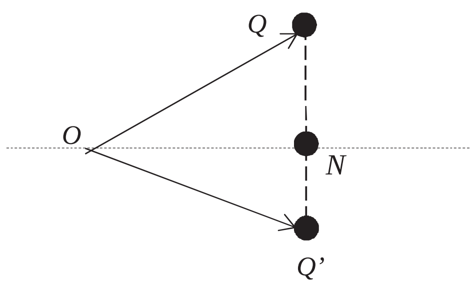
$$\overrightarrow{AQ}=\overrightarrow{OQ}-\overrightarrow{OA}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}-\begin{pmatrix}-1\\8\\3\end{pmatrix}=\begin{pmatrix}0\\-10\\0\end{pmatrix}$$

$$\left|\overrightarrow{AQ}\bullet\hat{\underset{\sim}{n}}\right|=\left|\begin{pmatrix}0\\-10\\0\end{pmatrix}\bullet\frac{\begin{pmatrix}-1\\2\\0\end{pmatrix}}{\sqrt{1^2+2^2}}\right|=\frac{20}{\sqrt{5}}=4\sqrt{5}$$

<table>
<tr><td></td><td>$$\therefore \overrightarrow{ON} = \begin{pmatrix} -1-\mu \\ -2+2\mu \\ 3 \end{pmatrix} = \begin{pmatrix} -1-4 \\ -2+8 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ 3 \end{pmatrix}$$
Shortest Distance
$= \left|\overrightarrow{QN}\right| = \left|\overrightarrow{ON} - \overrightarrow{OQ}\right|$
$= \left|\begin{pmatrix} -5 \\ 6 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}\right|$
$= \left|\begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}\right|$
$= \sqrt{4^2 + 8^2}$
$= \sqrt{80}$
$= 4\sqrt{5}$</td><td></td></tr>
<tr><td>Shortest distance</td><td>(iii)

Shortest distance between Q and l_1</td><td>There were some students who identified the lengths to be found for (ii) and (iii) wrongly (i.e. gave (iii) answer for (ii) and vice versa). They were still given the credit.

Alternative methods (e.g. using length of projection) were accepted.</td></tr>
</table>

	$=\sqrt{\left(4\sqrt{5}\right)^2+\left(\frac{5}{3}\sqrt{6}\right)^2}$ $=\sqrt{\left(\sqrt{80}\right)^2+\left(\frac{25}{9}\right)(6)}$ $=\sqrt{(80)+\left(\frac{150}{9}\right)}$ $=\sqrt{\frac{290}{3}}$ or $=\frac{1}{3}\sqrt{870}$ or 9.83 (3 s.f)	
Equation of line	**(iv)** Direction vector of l_2 $=\begin{pmatrix}2\\1\\5\end{pmatrix}\times\begin{pmatrix}-1\\2\\0\end{pmatrix}$ $=\begin{pmatrix}(0)-(2\times5)\\-\left[(0)-(-1\times5)\right]\\(4)-(-1)\end{pmatrix}$ $=\begin{pmatrix}-10\\-5\\5\end{pmatrix}=-5\begin{pmatrix}2\\1\\-1\end{pmatrix}$ Vector equation of the line l_2 is $\underset{\sim}{r}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}, t\in\mathbb{R}$	Most students were unable to visualize the relationship between l_2, l_1 and p . They tried to work backwards using the given Cartesian equation to obtain the vector equation of l_2 in an attempt to fulfill the question requirement of showing the process to find l_2.

	$x=-1+2t \qquad t=\dfrac{x+1}{2}$ $\Rightarrow\ y=-2+t \Rightarrow t=y+2$ $z=3-t \qquad t=3-z$ Cartesian equation of the line l_2 is $\dfrac{x+1}{2}=y+2=3-z$	
Reflection of line about a plane	**(v)**  Using Ratio Theorem  Since N is the mid-point of $\overrightarrow{OQ}$ and $\overrightarrow{OQ'}$, $\overrightarrow{ON}=\dfrac{1}{2}\left(\overrightarrow{OQ}+\overrightarrow{OQ'}\right)$ $\overrightarrow{OQ'}=2\overrightarrow{ON}-\overrightarrow{OQ}$	Most students were unable to visualize the situation when a line parallel to a plane is reflected in the plane. They did not realise that the line and reflected lines both share the same direction vector. Students who made mistakes in previous parts (e.g. position vector of foot of perpendicular) were given credit for their attempt to apply Ratio Theorem.

	$\overrightarrow{OQ'} = 2\begin{pmatrix}-5\\6\\3\end{pmatrix} - \begin{pmatrix}-1\\-2\\3\end{pmatrix} = \begin{pmatrix}-9\\14\\3\end{pmatrix}$ Vector equation of the line l_3 is $\underset{\sim}{r} = \begin{pmatrix}-9\\14\\3\end{pmatrix} + s\begin{pmatrix}2\\1\\-1\end{pmatrix}, s \in \mathbb{R}$	

Q10. Differential Equations

Assessment Objectives	Solution	Examiner's Feedback
Formulate of differential equation	**(i)** Growth rate $\propto \sqrt{L}$ Destruction rate $\propto L$ $\frac{\mathrm{d}L}{\mathrm{d}t} = a\sqrt{L} - bL$ When $L = 1$, $\frac{\mathrm{d}L}{\mathrm{d}t} = 1$ $a - b = 1 \quad -(1)$ When $L = 4$, $\frac{\mathrm{d}L}{\mathrm{d}t} = -2$ $a\sqrt{4} - 4b = -2$ $2a - 4b = -2 \quad -(2)$ Solving, $a = 3, b = 2$ $\frac{\mathrm{d}L}{\mathrm{d}t} = 3\sqrt{L} - 2L$ (shown)	Some students did not manage to formulate two equations with two unknowns as they set $a = b$ Most of those who managed to get the two equations with two unknowns proceeded to get the correct values of a and b. Many students did not attempt this part.
Solving differential equation	**(ii)** Using substitution: $L = y^2$ Differentiate with respect to *t*: $\frac{\mathrm{d}L}{\mathrm{d}t} = 2y\frac{\mathrm{d}y}{\mathrm{d}t}$ $2y\frac{\mathrm{d}y}{\mathrm{d}t} = 3y - 2y^2$ $2\frac{\mathrm{d}y}{\mathrm{d}t} = 3 - 2y$ $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3-2y}{2}$ Using substitution: $y = \sqrt{L}$ Differentiate with respect to *t*: $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}L^{-\frac{1}{2}}\frac{\mathrm{d}L}{\mathrm{d}t}$ $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}L^{-\frac{1}{2}}\left(3\sqrt{L} - 2L\right)$ $= \frac{3}{2} - \sqrt{L} = \frac{3}{2} - y$ $= \frac{3-2y}{2}$	Most students used $L = y^2$. The minority who used $y = \sqrt{L}$ are also able to get the correct equation.

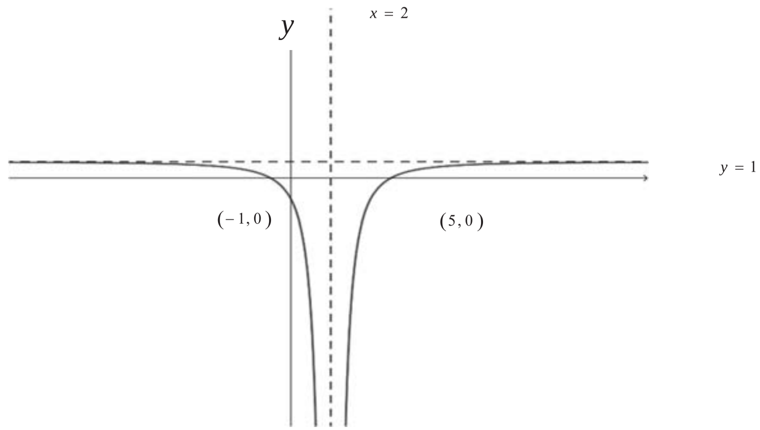
	$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{3-2y}{2}$ $\frac{2}{3-2y}\frac{\mathrm{d}y}{\mathrm{d}t}=1$	
	$\int\frac{2}{3-2y}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t=\int 1\,\mathrm{d}t$ $-\int\frac{-2}{3-2y}\,\mathrm{d}y=t+C$	Most students are able to identify this as a separation of variables DE.
	$-\ln\lvert 3-2y\rvert=t+C$ $\lvert 3-2y\rvert=\mathrm{e}^{-t-C}$ $3-2y=\pm\mathrm{e}^{-C}\mathrm{e}^{-t}$ $3-2y=A\mathrm{e}^{-t}$, where $A=\pm\mathrm{e}^{-C}$ $2y=3-A\mathrm{e}^{-t}$ $y=\frac{3-A\mathrm{e}^{-t}}{2}$ $y=\sqrt{L}=\frac{3-A\mathrm{e}^{-t}}{2}$	Many students did not include the modulus sign for the logarithm. Among those who did have the modulus sign, many removed the modulus and did not put $\pm$
	$L=\left(\frac{3-A\mathrm{e}^{-t}}{2}\right)^2$ When $t=0$, $L=1$, $1=\left(\frac{3-A\mathrm{e}^{-0}}{2}\right)^2$ Since $y>0$, $\frac{3-A}{2}=1\Rightarrow A=1$ $L=\left(\frac{3-\mathrm{e}^{-t}}{2}\right)^2$	Many students stopped here without finding A .

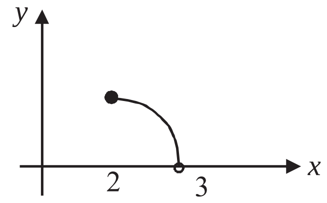
	(iii) $\frac{\mathrm{d}^2L}{\mathrm{d}t^2}=\frac{-2t}{\left(1+t^2\right)^2}$ Integrate both sides with respect to *t*, $\frac{\mathrm{d}L}{\mathrm{d}t}=\int\frac{-2t}{\left(1+t^2\right)^2}\,\mathrm{d}t$ $=-\int 2t\left(1+t^2\right)^{-2}\,\mathrm{d}t$ $=-\left[\frac{\left(1+t^2\right)^{-1}}{-1}\right]+C$ $=\frac{1}{1+t^2}+C$ When $t=0,\ L=1,\ \frac{\mathrm{d}L}{\mathrm{d}t}=1,$ $1=\frac{1}{1+0^2}+C$ $C=0$ $\therefore\frac{\mathrm{d}L}{\mathrm{d}t}=\frac{1}{1+t^2}$ Integrate again both sides with respect to *t*, $L=\int\frac{1}{1+t^2}\,\mathrm{d}t$ $=\tan^{-1}t+D$ When $t=0,\ L=1,$ $1=\tan^{-1}0+D$ $D=1$ $\therefore L=\tan^{-1}t+1$	Most students could not identify the form $\mathrm{f}'(x)\mathrm{f}(x)^n$ and thus could not integrate. Some students who proceeded either missed out or did not understand the phrase 'particular solution', thus they did not substitute in the initial values.

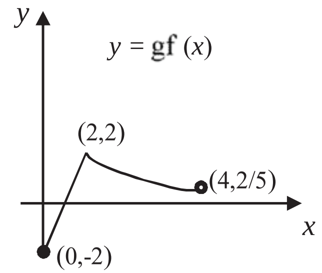
Limits	**(iv)** As $t \to \infty$, For Model A: $e^{-t} \to 0, L \to \left(\frac{3}{2}\right)^2 = \frac{9}{4}$ For Model B: $\tan^{-1} t \to \frac{\pi}{2}, L \to 1+\frac{\pi}{2}$	Common mistake was using 90 instead of $\frac{\pi}{2}$. Angles should be in radians for calculus questions.

Q1. System of Linear Equations		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Solve system of linear equations.	$\frac{13}{4}=a(1)+b(1)+\frac{1}{c}$ $97=a(64)+b(16)+\frac{4}{c}$ $642=a(512)+b(64)+\frac{8}{c}$ From G.C., $a=1$, $b=2$, $\frac{1}{c}=0.25$ $\therefore a=1$, $b=2$, $c=4$	Most candidates were successful with the question except those who did not know how to handle the transformed coordinates or the replacement of $1/c$ as another variable. There were also a lot of algebraic manipulation errors which reflected the lack in numeracy skills of the candidature. There is also a significant group of candidates who attempted to solve the equations manually instead of using the GC, and most of them were unsuccessful.

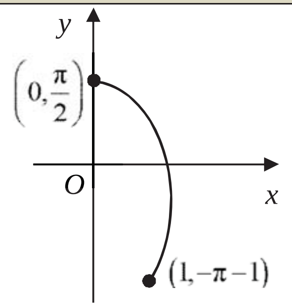
Q2. Vectors (Basic)		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Concept of parallel vectors	**(i)** $k\mathbf{p} = m\mathbf{q}$ where $m = (\mathbf{p}\cdot\mathbf{q}) \in \mathbb{R}\setminus\{0\}$ $\mathbf{p} = \frac{m}{k}\mathbf{q} = n\mathbf{q}$ where $n = \frac{m}{k} \in \mathbb{R}\setminus\{0\}$ Since $\mathbf{p} = n\mathbf{q}$, $\mathbf{p}$ and $\mathbf{q}$ are parallel vectors.	This basic question proved to be difficult for most candidates, where only a handful provided a complete solution with explanation. Majority of the accepted answers simply contain the key word "parallel" with ambiguous statements and were given the benefit of doubt.
Use of scalar product	**(ii)** $k\lvert\mathbf{p}\rvert = \lvert\mathbf{p}\bullet\mathbf{q}\rvert\lvert\mathbf{q}\rvert$ Since $\mathbf{p}$ and $\mathbf{q}$ are parallel vectors from part **(i)**, $\theta = 0°$ or $\theta = 180°$ so $\cos\theta = 1$, so $\lvert\mathbf{p}\bullet\mathbf{q}\rvert = \lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert$. $k\lvert\mathbf{p}\rvert = (\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert)\lvert\mathbf{q}\rvert$ $k = \lvert\mathbf{q}\rvert^2$ since $\lvert\mathbf{p}\rvert \neq 0$ $\lvert\mathbf{q}\rvert^2 = k$ $\lvert\mathbf{q}\rvert = \sqrt{k}$ since $k > 0$.	This question was either not attempted or very badly attempted by the candidature. Almost every candidate performed division on the vectors directly, which is conceptually wrong, instead of applying modulus throughout to reduce the equation to that consisting of only scalars and the usual algebraic manipulation can be applied. Other errors include assuming that the vectors are in the same direction (excluding opposite as a possibility) and comparing coefficients.

Q3. Applications of Differentiation (f' graph)		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Sketching graph of $\mathrm{f}'(x)$	**(i)** 	Generally students did quite ok for this part. Some did not present the turning points in coordinates form.
Graphical interpretation of $\mathrm{f}'(x)<0$	**(ii)(a)** $-1<x<2$ or $2<x<5$	Quite badly attempted.
Graphical interpretation of $\mathrm{f}''(x)>0$	**(ii)(b)** $x>2$	Quite badly attempted.

Q4. Functions		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Condition for existence of inverse function	**(i)**  Any horizontal line cuts the graph of f at most once, f is a one-one function, f^{-1} exists.	Badly attempted. Common mistakes: Many used only one counter example to show that f is one-one.
Able to resolve the modulus based on the domain and able to find the inverse function.	**(ii)** $y = x(3-x)$ $y = 3x - x^2$ $x^2 - 3x + y = 0$ $\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + y = 0$ $x = \frac{3}{2} \pm \sqrt{\frac{9}{4} - y}$ Since $x > 2$, $x = \frac{3}{2} + \sqrt{\frac{9}{4} - y}$ $f^{-1}(x) = \frac{3}{2} + \sqrt{\frac{9}{4} - x}$, $D_{f^{-1}} = (0, 2]$	Badly done. Many did not know how to resolve the modulus by looking at the domain. Some made algebraic slip and hence not able to perform completing the square to make x the subject. However majority were able to find the domain.

Sketch of piece wise functions	**(iii)** 	Most students attempted this part and were able to get either 1 or 2 marks. Students need to pay more attention to the presentation such as to label the coordinates of the end points, inclusive or exclusive.
Composite function with piece wise.	**(iv)** $\text{gf}(x) = 2\big(x(3-x)\big) - 2$ $= 2\left(3x - x^2 - 1\right)$	Badly attempted. Many students did not seem to understand the condition of the existence of composite function.
Range of composite function.	**(v)** $\text{R}_{\text{gf}} = (-2, 2]$	Badly done. Students who used the mapping method were able to get the answer easily.

Q5. Parametric Equations

Assessment Objectives	Solution	Examiner's Feedback
Sketch parametric graphs	**(i)**  At $\theta=-\pi$, $x=1-\sin(-\pi)=1$, $y=-\pi+\cos(-\pi)=-\pi-1$ $\therefore(1,-\pi-1)$ At $\theta=\frac{\pi}{2}$, $x=1-\sin\left(\frac{\pi}{2}\right)=0$, $y=\frac{\pi}{2}+\cos\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$ $\therefore\left(0,\frac{\pi}{2}\right)$	Many candidates failed to give the coordinates in the **exact** form, as required by the question. They are also unable to sketch the curve in the specified domain. Endpoints must be clearly labelled with solid circles since both endpoints are included.
Gradient of parametric equations	(ii) $x=1-\sin\theta$, $y=\theta+\cos\theta$, $\frac{dx}{d\theta}=-\cos\theta$, $\frac{dy}{d\theta}=1-\sin\theta$ $\frac{dy}{dx}=\frac{1-\sin\theta}{-\cos\theta}=\frac{\sin\theta-1}{\cos\theta}$ As $\theta\to-\frac{\pi}{2}$, $\cos\left(-\frac{\pi}{2}\right)\to0$, $\frac{dy}{dx}=\frac{1-\sin\theta}{\cos\theta}\to\infty$. The tangent is vertical. **OR** The tangent is parallel to y-axis.	Finding $\frac{dy}{dx}$ was generally well-attempted. However some candidates are still confused over differentiation of simple trigo terms and carelessness. The last part was badly attempted. Candidates found the value of $\frac{dy}{dx}$ but did not proceed to explain "what happened to the tangent". Those who wrote "gradient of tangent tends to infinity" were given the credit. However, "tangent tends to infinity" is not accepted as tangent is a line!

<table>
<tr><td></td><td></td><td>Candidates should learn how to spell “vertical” properly.</td></tr>
<tr><td>Equation of normal</td><td>(iii) At P, $\theta = p$, $(1-\sin p, p+\cos p)$

Gradient of normal $= \dfrac{\cos p}{1-\sin p}$

Equation of normal at P:

$y-(p+\cos p)=\dfrac{\cos p}{1-\sin p}(x-(1-\sin p))$

At y-axis, $x=0$:

$y-(p+\cos p)=\dfrac{\cos p}{1-\sin p}(0-(1-\sin p))$
$y-p-\cos p=-\cos p$
$y=p$

$\therefore Q(0, p)$</td><td>Majority of the candidates could give the gradient of normal based on their results in (ii), while some still struggled.

Many went on to assume values of p to be 0, $\frac{\pi}{4}$ or $\frac{\pi}{2}$ which is not needed. They merely used the results $(0, p)$ to verify instead of showing.

Algebraic manipulation needs to be further strengthened.</td></tr>
<tr><td>Area involving parametric equations</td><td>(iv) Integrating abt y-axis:

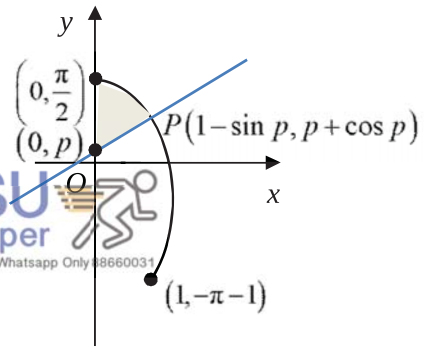

</td><td>This was badly attempted, showing that candidates have very weak foundation in finding areas involving parametric equations.

Diagram has to be aided to solve this part.</td></tr>
</table>

	area $=\frac{1}{2}(p+\cos p-p)(1-\sin p)+\int_{p+\cos p}^{\frac{\pi}{2}} x \, \mathrm{d}y$ $=\frac{1}{2}(\cos p)(1-\sin p)+\int_{p}^{\frac{\pi}{2}} x \frac{\mathrm{d}y}{\mathrm{d}\theta} \, \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin p\cos p}{2}+\int_{p}^{\frac{\pi}{2}}(1-\sin\theta)^2 \, \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\int_{p}^{\frac{\pi}{2}} 1-2\sin\theta+\sin^2\theta \, \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\int_{p}^{\frac{\pi}{2}} 1-2\sin\theta+\frac{1-\cos 2\theta}{2} \, \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\int_{p}^{\frac{\pi}{2}} \frac{3}{2}-2\sin\theta-\frac{\cos 2\theta}{2} \, \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\left[\frac{3}{2}\theta+2\cos\theta-\frac{\sin 2\theta}{4}\right]_{p}^{\frac{\pi}{2}}$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\left[\frac{3}{2}\cdot\frac{\pi}{2}+2\cos\frac{\pi}{2}-\frac{\sin\pi}{4}-\left(\frac{3}{2}p+2\cos p-\frac{\sin 2p}{4}\right)\right]$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\frac{3\pi}{4}-\frac{3}{2}p-2\cos p+\frac{\sin 2p}{4}$ $=\frac{3\pi}{4}-\frac{3}{2}p-\frac{3}{2}\cos p$	For those who attempted, they wrote the limits wrongly: $\int_{p}^{\frac{\pi}{2}} x \, \mathrm{d}y$ instead of $\int_{p+\cos p}^{\frac{\pi}{2}} x \, \mathrm{d}y$ due to confusion between y-limits and θ-limits. Some did not include the area of the triangle. Do not change the parametric equation into Cartesian equation unless you are specifically told to do so!

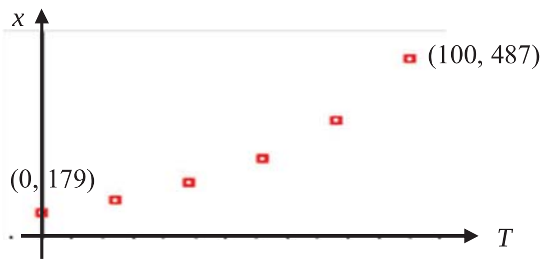
	Integrating abt x-axis: area $=\int_0^{1-\sin p} y\ \mathrm{d}x-\frac{1}{2}(p+\cos p+p)(1-\sin p)$ $=\int_{\frac{\pi}{2}}^{p} y\ \frac{\mathrm{d}x}{\mathrm{d}\theta}\ \mathrm{d}\theta-\frac{1}{2}(2p+\cos p)(1-\sin p)$ $=\int_{\frac{\pi}{2}}^{p}(\theta+\cos\theta)(-\cos\theta)\ \mathrm{d}\theta-p-\frac{1}{2}\cos p+p\sin p+\frac{1}{2}\sin p\cos p$ $=-\int_{\frac{\pi}{2}}^{p}\left(\theta\cos\theta+\cos^2\theta\right)\ \mathrm{d}\theta-p-\frac{1}{2}\cos p+p\sin p+\frac{\sin 2p}{4}$ $=-\left[[\theta\sin\theta]_{\frac{\pi}{2}}^{p}-\int_{\frac{\pi}{2}}^{p}\sin\theta\ \mathrm{d}\theta\right]-\int_{\frac{\pi}{2}}^{p}\cos^2\theta\ \mathrm{d}\theta-p-\frac{1}{2}\cos p+p\sin p+\frac{\sin 2p}{4}$ $=-\left[p\sin p-\frac{\pi}{2}\sin\frac{\pi}{2}+[\cos\theta]_{\frac{\pi}{2}}^{p}\right]-\int_{\frac{\pi}{2}}^{p}\frac{1+\cos 2\theta}{2}\ \mathrm{d}\theta-p-\frac{1}{2}\cos p+p\sin p+\frac{\sin 2p}{4}$ $=-\left[p\sin p-\frac{\pi}{2}+\cos p-\cos\frac{\pi}{2}\right]-\left[\frac{1}{2}\theta+\frac{\sin 2\theta}{4}\right]_{\frac{\pi}{2}}^{p}-p-\frac{1}{2}\cos p+p\sin p+\frac{\sin 2p}{4}$ $=-p\sin p+\frac{\pi}{2}-\cos p-\left[\frac{1}{2}p+\frac{\sin 2p}{4}-\left(\frac{1}{2}\cdot\frac{\pi}{2}+\frac{\sin\pi}{4}\right)\right]-p-\frac{1}{2}\cos p+p\sin p+\frac{\sin 2p}{4}$ $=\frac{\pi}{2}-\frac{3}{2}\cos p-\frac{1}{2}p-\frac{\sin 2p}{4}+\frac{\pi}{4}-p+\frac{\sin 2p}{4}$ $=\frac{3\pi}{4}-\frac{3}{2}p-\frac{3}{2}\cos p$	Candidates who chose to integrate about the x-axis often forgot that they need to subtract away a "trapezium". Again, they wrote their limits wrongly.

<table>
<tr>
<td>Locus</td>
<td>(v) At R, $y=0$,
$0-(p+\cos p)=\dfrac{\cos p}{1-\sin p}\left(x-(1-\sin p)\right)$
$-p-\cos p=\dfrac{\cos p}{1-\sin p}x-\cos p$
$\dfrac{\cos p}{1-\sin p}x=-p$
$x=\dfrac{p(\sin p-1)}{\cos p}$
$\therefore R\left(\dfrac{p(\sin p-1)}{\cos p},0\right)$
Midpoint of QR:
$\left(\dfrac{0+\dfrac{p(\sin p-1)}{\cos p}}{2},\dfrac{p+0}{2}\right)$
$\left(\dfrac{p(\sin p-1)}{2\cos p},\dfrac{p}{2}\right)$
$x=\dfrac{p(\sin p-1)}{2\cos p}$, $y=\dfrac{p}{2}\Rightarrow p=2y$
$x=\dfrac{y(\sin 2y-1)}{\cos 2y}$</td>
<td>Many left this part blank, simply because they are unsure of the word “locus”. However, they are still encouraged to continue finding the midpoint and gain some credit.

There is no need to bring in vectors here to find the midpoint.

The cohort needs to go and revise what is meant by “locus”!</td>
</tr>
</table>

Q6. Discrete Random Variables			
Assessment Objectives	**Solution**		**Examiner's Feedback**
The ability to list all possible outcomes and find their respective probabilities	(i)	$\begin{array}{|c|c|c|c|c|c|}\hline x & \$10 & \$20 & \$40 & \$80 & \$160 \\ \hline \mathrm{P}(X=x) & \frac{1}{9} & \frac{2}{9} & \frac{1}{3} & \frac{2}{9} & \frac{1}{9} \\ \hline \end{array}$	A handful of students are confused over table of outcomes and probability distribution table.
HOT – to identify and use the concept of expectation as long-term average loss in this context	(ii)	$\mathrm{E}(X)=10\left(\frac{1}{9}\right)+20\left(\frac{2}{9}\right)+40\left(\frac{1}{3}\right)+80\left(\frac{2}{9}\right)+160\left(\frac{1}{9}\right)=\$54.44$ Since \$54.44 should be the profit, which is 40% of \$$R$, Therefore, $0.4R=54.444$, Hence, need to set R to be \$136	Most students were able to find expectation but evaluated it wrongly due to sheer carelessness in pressing the calculator.
Finding the variance of a d.r.v	(iii)	$\mathrm{Var}(X)$ $=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ $=10^2\left(\frac{1}{9}\right)+20^2\left(\frac{2}{9}\right)+40^2\left(\frac{1}{3}\right)+80^2\left(\frac{2}{9}\right)+160^2\left(\frac{1}{9}\right)-\left(\frac{490}{9}\right)^2$ $=1935.8$ Hence, $\sigma=\sqrt{1935.8}=44.0$ $\therefore \mathrm{P}(X<\sigma)=\mathrm{P}(X<44.0)=\frac{2}{3}$	Students recalled the variance formula wrongly. For those who did so correctly, the numerical value was often wrong. Many students assumed normal distribution for X and went on to find $\mathrm{P}(X<\sigma)$. They failed to recognize that this question involves discrete random variable.

Q7. Correlation and Regression			
Assessment Objectives	**Solution**		**Examiner's Feedback**
Understand that $(\bar{T},\bar{x})$ always lie on the regression line		$x=2.94857T+146.238$ $\frac{1474+k}{6}=2.94857(50)+146.238$ $k=288 \quad$ (3sf)	Common mistake is to find k by substituting $T = 60$ into the given regression line.
Scatter diagram.	(i)		Common mistake is to take x as the horizontal axis (independent variable). A lot of students also did not label the axes nor the greatest and smallest x and y values.
Linear transformation	(ii)	Least square estimate of a is 5.12898 = 5.13 (3 sf) Least square estimate of b is 0.0098734 = 0.00987 (3 sf) r between T and ln x is 0.990 (3 sf)	Quite a number of students did not realised that they can/must use $k = 288$ to find the least square estimates of a and b, as well as the value of r between T and ln$x$. Hence not being able to get the correct answers.
Able to compare the models based on the scatter diagram and the r value.	(iii)	In (i), as T increases, x increases at an increasing rate instead of constant rate. In part (ii), the r value between T and ln x as compared to the r value between T and x is closer to 1. Hence $\ln x=0.0098734T+5.12898$ is the better model.	A lot of students only managed to get the part on comparing the r values between T and ln$x$ and between T and x. While only some managed to describe that "As T increases, x increases at an increasing rate instead of constant rate." on the graph of (i).

Able to use the appropriate line to do the prediction and knowing the factors of the reliability.	(iv)	$\ln(300)=0.0098734T+5.12898$ $T=58.2$ (3sf) The prediction is reliable as $x=300$ is within the data range of x and r value is close to 1.	Quite a number of students uses 3s.f. for their intermediate steps for calculations of the estimate for T and end up with a less accurate answer. A number of students wrote "both x and T are in the data range hence it is a reliable prediction." Instead of "$x=300$ is within the data range of x". A lot of students also missed out "r value is close to 1" as part of the reasons for reliability of the prediction.

Q8. Binomial Distribution

<table>
<tr><th>Assessment Objectives</th><th></th><th>Solution</th><th>Examiner's Feedback</th></tr>
<tr><td>Identifying binomial distribution</td><td>(i)</td><td>Let X be the number of bullseyes in a series.
$X \sim \mathrm{B}(15, 0.7)$

P(good series)
$= \mathrm{P}(X \geq 11)$
$= 1 - \mathrm{P}(X \leq 10)$
$= 0.515$</td><td>This part was generally well attempted.

Some students did not know how to define a binomial random variable. Some had defined it wrongly as the probability of getting a bullseyes.

Some students mixed up equivalent events with complementary events. The probability of getting at least 11 bullseyes is equal to the probability of getting at most 4 non-bullseyes. In other words, $\mathrm{P}(X \geq 11) \neq 1 - \mathrm{P}(X \leq 4)$.

Some students considered the complementary events wrongly. A common mistake is $\mathrm{P}(X \geq 11) = 1 - \mathrm{P}(X \leq 11)$.

Some students considered the probabilities for the outcomes of the shots but failed to consider the ${}^nC_r$.</td></tr>
<tr><td>Ability to interpret question correctly and setting up the relevant mathematical representation of the situation.

Efficient use of GC in obtaining answer</td><td>(ii)</td><td>Let Y be the number of good series in n attempts
$Y \sim \mathrm{B}(n, 0.51549)$

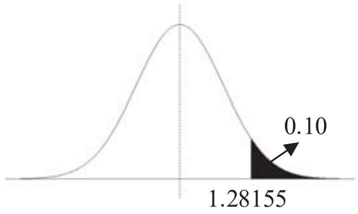
$\mathrm{P}(Y \geq 6) \geq 0.99$
$1 - \mathrm{P}(Y < 6) \geq 0.99$
$\mathrm{P}(Y \leq 5) \leq 0.01$
Using GC,
<table><tr><th>n</th><th colspan="2">$\mathrm{P}(Y \geq 6)$</th></tr><tr><td>20</td><td>0.9853</td><td>< 0.99</td></tr><tr><td>21</td><td>0.9908</td><td>> 0.99</td></tr><tr><td>22</td><td>0.9943</td><td>> 0.99</td></tr></table></td><td>This part was generally well attempted.

Students should state the mathematical inequality clearly which they should solve. This, in turn, would help them to write the correct headers for GC table. A significant percentage of students were not able to do write appropriate headers in GC table.

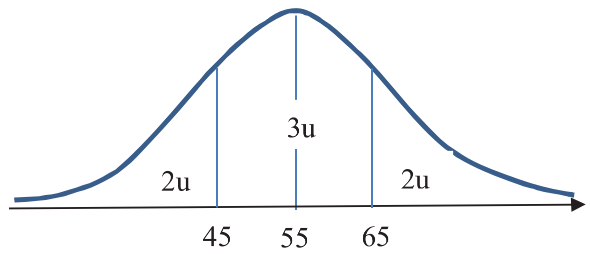
Students should also present a GC table to support their workings so that credit might be given when the wrong value of n was found.</td></tr>
</table>

		Hence, least $n = 21$	Some looked at the wrong range of n as they had compared the probabilities against 0.001 instead of 0.01.
Identifying the need to apply the central limit theorem and obtain the correct parameters	(iii)	From (i), Let X be the number of bullseyes in a series. **Method 1:** Let $\overline{X}$ be the average number of bullseyes per series. Since sample size $n = 40$ is sufficiently large, $\overline{X} \sim \text{N}\left(15\times0.7 , \dfrac{15\times0.7\times0.3}{40}\right)$ approximately by Central Limit Theorem Required probability $= \text{P}(\overline{X} > 10)$ $= 0.963$ **Method 2:** Let $X_1 + X_2 + \ldots + X_{40}$ be the total number of bullseyes for 40 series. Since sample size $n = 40$ is sufficiently large, $X_1 + X_2 + \ldots + X_{40} \sim \text{N}(40(15\times0.7) , 40(15\times0.7\times0.3))$ approximately by Central Limit Theorem Required probability $= \text{P}(X_1 + X_2 + \ldots + X_{40} > 40\times10)$ $= 0.963$	This part was badly done. Students should pick out the keywords 'estimate' and 'on average' in the question. A large percentage of students failed to • recognize the use of Central Limit Theorem for the distribution of the sample mean, • identify the sample size 40 correctly, • identify the success event (getting a bullseye) and its probability of 0.7 correctly. A significant number of students wrote $X \sim \text{N}(10.5, 3.15)$. Students should recognize that the random variable X does not follow a normal distribution. The Central Limit Theorem is used to approximate the distribution of sample mean $\overline{X}$, not X. A minority of students identified the total number of bullseyes as a binomial distribution and proceeded to find the required probability. They overlooked the keyword 'estimate' in this question part and thus were not given full credit.

Q9. Hypothesis Testing			
Assessment Objectives	**Solution**		**Examiner's Feedback**
Unbiased estimate of population mean and variance	(i)	Let X be the random variable denoting the speed of a car in the school compound. Unbiased estimate of population mean $=\frac{1325}{50}=26.5$ Unbiased estimate of population variance $=\frac{50}{49}\left[7.75^2\right]=61.288=61.3\text{ (3sf)}$	Almost all candidates could calculate unbiased est. of pop. Mean Most candidates could not recall the formula to link sample variance to unbiased ets. Of pop. Variance.
Apply to carry out the hypothesis testing with the concept of CLT.	(ii)	$H_0: \mu = 25$ $H_1: \mu > 25$, where μ is the population mean of X. Under H_0, Since n is large, by CLT, $\bar{X} \sim N\left(25, \frac{61.288}{50}\right)$ approx. $z_{test}=\frac{26.5-25}{\sqrt{61.288/50}}$ Since p-value $=0.0877345>0.05$, we do not reject H_0 and conclude that we have insufficient evidence at 5% level of significance that the mean speed of the cars is more than 25.	Many candidates used 26.5 instead of 25 for the population mean. Many candidates did not get the correct phrasing of the conclusion, missing out key points such as 5% level of significance/ do not reject H_0 etc
Able to use the critical approach to find the unknown n given the conclusion of the test.	(iii)	$X \sim N(25,36)$ $\bar{X} \sim N\left(25, \frac{36}{n}\right)$ $z_{test}=\frac{26.5-25}{\sqrt{36/n}}=0.25\sqrt{n}$	Many candidates are able to obtain critical value 1.28155, but could not form correct inequality

		 Since H_0 is rejected at 10 % level of significance, $z_{test} \geq 1.28155$, $0.25\sqrt{n} \geq 1.28155$ $n \geq 26.278$ $\{n : n \in \mathbb{Z}, n \geq 27\}$	Candidates need to note that n is an integer, and also the answer should be given in a set of values
Able to interpret the question and able to find the unknown given the probability.	(iv)	$X \sim \mathrm{N}(25, 36)$ $\mathrm{P}(X > t) = 0.75$ $t = 21.0$ (3 s.f.)	Many candidates were not able to interpret the question.

Q10. Normal Distribution

Assessment Objectives	Solution		Examiner's Feedback
Use of symmetrical properties of the normal distribution curve Finding unknown parameters by means of standardization	(i)	 $2\text{P}(X > 45) = 5\text{P}(X > 65)$ $\frac{\text{P}(X > 45)}{\text{P}(X > 65)} = \frac{5}{2}$ $\therefore$ From diagram, $\text{P}(X < 45) = \frac{2}{7}$ $\text{P}(Z < \frac{45-55}{\sigma}) = \frac{2}{7}$ $\frac{-10}{\sigma} = -0.56595$ $\sigma = 17.7$	Many students were unable to observe the relationship and symmetry between $\text{P}(X > 65)$ and $\text{P}(X < 45)$ A significant number of students were not able to present the standardization steps correctly with errors and omissions in presentation. Students need to learn to use graphical representation for probabilities under Normal Distribution.
Proper combining of random variables, and finding the associated parameters Identifying the need to use the modulus in the setup and manipulate it properly in the context of finding probability	(ii)	$(X_1 + X_2 + X_3 - 4Y) \sim \text{N}(3\times55 - 4\times45\ ,\ 3\times17.669^2 + 4^2\times10^2)$ Required probability $= \text{P}(\lvert X_1 + X_2 + X_3 - 4Y\rvert \le 10)$ $= \text{P}(-10 \le (X_1 + X_2 + X_3 - 4Y) \le 10)$ $= 0.151$	Most common mistake by students was the inability to interpret the question and distinguish between $X_1 + X_2 + X_3$ and $3X$ which led to the wrong calculation of variance. Also, majority of students did not take the modulus as required by the question.

HOT- cannot simply just quote standard distribution of sample mean because now 2 different random variables are involved in the sample	(iii)	$\frac{X_1+X_2+X_3+Y_1+Y_2+Y_3+Y_4}{7} \sim \mathrm{N}\left(\frac{3\times 55+4\times 45}{7}, \frac{3\times 17.669^2+4\times 10^2}{7^2}\right)$ Required probability $=\mathrm{P}\left(\frac{X_1+X_2+X_3+Y_1+Y_2+Y_3+Y_4}{7}\geq 50\right)$ $=0.446$	Many students were unable to interpret the question to obtain the random variable denoting the "mean score": $\frac{X_1+X_2+X_3+Y_1+Y_2+Y_3+Y_4}{7}$
Theory	(iv)	The random variables X and Y are independent of each other.	Many students were not aware that the conditions for summing random variables that are Normally distributed. There were some students who listed the conditions for a Binomial distribution instead.
Theory	(v)	Let C be the score of a randomly chosen student in Group C. If normal distribution is assumed, $\mathrm{P}(C<0)=0.0548$ A significant proportion would fall under an inadmissible area, hence normal distribution is not suitable as a model.	Most students gave vague answers about the large variance without substantiation.

Q11. Probability			
Assessment Objectives	**Solution**		**Examiner's Feedback**
Systematic consideration/application of P&C techniques in given situation	(i)	No of ways to form the string unrestricted = $4^5 - 4 = 1020$ No of ways to form the string using 2 letters = ${}^4C_2 \times (2^5 - 2) = 180$ Hence $P(A) = \frac{180}{1020} = \frac{3}{17} = 0.176$ No of ways to have a palindrome = $4^3 - 4 = 60$ Hence $P(B) = \frac{60}{1020} = \frac{1}{17} = 0.0588$	Poorly attempted. Many candidates forgot that they are supposed to find a probability, not the number of ways, so end up with answers like $P(A) = 24768737542$ or $P(B) = -0.446$
Probability of intersection of two events is not necessarily the product of their probabilities Mathematical proof of independence	(ii)	No of ways to form palindrome of 2 letters = ${}^4C_2 \times (2^3 - 2)$ or ${}^4C_2 \times \frac{3!}{2!} \times 2 = 36$ Hence $P(A \cap B) = \frac{36}{1020} = \frac{3}{85} = 0.0353$ $P(A) \times P(B) = 0.0104$ Since $P(A \cap B) \neq P(A) \times P(B)$, A and B are not independent.	Some candidates have incorrect conceptual understanding. The tests they used were $P(A \cap B) \neq 0$ $P(A \cap B) \neq P(A) + P(B)$
Ability to identify that the question is describing the union of two events, and use the relevant formula to find the probability	(iii)	$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{17} + \frac{1}{17} - \frac{3}{85} = \frac{1}{5} = 0.2$	Some candidates have incorrect conceptual understanding. They used $P(A \cup B) = P(A) + P(B) + P(A \cap B)$ Some did not see that this is asking for the union in the first place.
Finding conditional probability	(iv)	$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{3/85}{1/17} = \frac{3}{5} = 0.6$	Some candidates have incorrect conceptual understanding. They assumed that $P(A \cap B) = P(A) \cdot P(B)$

CJC 2019 H2 Mathematics

Prelim P2

Section A: Pure Mathematics (40 Marks)

1 The function h is given by $\mathrm{h}(x) = ax^3 + bx^2 + \dfrac{x}{c}$, $x \in \mathbb{R}$, where a, b and c are real constants.

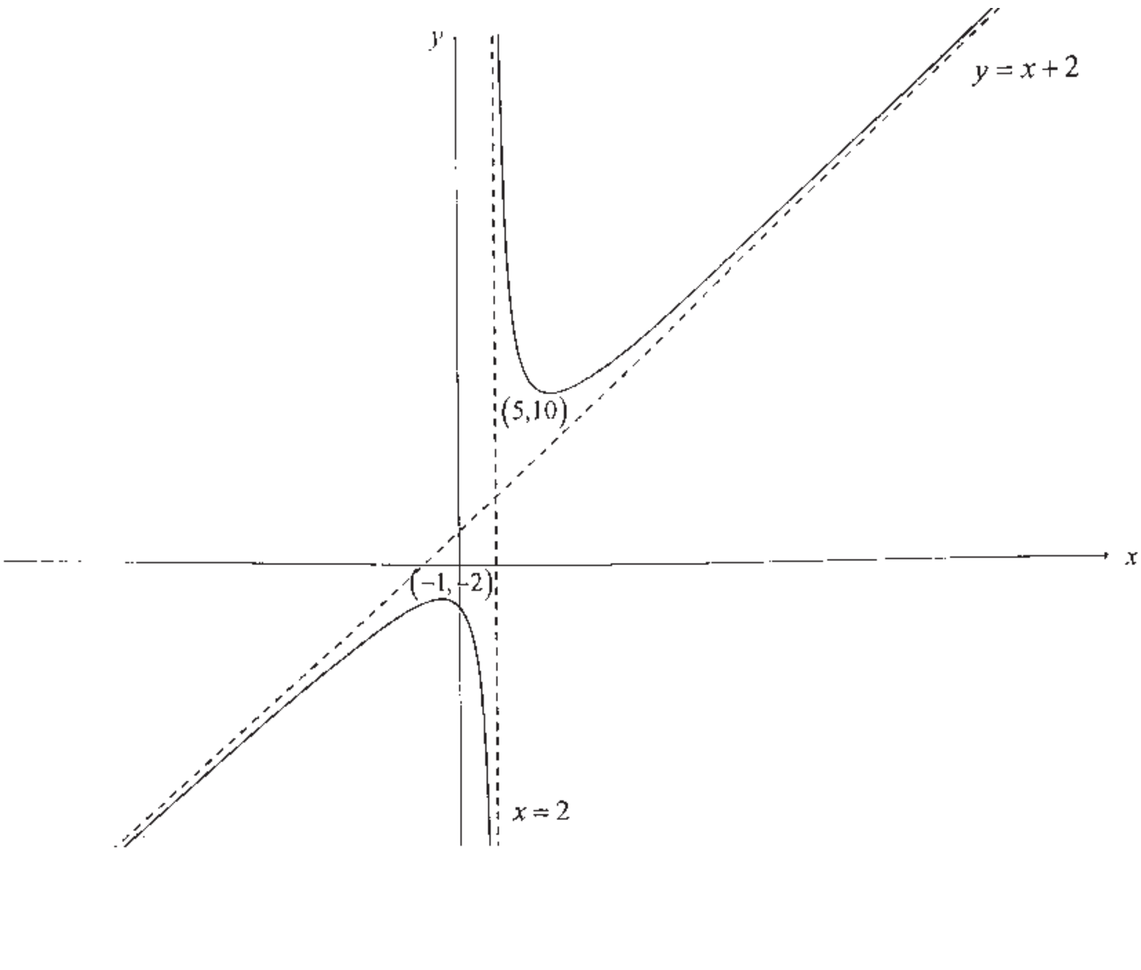
The graph of $y = \mathrm{h}(x)$ passes through the point $\left(1, \dfrac{13}{4}\right)$. The point $(-8, 642)$ lies on the graph of $y = \mathrm{h}(|x|)$ and the point $\left(4, \dfrac{1}{97}\right)$ lies on the graph of $y = \dfrac{1}{\mathrm{h}(x)}$. Find the values of a, b and c. [4]

2 Given that $k\mathbf{p} = (\mathbf{p}.\mathbf{q})\mathbf{q}$ where k is a positive constant, $\mathbf{p}$ and $\mathbf{q}$ are non – zero vectors.

(i) What is the geometrical relationship between $\mathbf{p}$ and $\mathbf{q}$? [1]

(ii) Find $|\mathbf{q}|$ in terms of k. [3]

3 The diagram below shows the graph of $y = \mathrm{f}(x)$. The curve has a minimum point $(5,10)$ and a maximum point $(-1,-2)$. The lines $x = 2$ and $y = x + 2$ are asymptotes of the graph.



(i) Sketch the curve $y = \mathrm{f}'(x)$, indicating clearly the coordinates of the points where the graph crosses the x axis and the equations of any asymptotes. [3]

(ii) State the range of values of x for which the graph of $y = \mathrm{f}(x)$ is

(a) strictly decreasing, [1]

(b) concave upwards. [1]

4 The function f is defined by $\mathrm{f} : x \to x|x-3|, \quad x \in \mathbb{R}, 2 \le x < 3$.

(i) Explain, with the aid of a sketch, why the inverse function f^{-1} exists.

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

It is given that $\mathrm{g}(x) = \begin{cases} 2x-2, & 0 \le x \le 2 \\ \dfrac{10-2x}{x+1}, & 2 < x < 4. \end{cases}$

(iii) Sketch the graph of $y = \mathrm{g}(x)$. [2]

(iv) Find gf (x). [2]

(v) Find the range of gf. [1]

5 A curve C has parametric equations $x = 1 - \sin\theta, \quad y = \theta + \cos\theta$, where $-\pi \le \theta \le \dfrac{\pi}{2}$.

(i) Sketch the graph of C, stating the exact coordinates of the end points. [2]

(ii) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ. What can be said about the tangent to C as $\theta \to -\dfrac{\pi}{2}$?

[3]

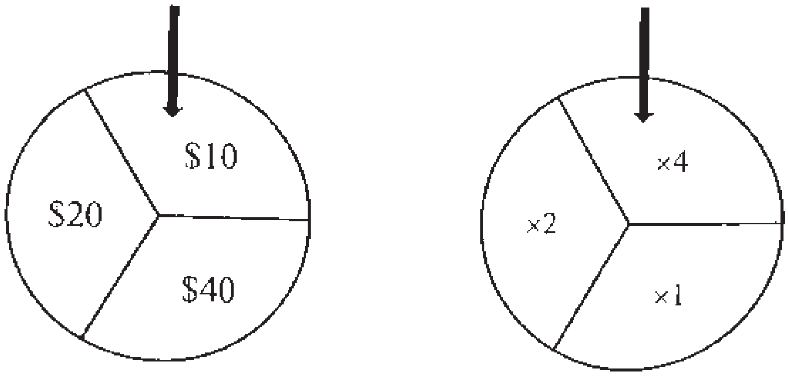
(iii) A point P on C has parameter p, where $0 < p < \dfrac{\pi}{2}$. Show that the normal to C at P crosses the y axis at point Q with coordinates $(0, p)$. [3]

(iv) Show that the area of region bounded by C, the normal to C at point P and the y axis is given by $a\pi + bp + c\cos p$, where a, b and c are to be determined. [6]

(v) The normal to C at P also crosses the x axis at point R. Find a Cartesian equation of the locus of the midpoint of QR as p varies. [3]

Section B: Probability and Statistics (60 marks)

6 To raise its profile, ABC Supermart devised a publicity where, if a customer has purchases of $\$R$ or more in a single receipt, he is qualified to participate in a sure-win "Lucky Spin" game. In the game, the customer spins the first wheel (which is divided into 3 equal parts) to see how much money he wins, then he spins a second wheel to see how much his winning is multiplied by.



First wheel   Second wheel

For example, based on the illustration above, if the result from the first spin is "$10", and the result of the second spin is " $\times$ 4", then the person playing the game would have received $40.

Let X denote the amount won from playing the game.

(i) Tabulate the probability distribution of X. [2]

(ii) ABC Supermart makes a profit of 40% on all purchases made by customers. It wishes to use the profits generated from every qualifying receipt to offset the amounts given away in the game. Determine the value of $\$R$ that the Supermart needs to set for a customer to qualify to play the game. Give your answer correct to the nearest dollar. [3]

(iii) Find the value of σ, the standard deviation of X. Hence find $P(X < \sigma)$. [3]

7 An experiment is being carried out to study the correlation between the solubility of sugar in water and the temperature of the water. The amount of sucrose x (in grams) dissolved in 100 ml of water for different water temperatures T (in degree Celsius) is recorded. The results shown in the table. Unfortunately, one of the values of x was accidentally deleted from the records later on and it is indicated by k as shown below.

T	0	20	40	60	80	100
X	179	204	241	k	363	487

It is given that the equation of the regression line of x on T is $x = 2.94857T + 146.238$. Show that $k = 288$.

(i) Draw a scatter diagram to illustrate the data, labelling the axes clearly. [1]

The value of the product moment correlation coefficient between x and T is 0.959. It is thought that a model given by the formulae $\ln x = a + bT$ may also be a suitable fit to the data where a and b are constants.

(ii) Calculate the least square estimates of a and b and find the value of the product moment correlation coefficient between T and $\ln x$. [2]

(iii) Use your answers to parts (i) and (ii) to explain which of $x = 2.94857T + 146.238$ or

$\ln x = a + bT$ is the better model. [2]

(iv) Hence, predict the value of T for which $x = 300$. Comment on the reliability of your prediction. [2]

8 Cynthia is a skilled pistol shooter who hits the bullseye of the target 70% of the time. During her training sessions, she shoots a series of 15 shots on a target before changing to a new target.

A series is considered "good" if she is able to hit the bullseye at least 11 times on the target.

(i) Find the probability that Cynthia is able to obtain a good series. [2]

Cynthia is able to end the training session early if she is able to obtain at least 6 good series in n attempts.

(ii) Find the least value of n such that she can be at least 99% certain of being able to end the training session early. [4]

Suppose Cynthia has completed 40 of the fifteen-shot series,

(iii) estimate the probability that she has, on average, hit the bullseye more than 10 times per series. [3]

9 The security guard of a particular school claims that the average speed of the cars in the school compound is greater than the speed limit of 25 km/h. To investigate the security guard's claim, the traffic police randomly selected 50 cars and the speed was recorded. The total speed and the standard deviation of the 50 cars are found to be 1325 km/h and 7.75 km/h respectively.

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) Test at 5% level of significance whether there is sufficient evidence to support the security guard's claim. [4]

It is now known that the speed of the cars is normally distributed with mean 25 km/h and standard deviation of 6 km/h.

(iii) A new sample of n cars is obtained and the sample mean is found to be unchanged. Using this sample, the traffic police conducts another test at 10% level of significance and concludes that the security guard's claim is valid. Find the set of values n can take. [3]

(iv) What is the speed exceeded by 75% of the cars? [1]

10 The Mathematics examination score of a randomly chosen student in Group A is X, where X follows a normal distribution with mean 55 and standard deviation σ. The Mathematics examination score of a randomly chosen student in Group B is Y, where Y follows a normal distribution with mean 45 and standard deviation 10.

(i) It is known that $2P(X > 45) = 5P(X > 65)$. Show that $\sigma = 17.7$, correct to 3 significance figures. [3]

(ii) Find the probability that total score of 3 randomly selected students from Group A differ from 4 times the score of a randomly selected student from Group B by at most 10. [4]

(iii) Find the probability that the mean score of 3 randomly selected students from Group A and 4 randomly selected students from Group B is at least 50. [3]

The Mathematics examination scores of students in Group C are found to have a mean of 40 and standard deviation of 25. Explain why the examination scores of students in Group C are unlikely to be normally distributed. [1]

11 A palindrome is a string of letters or digits that is the same when you read it forwards or backwards. For example: HHCCHH, RACECAR, STATS are palindromes.

A computer is instructed to use any of the letters **R, O, F, L** to randomly generate a string of 5 letters. Repetition of any letter is allowed, but the string cannot contain only one letter. For example, RRRRR is not allowed.

Events A and B are defined as follows:

A: the string generated contains 2 distinct letters

B: the string generated is a palindrome

(i) Find $P(A)$ and $P(B)$. [5]

(ii) Find $P(A \cap B)$ and hence determine if A and B are independent. [3]

(iii) Find the probability that the string generated either contains 2 distinct letters, or that it is a palindrome, or both. [2]

(iv) Find the probability that the string generated contains 2 distinct letters, given that it is a palindrome. [2]

Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9758/01

Paper 1

September 2019

3 hours

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Total
Score													
Max Score	**4**	**5**	**6**	**7**	**7**	**7**	**8**	**9**	**10**	**13**	**12**	**12**	**100**

1 A confectionary bakes 20 banana cakes, 50 chocolate cakes and 30 durian cakes every day. The total price of 1 banana cake, 1 chocolate cake and 1 durian cake is \$29.50. On a particular day, at 7 pm, the confectionary has collected \$730 from the sales of the cakes, and there were half the banana cakes, one-tenth of the chocolate cakes and one third of the durian cakes left. In order to sell as many cakes as possible, all cakes were discounted by 40% from their respective selling price from 7 pm onwards. By closing time, all the cakes were sold and the total revenue for the entire day was \$880. Determine the selling price of each type of cake before discount. [4]

2 **(a)** Without using a calculator, solve $\dfrac{30-11x}{x^2-9} \le -2.$ [3]

(b) Solve $(a-3bx^2)\mathrm{e}^{ax-bx^3} < 0,$ where a and b are positive constants. [2]

3 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to origin O, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel.

(i) Given that B lies on the line segment AC such that $\overrightarrow{BC} = 5\mathbf{b} - \mu\mathbf{a}$, find the value of μ. Hence find $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(ii) The point N is the midpoint of OC. The line segment AN meets OB at point E. Find the position vector of E. [4]

4 The function f is defined as follows:

$$\mathrm{f}(x) = x + \frac{1}{x-a}, \quad a < x \leq b$$

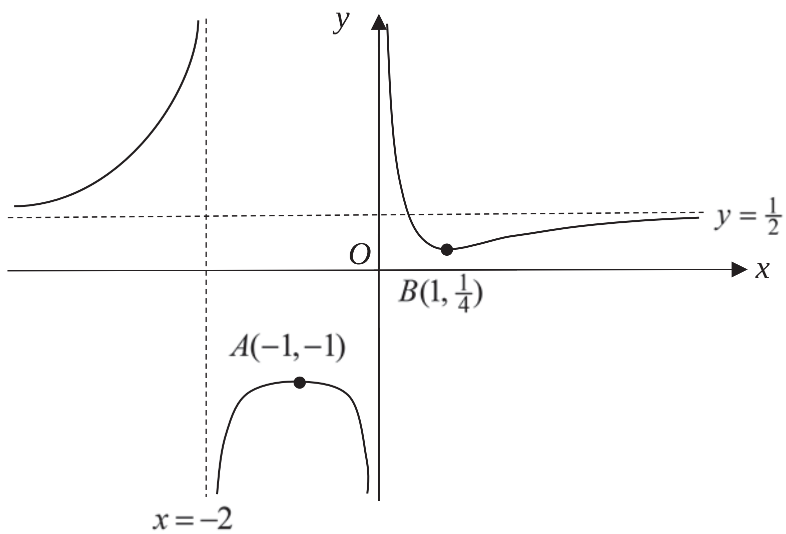
where a is a positive constant.

(i) Given that f^{-1} exist, show that $b \leq a+1$. [2]

(ii) Given that $a = 1$ and $b = 2$, find $\mathrm{f}^{-1}(x)$ and the domain of f^{-1}. [5]

5 **(a)** Describe a sequence of two transformations that maps the graph of $y = \ln\left(\dfrac{x^2}{x+1}\right)$ onto the graph of $y = \ln\left(\dfrac{2x+1}{4x^2}\right)$. [2]

(b) The diagram below shows the graph of $y = \mathrm{f}(x)$. It has a maximum point at $A(-1,-1)$ and a minimum point at $B(1, \frac{1}{4})$. The graph has asymptotes $y = \frac{1}{2}$, $x = 0$ and $x = -2$.



Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(2-x)$, [2]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

stating clearly the equations of any asymptotes, coordinates of any points of intersection with both axes and the points corresponding to A and B.

6 The sequence of complex numbers $\{w_n\}$ are defined as follows

$$w_n = \frac{\left[1+(n-1)\mathrm{i}\right]\left[1+(n+1)\mathrm{i}\right]}{(1+n\mathrm{i})^2} \text{ for } n \in \mathbb{Z}^+.$$

(i) Show that $\arg(w_n) = p\left[\arg(1+(n-1)\mathrm{i})\right] + q\left[\arg(1+n\mathrm{i})\right] + r\left[\arg(1+(n+1)\mathrm{i}\right]$, where p, q and r are constants to be determined. [1]

Consider a related sequence $\{z_n\}$ where $z_n = w_1 w_2 \ldots w_n$, the product of the first n terms of the above sequence.

(ii) Use the method of differences to show that $\arg z_n = -\frac{1}{4}\pi - \arg(1+n\mathrm{i}) + \arg[1+(n+1)\mathrm{i}]$. [4]

(iii) Deduce the limit of $\arg(z_n)$ as $n \to \infty$. Hence write down a linear relationship between $\text{Re}(z_n)$ and $\text{Im}(z_n)$ as $n \to \infty$. [2]

7 A curve C has parametric equations $x = 4\sin 2\theta - 2,\ y = 3 - 4\cos 2\theta$ for $0 \le \theta < \pi$.

(i) Find a cartesian equation of C. Give the geometrical interpretation of C. [3]

(ii) P is a point on C where $\theta = \frac{3}{8}\pi$. The tangent at P meets the y-axis at the point T and the normal at P meets the y-axis at the point N. Find the exact area of triangle NPT. [5]

8 The equations of two planes P_1 and P_2 are $x - 2y + 3z = 4$ and $3x + 2y - z = 4$ respectively.

(i) The planes P_1 and P_2 intersect in a line L. Find a vector equation of L. [2]

The equation of a third plane P_3 is $5x - ky + 6z = 1$, where k is a constant.

(ii) Given that the three planes have no point in common, find the value of k. [2]

Use the value of k found in part **(ii)** for the rest of the question.

(iii) Given Q is a point on L meeting the x-y plane, find the shortest distance from Q to P_3. [3]

(iv) By considering the plane containing Q and parallel to P_3 or otherwise, determine whether the origin O and Q are on the same or opposite side of P_3. [2]

9 It is given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\mathrm{e}^{-y} - 1$ and that $y = 0$ when $x = 0$.

(i) (a) Show that $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -\left(a + b\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, where a and b are constants to be determined. [2]

(b) Hence, find the first three non-zero terms in the Maclaurin series expansion for y. [2]

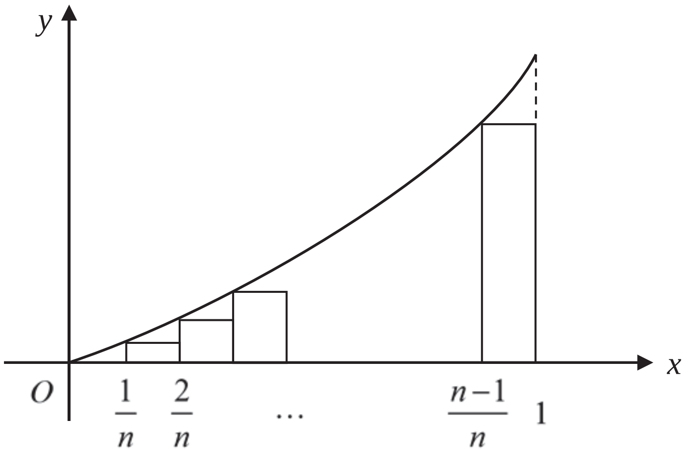
(ii) Find the particular solution of the differential equation, giving your answer in the form $y = \mathrm{f}(x)$. [4]

(iii) Denoting the answer in **(i)(b)** as $\mathrm{g}(x)$, for $x \geq 0$, find the set of values of x for which the value of $\mathrm{g}(x)$ is within ± 0.05 of the value of $\mathrm{f}(x)$. [2]

10 **(a)** Find $\int \frac{x}{\sqrt{2x-1}}\,\mathrm{d}x.$ [3]

(b) Using the substitution $t = \tan x,$ find $\int \frac{1}{4\cos^2 x + 9\sin^2 x}\,\mathrm{d}x.$ [4]

(c)



The diagram above shows part of the graph of $y = x^2 + 3x$, with rectangles approximating the area under the curve from $x = 0$ to $x = 1$. The area under the curve may be approximated by the total area, A, of $(n-1)$ rectangles each of width $\frac{1}{n}$. Given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, show that $A = \frac{(n-1)(11n-1)}{6n^2}$.

Explain briefly how the value of $\int_0^1 x^2 + 3x \, dx$ can be deduced from this expression, and hence find this value exactly without integration. [6]

11 The department of statistics of a country has developed two mathematical functions to analyse the foreign worker policy. The first function f models the amount of strain to the country's infrastructure (housing, transportation, utilities and access to medical care etc.) based on the number of foreign workers allowed into the country and is defined as follows:

$$\mathrm{f}(x) = (x-5)^3 + 200, \quad 0 \le x \le 15$$

where x denotes the number of foreign workers, in ten thousands, allowed into the country and $\mathrm{f}(x)$ denotes the amount strain to the country's infrastructure.

The second function g models the happiness index, from 0 (least happy) to 1 (most happy), of the country's local population based on the amount of strain to the country's infrastructure and is defined as follows:

$$\mathrm{g}(w) = \ln\left(\mathrm{e} - \frac{w}{1000}\right), \quad 0 \le w \le 1000(\mathrm{e}-1)$$

where w denotes the amount of strain to the country's infrastructure and $\mathrm{g}(w)$ denotes the happiness index of the country's local population.

(i) The composite function gf models the happiness index based on the number of foreign workers, show that this function exists. [2]

(ii) Find range of values for the happiness index of the country's local population if its government plans to allow 70,000 to 110,000 foreign workers into the country. [3]

(iii) Determine whether the happiness index increases or decreases as x increases. [3]

A third function h models the gross domestic product (GDP) of the country based on the number of foreign workers (in ten thousands), x, allowed into the country and is defined as follows:

$$\mathrm{h}(x) = 400 - (x-10)^2, \quad 0 \le x \le 15$$

where $\mathrm{h}(x)$ denotes the GDP in billions of dollars.

(iv) Find the range of values for the GDP if the government plans to have a happiness index from 0.7 to 0.9 in order to secure an electoral win for the coming elections. [4]

12 In a particular chemical reaction, every 2 grams of U and 1 gram of V are combined and converted to form 3 grams of W. Let u, v and w denote the mass (in grams) of U, V and W respectively present at time t (in minutes). According to the law of mass action, the rate of change of w with respect to t is proportional to the product of u and v. Initially, $u = 40$, $v = 50$ and $w = 0$.

(i) Show that $\frac{\mathrm{d}w}{\mathrm{d}t} = k(w-60)(w-150)$, where k is a positive constant. [3]

It is observed that when $t = 5,\ w = 10$.

(ii) Find w when $t = 20$, giving your answer to two decimal places. [7]

(iii) What happens to w for large values of t? [2]

Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9758/02

Paper 2

September 2019

3 hours

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

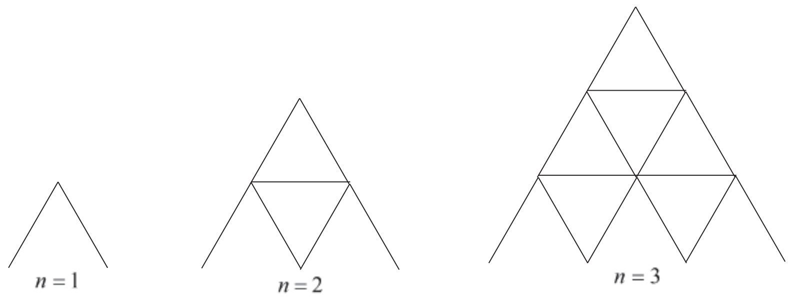
For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	**7**	**8**	**8**	**8**	**9**	**4**	**7**	**9**	**12**	**12**	**16**	**100**

 This question paper consists of **23** printed pages and **1** blank page.

Section A: Pure Mathematics [40 marks]

1 Ryan has playing cards which he stacks into pyramids. He will begin by stacking up cards to form a pyramid with 1 level, followed by another pyramid with 2 levels and so forth. The pyramids with n levels for different values of n are shown below:



Let S_n denotes the number of cards in a pyramid with n levels. It is given that $S_n = an^2 + bn + c$ for some constants a, b and c.

(i) Give an expression of the number of additional cards needed to form a pyramid of nth level from $(n-1)$th level. Leave your expression in terms of a, b and n. [2]

Commented [jo1]: APY

(ii) Find the values of a, b and c. [2]

(iii) Hence prove that S_n is the sum of an arithmetic progression and state the common difference. [2]

(iv) One pyramid of each level from 1 to 23 is formed. Find the total number of cards required to form these 23 pyramids. [1]

2 A curve C has equation $3x^2 - 2xy + 5y^2 = 14$.

Commented [jo2]: CHC – new qn

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x - y}{x - 5y}$. [2]

(ii) Find the exact x-coordinates of the points on the curve C at which the tangent is parallel to the y-axis. [3]

(iii) A point $P(x, y)$ moves along the curve C in such a way that y decreases at a constant rate of 7 units per second. Given that x increases at the instant when $y = 1$, find the corresponding rate of change in x. [3]

3 The complex number z is such that $az^2+bz+a=0$ where a and b are real constants. It is given that $z=z_0$ is a solution to this equation where $\mathrm{Im}(z_0)\neq 0$.

Commented [OMFJ3]: HG - ok

(i) Verify that $z=\dfrac{1}{z_0}$ is the other solution. Hence show that $|z_0|=1$. [4]

Take $\mathrm{Im}(z_0) = \frac{1}{2}$ for the rest of the question.

(ii) Find the possible complex numbers for z_0. [2]

(iii) If $\mathrm{Re}(z_0) > 0$, find b in terms a. [2]

4 The complex number w has modulus $\sqrt{2}$ and argument $\frac{1}{4}\pi$ and the complex number z has modulus $\sqrt{2}$ and argument $\frac{5}{6}\pi$.

Commented [jo4]: APY

(i) By first expressing w and z in the form $x+\mathrm{i}y$, find the exact real and imaginary parts of $w+z$. [3]

(ii) On the same Argand diagram, sketch the points P, Q, R representing the complex numbers z, w and $z+w$ respectively. State the geometrical shape of the quadrilateral $OPRQ$. [3]

(iii) Referring to the Argand diagram in part **(ii)**, find $\arg(w+z)$ and show that $\tan\left(\frac{11}{24}\pi\right)=\dfrac{a+\sqrt{2}}{\sqrt{6}+b}$ where a and b are constants to be determined. [2]

5 The curves C_1 and C_2 have equations $y = \dfrac{x-b}{x-a}$ and $y = \dfrac{x-b}{b}$ respectively, where a and b are constants with $1 < a < b$.

Commented [OMFJ5]: JO

(i) Show that the x-coordinates of the points of intersection of C_1 and C_2 are b and $a+b$. Hence sketch C_1 and C_2 on a single diagram, labelling any points of intersection with the axes and the equations of any asymptotes. [4]

(ii) Using the diagram, solve $\dfrac{x-b}{x-a} \geq \dfrac{x-b}{b}$. [2]

(iii) Let $a = 2$ and $b = 3$. The region bounded by C_1 and C_2 is rotated through 4 right angles about the y-axis to form a solid of revolution of volume V. Find the numerical value of V, giving your answer correct to 3 decimal places. [3]

Commented [jo6]: new qn to add in volume

Section B: Probability and Statistics [60 marks]

6 Nine gifts, three of which are identical and the rest are distinct, are distributed among five people without restrictions on the number of gifts a person can have. By first considering the number of ways to distribute the distinct gifts or otherwise, find the number of way that the nine gifts can be distributed. [4]

Commented [jo7]: check answer

7 In a school survey, a group of 80 students are asked about how much time per week (to nearest hour) they spend on their co-curricular activities (CCA). The readings are shown below:

Commented [jo8]: EC - ok

	CCA (hours)		
	3 or less	**4 to 6**	**7 or more**
Boy	17	20	10
Girl	18	$15-k$	k

A student is selected random from the group. Defining the events as follows:

G : The student is a girl.
L : The student spends 6 hours or less weekly.
M : The student spends 4 hours or more weekly.

Find the following probabilities in terms of k.

(i) $\mathrm{P}(L'\cup M')$ [2]

(ii) $\mathrm{P}(G \mid L')$ [1]

(iii) Given that $\mathrm{P}(L\cap M)=\dfrac{2}{5}$, find the value of k. Hence determine if L and M are independent, justifying your answer. [3]

(iv) If the events G and $(L\cap M)$ are mutually exclusive, find the value of k. [1]

8 Sharron who is an amateur swimmer has been attending swimming lessons. She records her time taken to swim 50 metres each month. Her best timing, t seconds, recorded each month x, for the first 7 months is as follows.

Month x	1	2	3	4	5	6	7
Time taken, t	115	87	75	67	62	61	55

(i) Draw a scatter diagram showing these timings. [1]

(ii) It is desired to predict Sharron's timings on future swims. Explain why, in this context, neither a linear nor a quadratic model is likely to be appropriate. [2]

Commented [jo9]: Nat – ok

It is decided to fit a model of the form $t = a + \frac{b}{x}$ where a and b are constants.

(iii) State with a reason whether each of a and b is positive or negative. [2]

(iv) Find the product moment correlation coefficient and the constants a and b. [2]

At the 8th month, Sharron recorded her best timing and calculated the regression line using all the data from the first 8 months to be $t = 48.28 + \frac{69.45}{x}$.

(v) Find her best timing, to the nearest second, at the 8th month. [2]

9 The time taken, T (in minutes), for a 17-year-old student to complete a 5-km run is a random variable with mean 30. After a new training programme is introduced for these students, a random sample of n students is taken. The mean time and standard deviation for the sample are found to be 28.9 minutes and 4.0 minutes respectively.

Commented [OMFJ10]: YCH – shared H1 (use as contextual)

(a) Find the unbiased estimate of the population variance in terms of n. [1]

(b) Using $n = 40$,

(i) carry out a test at the 10% significance level to determine if the mean time taken has changed. State appropriate hypotheses for the test and define any symbols you use. [4]

(ii) State what it means by the p-value in this context. [1]

(iii) Give a reason why no assumptions about the population are needed in order for the test to be valid. [1]

(c) The trainer claims instead that the new training programme is able to improve the mean of T, 30 minutes, by at least 5%. The school wants to test his claim.

Commented [jo11]: FM marker in case of t-test

(i) Write down the null and alternative hypothesis. [1]

(ii) Using the existing sample, the school carried out a test at 1% significance level and found that there was sufficient evidence to reject the trainer's claim. Find the set of values that n can take, stating any necessary assumption(s) needed to carry out the test. [4]

Commented [jo12]: check if want to mark down

10 The speeds of an e-scooter (X km/h) and a pedestrian (Y km/h) measured on a particular stretch of footpath are normally distributed with mean and variance as follows:

	mean	variance
X	12.3	9.9
Y	μ	σ^2

It is known that $\mathrm{P}(Y < 5.2) = \mathrm{P}(Y \geq 7.0) = 0.379$.

(i) State the value of μ and find the value of σ. [2]

(ii) Given that the speeds of half of the e-scooters measured are found to be within a km/h of the mean, find a. [2]

Commented [jo13]: EC – OK (use as contextual)

(iii) A LTA officer stationed himself at the footpath and measured the speeds of 50 e-scooters at random. Find the probability that the 50th e-scooter is the 35th to exceed LTA's legal speed limit of 10 km/h. [3]

(iv) On another day, the LTA officer randomly measured the speeds of 6 e-scooters and 15 pedestrians. Find the probability that the mean speed of the e-scooters is more than twice the mean speed of the pedestrians captured. [3]

(v) Find the probability that the mean speed of n randomly chosen e-scooters is more than 10 km/h, if n is large. [2]

11 **(a)** At a funfair, Alice pays \$3 to play a game by tossing a fair dice until she gets a '6'. Let X be the number of times that the player tosses a fair dice until he gets a '6'. The prize, S (in dollars), that the player may win is given by the following function:

$$S = \begin{cases} 8, & \text{if } X = 1, \\ 4, & \text{if } 2 \le X \le k, \\ 0, & \text{otherwise.} \end{cases}$$

where k is a positive integer.

(i) Show that $\mathrm{P}(2 \le X \le k) = (\frac{5}{6}) - (\frac{5}{6})^k$. Hence draw up a table showing the probability distributions of S. [4]

Commented [jo14]: CHC – ok (to update alternative solution)

(ii) Find the least value of k such that Alice is expected to earn a profit. [3]

(b) Alice uses a computer program to simulate 80 tosses of a biased coin. Let Y be the random variable denoting the number of heads obtained and p be the probability of obtaining a head. It is given that $80 + \mathrm{E}(Y) = 6\mathrm{Var}(Y)$.

(i) Find the exact value of p. [3]

(ii) Find the probability of obtaining at least 30 heads, given that the first 5 tosses are heads. [3]

(iii) Alice executes the program 50 times. Find the probability that the mean number of heads, $\overline{Y}$, is less than 25. [3]

2019 Year 6 H2 Math Prelim P1 Mark Scheme

Qn	Suggested Solution	

1

	Number sold before 7 pm	Number left after 7 pm
Banana	10	10
Chocolate	45	5
Durian	20	10

Let the selling price of banana cake, chocolate cake, durian cake before discount be \$$b$, \$$c$, \$$d$ respectively.

$$a+b+c=29.50 \quad \cdots(1)$$

$$10b+45c+20d=730$$
$$2b+9c+4d=146 \quad \cdots(2)$$

$$0.6(10b+5c+10d)=880-730$$
$$10b+5c+10d=250$$
$$2b+c+2d=50 \quad \cdots(3)$$

Solving (1), (2), (3) using GC, $a=8.50$, $b=9$, $c=12$
The selling price of banana cake, chocolate cake and durian cake is \$8.50, \$9 and \$12 respectively.

Qn	Suggested Solution	

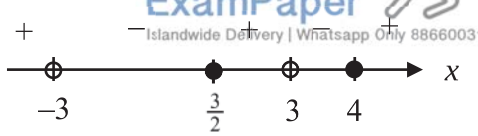
2(a)

$$\frac{30-11x}{x^2-9}\le -2$$

$$\frac{30-11x+2(x^2-9)}{x^2-9}\le 0$$

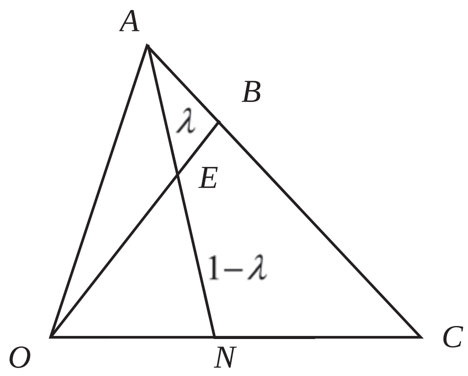
$$\frac{2x^2-11x+12}{x^2-9}\le 0$$

$$\frac{(2x-3)(x-4)}{(x-3)(x+3)}\le 0$$

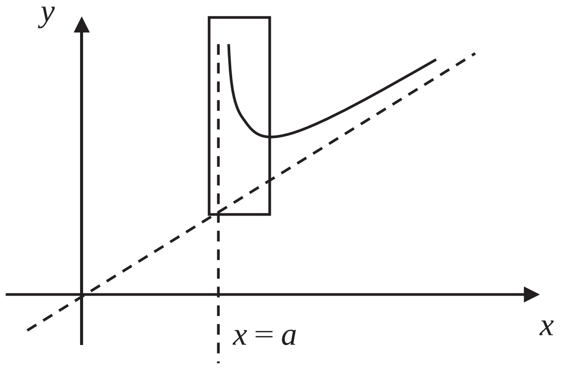


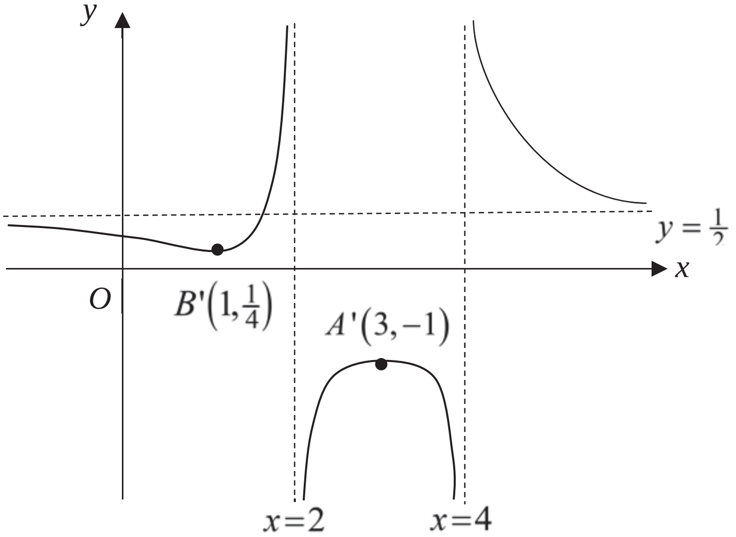
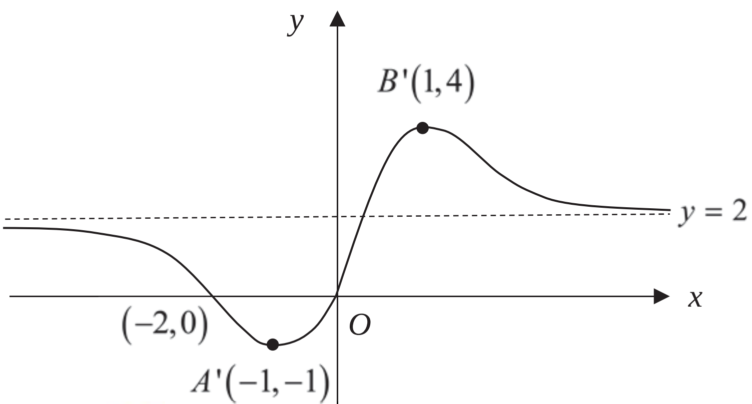
$$\therefore -3<x\le \tfrac{3}{2} \quad \text{or} \quad 3<x\le 4$$

(b)	$(a-3bx^2)e^{ax-bx^3}<0$ $a-3bx^2<0$ since $e^{ax-bx^3}>0$ for all x $x^2>\dfrac{a}{3b}$ $x>\sqrt{\dfrac{a}{3b}}$ or $x<-\sqrt{\dfrac{a}{3b}}$	

Qn	**Suggested Solution**	
3(i)	Since A, B and C are collinear and $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$ $\therefore \mu=5$ $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}$ $=\mathbf{b}+(5\mathbf{b}-5\mathbf{a})$ $=6\mathbf{b}-5\mathbf{a}$	
(ii)	 $\overrightarrow{OE}=k\mathbf{b}$ $\overrightarrow{OE}=\lambda\overrightarrow{ON}+(1-\lambda)\overrightarrow{OA}$ $=\dfrac{\lambda}{2}(6\mathbf{b}-5\mathbf{a})+(1-\lambda)\mathbf{a}$ $=3\lambda\mathbf{b}+\left(1-\dfrac{7}{2}\lambda\right)\mathbf{a}$ $1-\dfrac{7}{2}\lambda=0\Rightarrow\lambda=\dfrac{2}{7}\Rightarrow k=\dfrac{6}{7}$ $\therefore\overrightarrow{OE}=\mu\mathbf{b}=\dfrac{6}{7}\mathbf{b}$	

Qn	**Suggested Solution**	

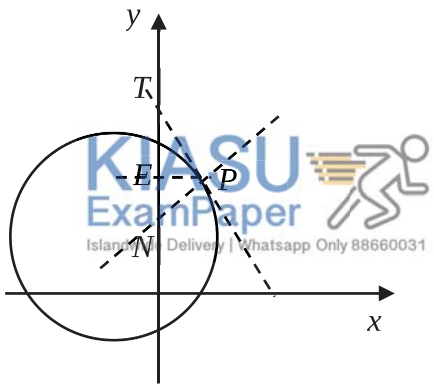
4 (i)	Since the shape of the curve is  For f to be 1-1, the largest b can take is the x-coordinate of the turning point. $\text{f}'(x)=1-\dfrac{1}{(x-a)^2}$ $1-\dfrac{1}{(x-a)^2}=0$ $x=a\pm 1$ x-coordinate of turning point is $a+1$, since $b>a$ For graph to be 1-1, $b\leq a+1$,	
(ii)	Let $y=\text{f}(x)$ $y=x+\dfrac{1}{x-1}$ $(x-1)y=x(x-1)+1$ $xy-y=x^2-x+1$ $x^2-(1+y)x+1+y=0$ $\left(x-\dfrac{(1+y)}{2}\right)^2-\dfrac{(1+y)^2}{4}+1+y=0$ $\left(x-\dfrac{(1+y)}{2}\right)^2=\dfrac{(y-1)^2}{4}-1$ $x=\dfrac{(1+y)}{2}\pm\sqrt{\dfrac{(y-1)^2}{4}-1}$ Since $\left(\dfrac{3}{2},\dfrac{7}{2}\right)$ is a point on the curve of $y=\text{f}(x)$, $x=\dfrac{(1+y)}{2}-\sqrt{\dfrac{(y-1)^2}{4}-1}$ $\text{f}^{-1}(x)=\dfrac{(1+x)}{2}-\sqrt{\dfrac{(x-1)^2}{4}-1}$ The domain of f^{-1} is the range of $\text{f}=[3,\infty)$.	

Qn	Suggested Solution	
5(a)	Series of transformations: $y = \ln\dfrac{x^2}{x+1}$ $\downarrow$ **1. Reflect in the x-axis** (replace y with $-y$) $y = -\ln\dfrac{x^2}{x+1} = \ln\dfrac{x+1}{x^2}$ $\downarrow$ **2. Scale by factor $\frac{1}{2}$ parallel to the x-axis** (replace x with $2x$) $y = \ln\dfrac{2x+1}{(2x)^2} = \ln\dfrac{2x+1}{4x^2}$	
(b)(i)		
(ii)		

Qn	Suggested Solution	
6	$\arg(w_n) = \arg[1+(n-1)\mathrm{i}] - 2\arg(1+n\mathrm{i}) + \arg[1+(n+1)\mathrm{i}]$	

(i)		
(ii)	$\arg z_n$ $= \arg(w_1 w_2 \ldots w_n)$ $= \arg(w_1) + \arg(w_2) + \ldots \arg(w_n)$ $= \displaystyle\sum_{k=1}^{n} \arg w_k$ $= \displaystyle\sum_{k=1}^{n} \arg\left(\frac{[1+(k-1)\mathrm{i}][1+(k+1)\mathrm{i}]}{(1+k\mathrm{i})^2}\right)$ $= \displaystyle\sum_{k=1}^{n}\left[\arg[1+(k-1)\mathrm{i}] - 2\arg(1+k\mathrm{i}) + \arg[1+(k+1)\mathrm{i}]\right]$ $= \left\{\begin{array}{lllll} & [\arg(1) & -\ 2\arg(1+\mathrm{i}) & + & \arg(1+2\mathrm{i})] \\ + & [\arg(1+\mathrm{i}) & -\ 2\arg(1+2\mathrm{i}) & + & \arg(1+3\mathrm{i})] \\ + & [\arg(1+2\mathrm{i}) & -\ 2\arg(1+3\mathrm{i}) & + & \arg(1+4\mathrm{i}) \\ & & \vdots & & \\ + & [\arg[1+(n-2)\mathrm{i}] & -\ 2\arg[1+(n-1)\mathrm{i}] & + & \arg(1+n\mathrm{i})] \\ + & [\arg[1+(n-1)\mathrm{i}] & -\ 2\arg(1+n\mathrm{i}) & + & \arg[1+(n+1)\mathrm{i}]] \end{array}\right.$ $= \arg(1) - \arg(1+\mathrm{i}) - \arg(1+n\mathrm{i}) + \arg[1+(n+1)\mathrm{i}]$ $= -\frac{1}{4}\pi - \arg(1+n\mathrm{i}) + \arg[1+(n+1)\mathrm{i}]$	
(iii)	As $n \to \infty$, $\arg(1+n\mathrm{i}) \to \frac{1}{2}\pi$ and $\arg[1+(n+1)\mathrm{i}] \to \frac{1}{2}\pi$ Hence $\arg z_n \to -\frac{1}{4}\pi$. (argand diagram with y = -x line to show argument) Thus $\mathrm{Re}(z_n) = -\mathrm{Im}(z_n)$	

Qn	Suggested Solution	
7 (i)	$4\sin 2\theta = x+2$ ---- (1) $16\sin^2 2\theta = (x+2)^2$ $4\cos 2\theta = 3-y$ ---- (2) $16\cos^2 2\theta = (3-y)^2$ (1) + (2) gives $(x+2)^2 + (y-3)^2 = 16$	

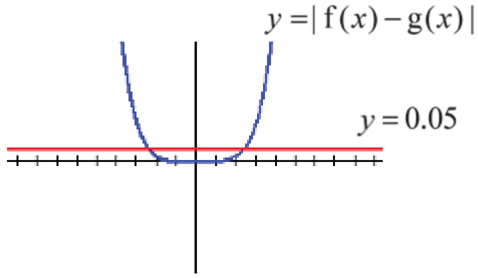
	Hence C is a circle with centre $(-2,3)$ and radius 4 units.	
(ii)	$\frac{dx}{d\theta}=8\cos 2\theta$ and $\frac{dy}{d\theta}=8\sin 2\theta$ gives $\frac{dy}{dx}=\tan 2\theta$ For $\theta=\frac{3}{8}\pi$, $x=2\sqrt{2}-2$ $y=3+2\sqrt{2}$ $\frac{dy}{dx}=-1$ Equation of tangent: $y-3-2\sqrt{2}=-1(x-2\sqrt{2}+2)$ Equation of normal: $y-3-2\sqrt{2}=x-2\sqrt{2}+2$ So $\mathrm{T}(0,1+4\sqrt{2})$ and $\mathrm{N}(0,5)$ Hence the area of triangle NPT $=\frac{1}{2}(4\sqrt{2}-4)(2\sqrt{2}-2)$ $=(2\sqrt{2}-2)(2\sqrt{2}-2)$ $=12-8\sqrt{2}$ units^2 Alternatively, Let E be the point closest to P along the y-axis. Since $\frac{dy}{dx}=-1$ at P, the triangle TPE is such that $ET=EP$ and $\measuredangle TEP=90^o$. 	

	The normal at P i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}=1$. the triangle NPE is such that $EN = EP$ and $\measuredangle NEP = 90^o$. Therefore the two triangles are congruent, and the area of triangle NPT $=2\left[\frac{1}{2}\left(2\sqrt{2}-2\right)\left(2\sqrt{2}-2\right)\right]$ $=\left(2\sqrt{2}-2\right)^2$ $=12-8\sqrt{2}$	

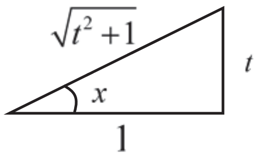
Qn	Suggested Solution	
8 (i)	$x-2y+3z=4$ ---- (1) $3x+2y-z=4$ ---- (2) Solving (1) and (2) using GC gives $x=2-0.5z$ $y=-1+1.25z$ $z=z$ Hence L : $\mathbf{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\5\\4\end{pmatrix}$, $\lambda\in\mathbb{R}$	
(ii)	P_3 : $\mathbf{r}\bullet\begin{pmatrix}5\\-k\\6\end{pmatrix}=1$ If the three planes have no point in common, $\begin{pmatrix}-2\\5\\4\end{pmatrix}\bullet\begin{pmatrix}5\\-k\\6\end{pmatrix}=0$ $\Rightarrow -10-5k+24=0$ $\therefore\ k=2.8$	

(iii)	$\overrightarrow{OQ}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$ Distance required $=\dfrac{\left\|1-\begin{pmatrix}2\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\-2.8\\6\end{pmatrix}\right\|}{\left\|\begin{pmatrix}5\\-2.8\\6\end{pmatrix}\right\|}$ $=\dfrac{\|1-12.8\|}{\sqrt{68.84}}=1.42$ units (3 s.f.) **Alternative** $\overrightarrow{OQ}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$ and let $\overrightarrow{OY}=\begin{pmatrix}0\\0\\1/6\end{pmatrix}$ where Y is a point on P_3 Shortest distance from Q to P_3 $=\dfrac{\left\|\overrightarrow{YZ}\bullet\begin{pmatrix}5\\-2.8\\6\end{pmatrix}\right\|}{\sqrt{5^2+(-2.8)^2+6^2}}=\dfrac{\left\|\begin{pmatrix}2\\-1\\-1/6\end{pmatrix}\bullet\begin{pmatrix}5\\-2.8\\6\end{pmatrix}\right\|}{\sqrt{68.84}}=1.42$ units	
(iv)	Plane containing Q and parallel to P_3 : $5x-2.8y+6z=d$ where $d=\begin{pmatrix}2\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\-2.8\\6\end{pmatrix}=5(2)-2.8(-1)+6(0)=12.8$ $\therefore 5x-2.8y+6z=12.8$ Since $12.8>1>0$, P_3 is in between the above plane and the origin. Thus O and Q are on the opposite sides of P_3.	

Qn	Suggested Solution	
9(i) (a)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\mathrm{e}^{-y}-1$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{2}\left(-\mathrm{e}^{-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ $=-\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^2}=-\left[\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]=-\left(1+2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	
(b)	$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=-\left[\left(1+2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]$ When $x=0,\ y=0$ (given) $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2},\quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{4},\quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0,\quad \frac{\mathrm{d}^4y}{\mathrm{d}x^4}=-\frac{1}{8}$ $y=-\frac{1}{2}x+\frac{\frac{1}{4}}{2!}x^2+0-\frac{\frac{1}{8}}{4!}x^3+\ldots$ $=-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4+\ldots$	
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\mathrm{e}^{-y}-1$ $\frac{1}{\frac{1}{2}\mathrm{e}^{-y}-1}\frac{\mathrm{d}y}{\mathrm{d}x}=1$ $\int\frac{1}{\frac{1}{2}\mathrm{e}^{-y}-1}\,\mathrm{d}y=\int 1\,\mathrm{d}x$ $\int\frac{\mathrm{e}^{y}}{\frac{1}{2}-\mathrm{e}^{y}}\,\mathrm{d}y=x+C$ $-\ln\lvert\frac{1}{2}-\mathrm{e}^{y}\rvert=x+C$ $\frac{1}{2}-\mathrm{e}^{y}=\pm\mathrm{e}^{-x+C}=A\mathrm{e}^{-x}$ $y=\ln(\frac{1}{2}-A\mathrm{e}^{-x})$ When $x=0,\ y=0$ $0=\ln(\frac{1}{2}-A\mathrm{e}^{0})$ $A=-\frac{1}{2}$ $\therefore y=\ln(\frac{1}{2}+\frac{1}{2}\mathrm{e}^{-x})$	

	Alternative (for integration) $\int \frac{1}{\frac{1}{2}e^{-y}-1}\,dy = x+C$ $\int \frac{1-\frac{1}{2}e^{-y}+\frac{1}{2}e^{-y}}{\frac{1}{2}e^{-y}-1}\,dy = x+C$ $\int -1-\frac{(-\frac{1}{2}e^{-y})}{\frac{1}{2}e^{-y}-1}\,dy = x+C$ $-y-\ln\lvert\frac{1}{2}e^{-y}-1\rvert = x+C$ $\ln e^{-y}-\ln\lvert\frac{1}{2}e^{-y}-1\rvert = x+C$ $\ln\left\lvert\frac{e^{-y}}{\frac{1}{2}e^{-y}-1}\right\rvert = x+C$ $\ln\left\lvert\frac{1}{\frac{1}{2}-e^{y}}\right\rvert = x+C$ $-\ln\lvert\frac{1}{2}-e^{y}\rvert = x+C$ $\vdots$	
(iii)	$\lvert f(x)-g(x)\rvert < 0.05$  From GC, $\{x\in\mathbb{R}: 0\le x<2.43\}$	

Qn	Suggested Solution	
10(i)	$\int \frac{x}{\sqrt{2x-1}}\,dx = \left[x\sqrt{2x-1}\right]-\int\sqrt{2x-1}\,dx$ $= x\sqrt{2x-1}-\frac{1}{3}\left((2x-1)^{\frac{3}{2}}\right)+C$ $=\sqrt{2x-1}\left(x-\frac{1}{3}(2x-1)\right)+C$ $=\frac{1}{3}\sqrt{2x-1}\left(x+1\right)+C$	

	$\int \frac{x}{\sqrt{2x-1}}\,dx = \frac{1}{2}\int \frac{2x-1+1}{\sqrt{2x-1}}\,dx$ $= \frac{1}{2}\int \sqrt{2x-1}\,dx + \frac{1}{2}\int \frac{1}{\sqrt{2x-1}}\,dx$ $= \frac{1}{2}\frac{(2x-1)^{\frac{3}{2}}}{\frac{3}{2}(2)} + \frac{1}{2}\frac{(2x-1)^{\frac{1}{2}}}{\frac{1}{2}(2)} + C$ $= \frac{1}{6}(2x-1)^{\frac{3}{2}} + \frac{1}{2}(2x-1)^{\frac{1}{2}} + C$	
(ii)	$x = \tan^{-1} t, \quad \frac{dx}{dt} = \frac{1}{1+t^2}, \quad \sin x = \frac{t}{\sqrt{t^2+1}}$  $\int \frac{1}{4\cos^2 x + 9\sin^2 x}\,dx$ $= \int \frac{1}{4+5\sin^2 x}\,dx$ $= \int \frac{1}{4+5\frac{t^2}{t^2+1}} \cdot \frac{1}{1+t^2}\,dt$ $= \int \frac{1}{4+9t^2}\,dt$ $= \frac{1}{9}\int \frac{1}{(\frac{2}{3})^2 + t^2}\,dt$ $= \frac{1}{6}\tan^{-1}\frac{3t}{2} + C = \frac{1}{6}\tan^{-1}\frac{3\tan x}{2} + C$	
(iii)	$A = \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n-1}{n}\right)\right)$ $= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n-1}{n}\right)^2 \right.$ $\left. + 3\left(\frac{1}{n}\right) + 3\left(\frac{2}{n}\right) + \ldots + 3\left(\frac{n-1}{n}\right)\right]$ $= \frac{1}{n}\left[\frac{1}{n^2}(1^2 + 2^2 + \ldots + (n-1)^2) + \frac{3}{n}(1 + 2 + \ldots + (n-1))\right]$ $= \frac{1}{n^3}\left(\frac{1}{6}(n-1)(n)(2n-1)\right) + \frac{3}{n^2}\left(\frac{n-1}{2}(n)\right)$ $= \frac{(n-1)(2n-1+9n)}{6n^2} = \frac{(n-1)(11n-1)}{6n^2}$ $A \to \int_0^1 x^2 + 3x\,dx \text{ as } n \to \infty$ in particular,	

	$\dfrac{(n-1)(11n-1)}{6n^2}=\dfrac{11n^2-12n+1}{6n^2}=\dfrac{11-\frac{12}{n}+\frac{1}{n^2}}{6}\to\dfrac{11}{6}$	

Qn	Suggested Solution	
11(i)	$R_f=[75,1200]$, $D_g=[0,1000(e-1)]$ Since $R_f \subset D_g$, the composite function gf exist.	
(ii)	$[7, 11] \xrightarrow{f} [208, 416] \xrightarrow{g} [0.834, 0.920]$ The range of values for the happiness index is [0.834, 0.920]	
(iii)	Since f is an increasing function and g is a decreasing function, the composite function gf will be a decreasing function. e.g. for $b>a$ f is an increasing function $\Rightarrow$ $f(b)>f(a)$ g is a decreasing function $\Rightarrow$ $gf(b)<gf(a)$ **Alternative** Differentiate and deduce negative gradient	
(iv)	$[8.8859, 12.961] \xleftarrow{f^{-1}} [258.68, 704.53] \xleftarrow{g^{-1}} [0.7, 0.9]$ The number of foreign workers allowed in the country can be from 88859 to 129610. $[8.8859, 12.961] \xrightarrow{h} [391, 400]$ Take note that h(x) is a quadratic expression, thus the range of GDP will be 391 billion to 400 billion dollars.	

Qn	Suggested Solution	
12(i)	Amount of U in time t $= 40 - \frac{2}{2+1}w = 40 - \frac{2}{3}w$ Amount of V in time t $= 50 - \frac{1}{3}w$ $\frac{dw}{dt} = k_1\left(40 - \frac{2}{3}w\right)\left(50 - \frac{1}{3}w\right),\ k_1 \in \mathbb{R}^+$ as amt. of $w \uparrow$ $= k_1\left(-\frac{2}{3}\right)(w-60)\left(-\frac{1}{3}\right)(w-150)$ $= k(w-60)(w-150), \quad k = \frac{2}{9}k_1$	
(ii)	$\frac{dw}{dt} = k(w-60)(w-150)$ $\frac{1}{(w-60)(w-150)}\frac{dw}{dt} = k$ $\frac{1}{w^2 - 210w + 9000}\frac{dw}{dt} = k$ $\frac{1}{(w-105)^2 - 45^2}\frac{dw}{dt} = k$ Integrating w.r.t. t: $\frac{1}{2(45)}\ln\left\|\frac{(w-105)-45}{(w-105)+45}\right\| = kt + C,\ k$ an arbitrary constant $\left\|\frac{w-150}{w-60}\right\| = e^{90C}e^{90kt}$ $\frac{w-150}{w-60} = Ae^{90kt}$, where $A = \pm e^{90C}$ When $t=0,\ w=0$: $\frac{-150}{-60} = A$ $\therefore A = \frac{5}{2}$ When $t=5,\ w=10$:	

	$\dfrac{10-150}{10-60}=\dfrac{5}{2}e^{90k(5)}$ $k=\dfrac{1}{450}\ln\dfrac{28}{25}$ $\therefore \dfrac{w-150}{w-60}=\dfrac{5}{2}e^{\left(\frac{1}{5}\ln\frac{28}{25}\right)t}=\dfrac{5}{2}\left(\dfrac{28}{25}\right)^{\frac{t}{5}}$ When $t=20$, $\dfrac{w-150}{w-60}=\dfrac{5}{2}\left(\dfrac{28}{25}\right)^{\frac{20}{5}}=3.93379$ $w(3.93379-1)=60(3.93379)-150$ $w=29.3229=29.32$ (2 d.p.)	
(iii)	$\dfrac{w-150}{w-60}=\dfrac{5}{2}\left(\dfrac{28}{25}\right)^{\frac{t}{5}}$ As $t\to\infty$, RHS $\to\infty$ i.e. $w-60\to 0$ $\therefore w\to 60$ Method 2: (remove from solution) Use graph of dw/dt vs w and deduce equilibrium (or equivalent deductions)	

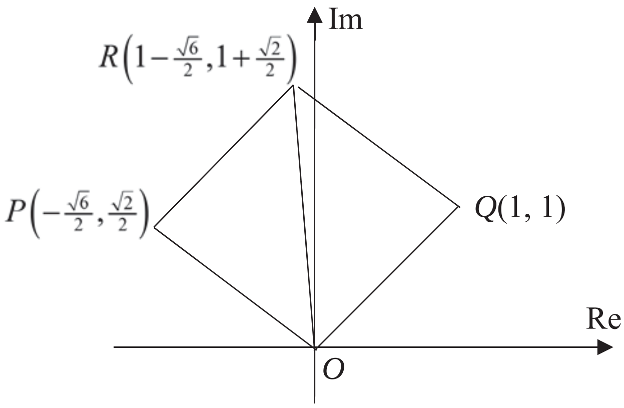
2019 Year 6 H2 Math Prelim P2 Mark Scheme

Qn	Suggested Solution	
1(i)	$S_n - S_{n-1}$ $= an^2 + bn + c - (a(n-1)^2 + b(n-1) + c)$ $= 2an - a + b$ Total number of additional cards need is $2an - a + b$	
(ii)	Additional cards to form 2nd level from 1st level $= 5$ $4a - a + b = 5 \Rightarrow 3a + b = 5$ --- (1) Additional cards to form 3rd level from 2nd level $= 8$ $6a - a + b = 8 \Rightarrow 5a + b = 8$ ---(2) Solving both (1) and (2), $a = \frac{3}{2}, b = \frac{1}{2}$. Using $S_1 = 2 \Rightarrow \frac{3}{2}(1)^2 + \frac{1}{2}(1) + c = 2 \Rightarrow c = 0.$ **Alternative** Substituting different values of n, $n = 1$: $a + b + c = 2$ $n = 2$: $4a + 2b + c = 7$ $n = 3$: $9a + 3b + c = 15$ From GC, $a = 1.5$, $b = 0.5$ and $c = 0$ **Alternative** n = 1, number of cards = 2 n = 2, number of cards = 2 + 5 n = 3, number of cards = 2 + 5 + 8 $S_n = \frac{n}{2}[2(2) + (n-1)(3)] = \frac{n}{2}(3n+1) = 1.5n^2 + 0.5n$ $\therefore$ $a = 1.5$, $b = 0.5$ and $c = 0$	
(ii)	$u_n = 3n - 1$ $u_n - u_{n-1} = (3n-1) - (3(n-1)-1) = 3$ (constant) Thus Sn is a sum of AP with common difference 3.	
(iii)	$\sum_{n=1}^{23} S_n = \sum_{n=1}^{23} (1.5n^2 + 0.5n) = 6624$	

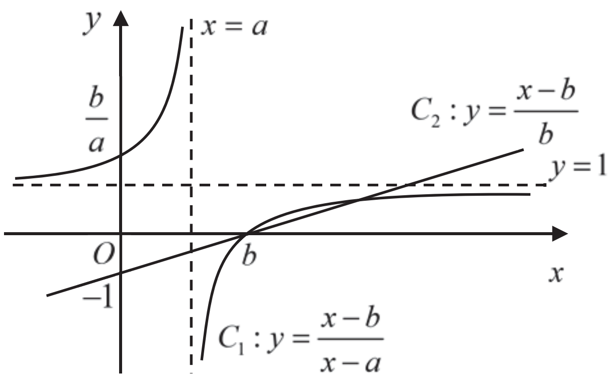
Qn	Suggested Solution	
2 (i)	$3x^2-2xy+5y^2=14$ ---- (1) Differentiate (1) implicitly wrt x: $6x-2x\frac{dy}{dx}-2y+10y\frac{dy}{dx}=0$ $(2x-10y)\frac{dy}{dx}=6x-2y$ $\frac{dy}{dx}=\frac{3x-y}{x-5y}$ (shown)	
(ii)	$x-5y=0 \Rightarrow y=0.2x$ Sub $y=0.2x$ into (1): $3x^2-2x(0.2x)+5(0.2x)^2=14$ $2.8x^2=14$ $x=\pm\sqrt{5}$	
(iii)	When $y=1$, $3x^2-2x-9=0$ Therefore, $x=-1.4305$ or $x=2.0972$ $\frac{dy}{dt}=(\frac{dy}{dx})(\frac{dx}{dt})$ $-7=(\frac{3x-1}{x-5})(\frac{dx}{dt})$ $\frac{dx}{dt}=\frac{7(5-x)}{3x-1}$ When $x=2.0972$, $\frac{dx}{dt}=3.84$ units per second (3 s.f.)	

Qn	Suggested Solution	
3(i)	LHS $=a\left(\frac{1}{z_0}\right)^2+b\left(\frac{1}{z_0}\right)+a$ $=\left(\frac{1}{z_0}\right)^2\left(a+bz_0+az_0^{\ 2}\right)$ $=0$ $\because a+bz_0+az_0^{\ 2}=0$ Thus $z=\frac{1}{z_0}$ is a solution. Since a and b are real constants,	

	$\frac{1}{z_0} = z_0^*$ $z_0 z_0^* = 1$ $\|z_0\|^2 = 1$ Since $\|z_0\| > 0,\ \|z_0\| = 1$ **Alternative for first part:** Let second root be z_1 product of roots $z_0 z_1 = \frac{a}{a} = 1$ $\therefore z_1 = \frac{1}{z_0}$	
(ii)	Let $z_0 = x_0 + \mathrm{i}y_0$ Since $\mathrm{Im}(z_0) = \frac{1}{2},\ y_0 = \frac{1}{2}$. From part (i), $\|z_0\| = 1$ $\sqrt{x_0^2 + y_0^2} = 1$ $\sqrt{x_0^2 + \left(\frac{1}{2}\right)^2} = 1$ $x_0 = \pm\frac{\sqrt{3}}{2}$ $z_0 = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2} \quad \text{or} \quad -\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$	
(iii)	Since $\mathrm{Re}(z_0) > 0,\ z_0 = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$. Subst into $az_0^2 + bz_0 + a = 0$, $a\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right)^2 + b\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right) + a = 0$ $a\left(\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) + b\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right) + a = 0$ $\left(\frac{3}{2}a + \frac{\sqrt{3}}{2}b\right) + \mathrm{i}\left(\frac{1}{2}b + \frac{\sqrt{3}}{2}a\right) = 0$ $\therefore b = -\sqrt{3}a$	

Qn	**Suggested Solution**	
4(i)	$w=\sqrt{2}\left(\cos\frac{1}{4}\pi+\mathrm{i}\sin\frac{1}{4}\pi\right)$ $=1+\mathrm{i}$ $z=\sqrt{2}\left(\cos\frac{5}{6}\pi+\mathrm{i}\sin\frac{5}{6}\pi\right)$ $=-\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}$ $w+z=\left(1-\frac{\sqrt{6}}{2}\right)+\left(1+\frac{\sqrt{2}}{2}\right)\mathrm{i}$	
(ii)	 *OPRQ* is a rhombus	
(iii)	Note that *OR* bisects the angle *POQ* since *OPRQ* is a rhombus. Thus $\arg(w+z)=\frac{1}{2}\left(\frac{1}{4}\pi+\frac{5}{6}\pi\right)=\frac{13}{24}\pi$. $\tan\left(\frac{11}{24}\pi\right)=\dfrac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{6}}{2}-1}$ $=\dfrac{2+\sqrt{2}}{\sqrt{6}-2}$ $\therefore a=2, b=-2$	

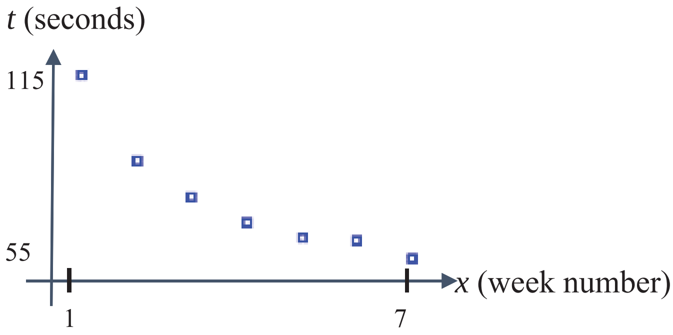
Qn	**Suggested Solution**	
5(i)	Graphs intersect at:	

	$\frac{x-b}{x-a}=\frac{x-b}{b}$ $b(x-b)=(x-b)(x-a)$ $(x-b)(x-a-b)=0$ $x=b \quad \text{or} \quad x=a+b$ 	
(ii)	$\therefore x<a \quad \text{or} \quad b\le x\le a+b$	
(iii)	From GC, point of intersection at $(5,\frac{2}{3})$ $V=\pi\int_0^{\frac{2}{3}} \underbrace{x_2^{\,2}}_{C_2}-\underbrace{x_1^{\,2}}_{C_1}\,\mathrm{d}y$ $=\pi\int_0^{\frac{2}{3}} (3y+3)^2-\left(\frac{2y-3}{y-1}\right)^2\,\mathrm{d}y$ $=5.742$ (3 d.p.)	

Qn	Suggested Solution	
6	For distinct gifts, 5^6 ways Now considering the distinct gifts, Case 1: 3 person get 1 gift No of ways = ${}^5C_3\times 5^6=156250$ Case 2: 1 person get 1 gift, another person gets 2 gifts No of ways = ${}^5C_2\,(2)\times 5^6=312500$ Case 3: 1 person get 3 gifts No of ways = ${}^5C_1\times 5^6=78125$ Total number of ways $=156250+312500+78125=546875$ **Alternative**	

	<u>Stage 1: Distribute 6 distinct gifts among 5 people</u> No of ways = 5^6 <u>Stage 2: Distribute 3 identical gifts among 5 people</u> Case 1: 3 person get 1 gift No of ways = ${}^5C_3 = 10$ Case 2: 1 person get 1 gift, another person gets 2 gifts No of ways = ${}^5C_2(2) = 20$ Case 3: 1 person get 3 gifts No of ways = ${}^5C_1 = 5$ Total number of ways = $(10+20+5)5^6 = 546875$	

Qn	Suggested Solution (updated 26 Sep)	
7(i)	$\mathrm{P}(L'\cup M') = \dfrac{80-n(L\cap M)}{80}$ 4 to 6 hours $= \dfrac{80-(35-k)}{80} = \dfrac{45+k}{80}$ **ALT** $\mathrm{P}(L'\cup M') = \mathrm{P}(L) + P(M') - \mathrm{P}(L'\cap M')$ $= \dfrac{10+k}{80} + \dfrac{35}{80} - 0$ $= \dfrac{45+k}{80}$	
(ii)	$\mathrm{P}(G \mid L') = \dfrac{P(G\cap L')}{P(L')} = \dfrac{k}{k+10}$	
(iii)	Given $\mathrm{P}(L\cap M) = \dfrac{2}{5}$ From table: $\mathrm{P}(L\cap M) = \dfrac{20+(15-k)}{80} = \dfrac{35-k}{80}$ Solving: $k=3$ $P(L)P(M) = \dfrac{67}{80}\times\dfrac{45}{80} = \dfrac{603}{1280} \neq \dfrac{2}{5}$ Since $P(L\cap M) \neq P(L)P(M)$, L and M are **<u>NOT</u>** independent **ALT** $P(L) = \dfrac{70-k}{80} = \dfrac{67}{80}$ $\mathrm{P}(L \mid M) = \dfrac{35-k}{45} = \dfrac{32}{45} \neq \dfrac{67}{80}$ Since $P(L) \neq \mathrm{P}(L \mid M)$, L and M are **<u>NOT</u>** independent	
(iv)	Since $\mathrm{P}(G\cap(L\cap M)) = 0$ $\Rightarrow 15-k=0$ $\therefore k=15$	

Qn	Suggested Solution	
8 (i)		
(ii)	A linear model would predict her timing to decrease at a constant rate and eventually negative, which is not possible as there is a limit to how fast a person can swim. A quadratic model would predict that her timings would have a minimum and then increase at an increasing rate, which is also not appropriate.	
(iii)	Based on the scatter diagram and the model, as x increases t decreases at a decreasing rate, therefore b is positive. a has to be positive as it represents the best possible timing that Sharron can swim in the long run.	
(iv)	From GC, $r = 0.991$ $b = 67.69$ $a = 49.50$	
(v)	Let m be the best timing Sharron has at the 8$^{\text{th}}$ month. $\overline{\left(\frac{1}{x}\right)} = 0.33973$ We know that $\left(\overline{\frac{1}{x}}, \bar{t}\right)$ is on the regression line $t = 48.28 + 69.45\left(\frac{1}{x}\right)$. $\bar{t} = 48.28 + 69.45(0.33973) = 71.874$ $\frac{522+m}{8} = 71.874$ $m = 52.992$ Sharron best timing is 53 seconds at the 8th month	

Qn	Suggested Solution	
9 (a)	An unbiased estimate for the population variance : $s^2 = \frac{n}{n-1}\left(4^2\right) = \frac{16n}{n-1}$ minutes^2	
(b) (i)	Let μ be the population mean time taken for a 17-year-old student to complete a 5 km run. To test at 10 % significance level, $H_0 : \mu = 30.0$ min $H_1 : \mu \neq 30.0$ min For $n = 40$, $s^2 = \frac{16(40)}{39} = \frac{640}{39}$ Test Statistic: Under H_0 , $\overline{T} \sim \mathrm{N}\left(30.0, \frac{640/39}{40}\right)$ approximately by Central Limit Theorem since n is large p-value $= 2\mathrm{P}\left(\overline{T} \leq 28.9\right) = 0.0859 \leq 0.10$, we reject H_0 and conclude that there is sufficient evidence at the 10 % significance level that the population mean time taken has changed.	
(ii)	The p-value is the probability of obtaining a sample mean at least as extreme as the given sample, assuming that the population mean time taken has not changed from 30.0 min. **OR** The p-value is the smallest significance level to conclude that the population mean time has changed from 30.0 min.	
(iii)	Since the sample size of 40 is large, by Central Limit Theorem, $\overline{T}$ follows a normal distribution approximately. Thus no assumptions are needed.	
(c) (i)	New population mean timing = $0.95 \times 30 = 28.5$ min To test at 5 % significance level, $H_0 : \mu = 28.5$ min $H_1 : \mu > 28.5$ min	
(ii)	Assumption: n is large for Central Limit Theorem to apply. Test Statistic: Under H_0, $\overline{T} \sim \mathrm{N}\left(28.5, \frac{4.0^2}{n-1}\right)$ approximately by Central Limit Theorem	

	For H_0 to be rejected, we need $\mathrm{P}\left(\overline{T} \geq 28.9\right) \leq 0.01$ $\mathrm{P}\left(Z \geq \frac{28.9-28.5}{\frac{4}{\sqrt{n-1}}}\right) \leq 0.01$ $\mathrm{P}\left(Z \geq \frac{\sqrt{n-1}}{10}\right) \leq 0.01$ $\frac{\sqrt{n-1}}{10} \geq 2.3263$ $n \geq 542.2$ Thus required set = $\left\{n \in \mathbb{Z} : n \geq 543\right\}$	

Qn	Suggested Solution	
10 (i)	By symmetry, $\mu = \dfrac{5.2+7.0}{2} = 6.1$ $\text{P}(Y<5.2) = \text{P}(Y \geq 7.0) = 0.379$ $\text{P}\left(Z < \dfrac{5.2-6.1}{\sigma}\right) = 0.379 \Rightarrow \dfrac{-0.9}{\sigma} = -0.308108$ $\sigma = 2.92105 = 2.92$ (3sf)	
(ii)	$X \sim N(12.3, 9.9)$ $\text{P}(\lvert X-12.3 \rvert < a) = 0.5$ $\text{P}(12.3-a < X < 12.3+a) = 0.5$ From GC, $12.3 - a = 10.1777$ $a = 2.1223 = 2.12$ (3sf) **Alternative** $\text{P}(\lvert X-12.3 \rvert < a) = 0.5$ $\text{P}(\lvert Z \rvert < \dfrac{a}{\sqrt{9.9}}) = 0.5$ $\text{P}(Z < -\dfrac{a}{\sqrt{9.9}}) = 0.25 \Rightarrow \quad -\dfrac{a}{\sqrt{9.9}} = -0.674489$ $a = 2.12$ (3sf)	
(iii)	$\text{P}(X > 10 = 0.76761$ Let W = number of e-scooters that exceed speed limit, out of 49 $W \sim \text{B}(49, \text{P}(X>10))$ i.e. $W \sim \text{B}(49, 0.76761)$ Probability required $= \text{P}(W=34) \times 0.76761$ $= 0.61022 \times 0.76761$ $= 0.046840 = 0.0468$ (3sf)	
(iv)	Want: $\text{P}\left(\dfrac{X_1+\ldots+X_6}{6} > 2\left(\dfrac{Y_1+\ldots+Y_{15}}{15}\right)\right)$ $= \text{P}(\overline{X} - 2\overline{Y} > 0)$ $\overline{X} - 2\overline{Y} \sim \text{N}\left(12.3-2(6.1), \dfrac{9.9}{6} + \dfrac{4}{15}(2.92105^2)\right)$ i.e. $\overline{X} - 2\overline{Y} \sim \text{N}(0.1, 3.92533)$ $\therefore \text{P}(\overline{X} - 2\overline{Y} > 0) = 0.520$ (3sf)	

(v)	Let T = Total speed of n e-scooters $\overline{T} \sim \mathrm{N}(12.3, \frac{9.9}{n})$ $\mathrm{P}(\overline{T} > 10) = \mathrm{P}(Z > \frac{10-12.3}{\sqrt{\frac{9.9}{n}}})$ $= \mathrm{P}(Z > -0.73098\sqrt{n}) = 1$ (since n is large) **<u>Alternative</u>** As n gets larger, $\overline{x} \to \mu = 12.3 > 10$ Thus mean speed of these n e-scooters > 10 with probability 1	

Qn	Suggested Solution	
11 (a)(i)	**Method 1: direct computation** $\mathrm{P}(2 \le X \le k)$ $= \mathrm{P}(X=2)+\mathrm{P}(X=3)+\mathrm{P}(X=4)+\ldots+\mathrm{P}(X=k)$ $= (\frac{5}{6})(\frac{1}{6})+(\frac{5}{6})^2(\frac{1}{6})+(\frac{5}{6})^3(\frac{1}{6})+(\frac{5}{6})^4(\frac{1}{6})+\ldots+(\frac{5}{6})^{k-1}(\frac{1}{6})$ $= (\frac{1}{6})\left[(\frac{5}{6})+(\frac{5}{6})^2+(\frac{5}{6})^3+(\frac{5}{6})^4+\ldots+(\frac{5}{6})^{k-1}\right]$ $= (\frac{1}{6})\left[\frac{(\frac{5}{6})(1-(\frac{5}{6})^{k-1})}{1-(\frac{5}{6})}\right]$ $= (\frac{5}{6})-(\frac{5}{6})^k$ **Method 2: complement method** $\mathrm{P}(2 \le X \le k)$ $= 1-\mathrm{P}(X=1)-\underbrace{\mathrm{P}(X>k)}_{\text{first } k \text{ are not } 6's}$ $= 1-\frac{1}{6}-(\frac{5}{6})^k$ $= \frac{5}{6}-(\frac{5}{6})^k$ <table><tr><td>s</td><td>8</td><td>4</td><td>0</td></tr><tr><td>$\mathrm{P}(S=s)$</td><td>$\frac{1}{6}$</td><td>$(\frac{5}{6})-(\frac{5}{6})^k$</td><td>$(\frac{5}{6})^k$</td></tr></table>	
(ii)	From GC, $\mathrm{E}(S) = \frac{8}{6}+4\left(\frac{5}{6}-\left(\frac{5}{6}\right)^k\right) = \frac{14}{3}-4\left(\frac{5}{6}\right)^k$ $\mathrm{E}(\text{Profit}) = \frac{14}{3}-4\left(\frac{5}{6}\right)^k-3>0$ $\frac{14}{3}-4\left(\frac{5}{6}\right)^k-3>0$ $\left(\frac{5}{6}\right)^k<\frac{5}{12}$ $k>4.802$ Least value of k is 5.	
(b)(i)	$Y \sim \mathrm{B}(80, p)$ $80+80p = 480p(1-p)$ $1+p = 6p-6p^2$ $6p^2-5p+1=0$ $p=\frac{1}{3}$ or $p=\frac{1}{2}$ (rejected as coin is not fair)	

(ii)	Let W be the number of heads obtained in the last 75 tosses $W \sim \mathrm{B}(75, \frac{1}{3})$ Required probability $= \mathrm{P}(W \geq 25)$ $= 1 - \mathrm{P}(W \leq 24)$ $= 0.543$ **Alternative** Use conditional probability	
(iii)	$\overline{Y} \sim \mathrm{N}(\frac{80}{3}, \frac{16}{45})$ approximately by central limit theorem since the sample size of 50 is large $\mathrm{P}(\overline{Y} < 25) = 0.00259 \quad (3 \text{ s.f.})$	

EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2019

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME

CLASS | INDEX NO.

MATHEMATICS 9758/01

Paper 1 [100 marks]

04 September 2019

3 hours

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.
Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **27** printed pages (including this cover page) and **1** blank page.

For markers' use:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total

1 Electricity cost per household is calculated by multiplying the electricity consumption (in kWh), by the tariff (in cents/kWh). The tariff is set by the government and reviewed every 4 months.

The amount of electricity used by each household for each 4-month period, together with the total electricity cost for each household in the year, are given in the following table.

	Jan – April (in kWh)	May – Aug (in kWh)	Sept – Dec (in kWh)	Total electricity cost in the year ($)
Household 1	677	586	699	529.53
Household 2	1011	871	1048	790.63
Household 3	1349	1174	1417	1063.28

Write down and solve equations to find the tariff, in cents/kWh, to 2 decimal places, for each 4-month period. [4]

2 A string of fixed length l is cut into two pieces. The first piece is used to form a square of side s and the second piece is used to form a circle of radius r. Find the ratio of the length of the first piece to the second piece that gives the smallest possible combined area of the square and circle. [6]

3 A geometric progression has first term a and common ratio r, and an arithmetic progression has first term a and common difference d, where a and d are non-zero. The sums of the first 2 and 4 terms of the arithmetic progression are equal to the respective sums of the first 2 and 4 terms of the geometric progression.

(i) By showing $r^3 + r^2 - 5r + 3 = 0$, or otherwise, find the value of the common ratio. [5]

(ii) Given that $a < 0$ and the nth term of the geometric progression is positive, find the smallest possible value of n such that the nth term of the geometric progression is more than 1000 times the nth term of the arithmetic progression. [3]

4 **(i)** Find the series expansion for $(1+ax)^n$ in ascending powers of x, up to and including the term in x^3, where a is non-zero and $|a| < 1$. [1]

(ii) It is given that the coefficients of the terms in x, x^2, and x^3 are three consecutive terms in a geometric progression. Show that $n = -1$. [2]

(iii) Show that the coefficients of the terms in the series expansion of $(1+ax)^{-1}$ form a geometric progression. [3]

(iv) Evaluate the sum to infinity of the coefficients of the terms in x of odd powers. [2]

5 **(a)** It is given that the equation $f(x) = a$ has three roots x_1, x_2, x_3 where $x_1 < 0 < x_2 < x_3$, and a is a constant.

(i) How many roots does the equation $f(|x|) = a$ have? With the aid of a diagram, or otherwise, explain your answer briefly. [2]

(ii) How many roots does the equation $f(x-a) = a$ have? With the aid of a diagram, or otherwise, explain your answer briefly. [2]

(b) Solve the inequality $\dfrac{2\ln 2}{3\pi}x \leq |\ln(1-\sin x)|$, where $0 \leq x < 2\pi$. [4]

6 The curve C has equation $y = \dfrac{x^2+5x+3}{x+1}$.

(i) Show algebraically that the curve C has no stationary points. [2]

(ii) Sketch the curve C, indicating the equations of any asymptotes, and the coordinates of points where C intersects the axes. [4]

(iii) Region S is bounded by C, the y-axis, and the line $y = \dfrac{9}{2}$. Find the volume of the solid formed when region S is rotated about the x-axis completely. [3]

7 In the Argand diagram, the points P_1 and P_2 represent the complex numbers z and z^2 respectively, where $z = \sqrt{3} + \mathrm{i}\sqrt{3}$.

(i) Find the exact modulus and argument of z. [2]

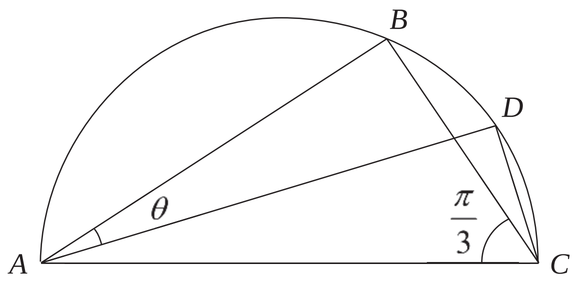
(ii) Mark the points P_1 and P_2 on an Argand diagram and find the area of the triangle OP_1P_2, where O represents the complex number 0. [3]

Let $w = 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$.

(iii) Find the set of integer values n such that $\arg(w^n z^3) = -\dfrac{\pi}{4}$. [4]

8 **(a)** Given that $2^y = 2 + \sin 2x$, use repeated differentiation to find the Maclaurin series for y, up to and including the term in x^2. [5]

(b)



The points *A*, *B*, *C*, and *D* lie on a semi-circle with *AC* as its diameter. Furthermore, angle $DAB = \theta$, and angle $ACB = \dfrac{\pi}{3}$.

(i) Show that $\dfrac{BC}{DC} = \dfrac{1}{\cos\theta - \sqrt{3}\sin\theta}$. [3]

(ii) Given that θ is a sufficiently small angle, show that

$$\frac{BC}{DC} \approx 1 + a\theta + b\theta^2,$$

for constants a and b to be determined. [3]

9 **(a)** **(i)** Find $\int 2\sin x \cos 3x \, dx$. [3]

(ii) Hence, show that $\int 2x \sin x \cos 3x \, dx = \dfrac{1}{16}\left[-4x\cos 4x + 8x\cos 2x + \sin 4x - 4\sin 2x\right] + C$, where C is an arbitrary constant. [3]

(b) The curve *C* has parametric equations

$$x = \theta^2,\ y = \sin\theta\cos 3\theta,\ \text{where } 0 \le \theta \le \frac{\pi}{2}.$$

(i) Sketch the curve *C*, giving the exact coordinates of the points where it intersects the *x*-axis. [2]

(ii) By using the result in **(a)(ii)**, find the exact total area of the regions bounded by the curve *C* and the *x*-axis. [4]

10 An object is heated up by placing it on a hotplate kept at a high temperature. A simple model for the temperature of the object over time is given by the differential equation

$$\frac{dT}{dt} = k(T_H - T),$$

where T is the temperature of the object in degrees Celsius, T_H is the temperature of the hotplate in degrees Celsius, t is time measured in seconds and k is a real constant.

(i) State the sign of k and explain your answer. [1]

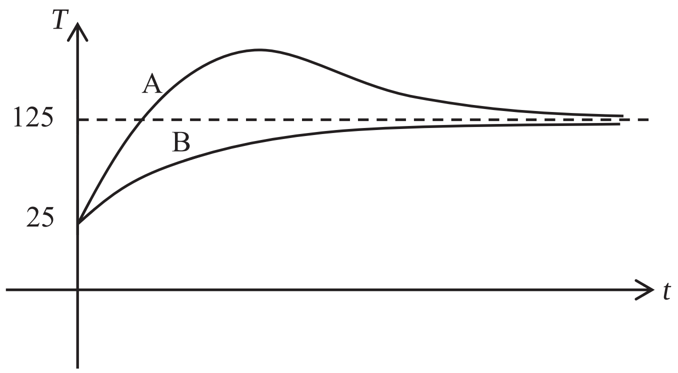
(ii) It is given that the temperature of the object is 25 degrees Celsius at $t = 0$, and the temperature of the hotplate is kept constant at 275 degrees Celsius. If the temperature of the object is 75 degrees Celsius at $t = 100$, find T in terms of t, giving the value of k to 5 significant figures. [6]

The model is now modified to account for heat lost by the object to its surroundings. The new model is given by the equation

$$\frac{dT}{dt} = k(T_H - T) - m(T - T_S),$$

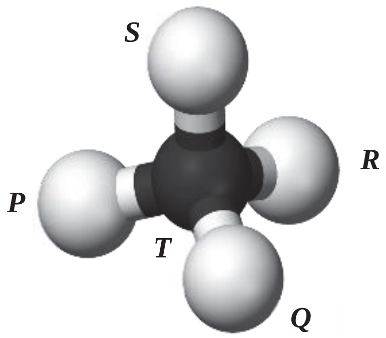
where T_S is the temperature of the surrounding environment in degrees Celsius and m is a positive real constant.

(iii) It is given that the object eventually approaches an equilibrium temperature of 125 degrees Celsius, and that the surrounding environment has a constant temperature which is lower than 125 degrees Celsius. One of the two curves (A and B) shown below is a possible graph of the object's temperature over time. State which curve this is, and explain clearly why the other curve cannot be a graph of the object's temperature over time. [2]



(iv) Using the same value of k as found in part **(ii)** and assuming $T_S = 25$, find the value of m. (You need not solve the revised differential equation.) [3]

11



Methane (CH_4) is an example of a chemical compound with a tetrahedral structure. The 4 hydrogen (H) atoms form a regular tetrahedron, and the carbon (C) atom is in the centre.

Let the 4 H-atoms be at points *P*, *Q*, *R*, and *S* with coordinates $(9,2,9)$, $(9,8,3)$, $(3,2,3)$, and $(3,8,9)$ respectively.

(i) Find a Cartesian equation of the plane Π_1 which contains the points *P*, *Q* and *R*. [4]

(ii) Find a Cartesian equation of the plane Π_2 which passes through the midpoint of *PQ* and is perpendicular to $\overrightarrow{PQ}$. [2]

(iii) Find the coordinates of point *F*, the foot of the perpendicular from *S* to Π_1. [4]

(iv) Let *T* be the point representing the carbon (C) atom. Given that point *T* is equidistant from the points *P*, *Q*, *R* and *S*, find the coordinates of *T*. [3]

End of Paper

EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2019

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME	

CLASS		INDEX NO.	

MATHEMATICS 9758/02

18 September 2019

Paper 2 [100 marks]

3 hours

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.
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The number of marks is given in brackets [ ] at the end of each question or part question.

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For markers' use:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total

Section A: Pure Mathematics [40 marks]

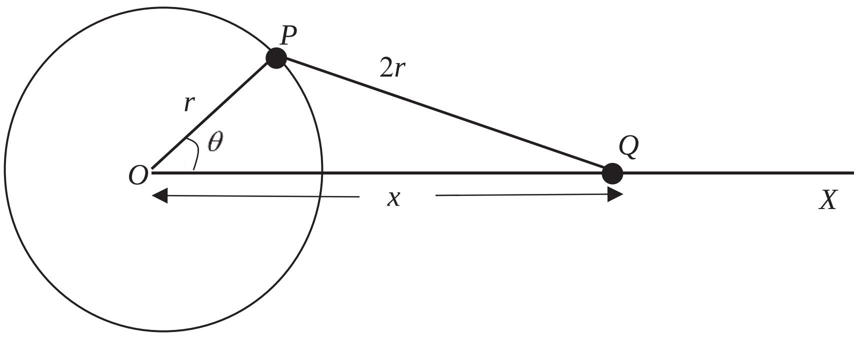
1 **(a)** Find the complex numbers z and w that satisfy the equations

$$\frac{z}{w} = 2 + 2\mathrm{i},$$

$$(1 - 2\mathrm{i})z = 39 - (11\mathrm{i})w. \qquad [3]$$

(b) It is given that $(1 + \mathrm{i}c)^3$ is real, where c is also real. By first expressing $(1 + \mathrm{i}c)^3$ in Cartesian form, find all possible values of c. [3]

2



The diagram shows a mechanism for converting rotational motion into linear motion. The point P is on the circumference of a disc of fixed radius r which can rotate about a fixed point O. The point Q can only move on the line OX, and P and Q are connected by a rod of length $2r$. As the disc rotates, the point Q is made to slide along OX. At time t, angle POQ is θ and the distance OQ is x.

(i) State the maximum and minimum values of x. [1]

(ii) Show that $x = r\left(\cos\theta + \sqrt{4 - \sin^2\theta}\right)$. [2]

(iii) At a particular instant, $\theta = \frac{\pi}{6}$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.3$. Find the numerical rate at which point Q is moving towards point O at that instant, leaving your answer in terms of r. [3]

3 The position vectors of points P and Q, with respect to the origin O, are $\mathbf{p}$ and $\mathbf{q}$ respectively. Point R, with position vector $\mathbf{r}$, is on PQ produced, such that $3\overrightarrow{PR} = 5\overrightarrow{PQ}$.

(i) Given that $|\mathbf{p}| = \sqrt{29}$ and $\mathbf{p.r} = 11$, find the length of projection of $\overrightarrow{OQ}$ onto $\overrightarrow{OP}$. [4]

(ii) S is another point such that $\overrightarrow{PS} = \mathbf{r}$. Given that $\mathbf{p} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ and $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, find the area of the quadrilateral $OPSR$. [3]

4 The ceiling function maps a real number x to the least integer greater than or equal to x. Denote the ceiling function as $\lceil x \rceil$. For example, $\lceil 2.1 \rceil = 3$ and $\lceil -3.8 \rceil = -3$.

The function f is defined by

$$\mathrm{f}(x) = \begin{cases} \lceil x \rceil & \text{for } x \in \mathbb{R}, \quad -2 < x \le 1, \\ 0 & \text{for } x \in \mathbb{R}, \quad 1 < x \le 2. \end{cases}$$

(i) Find the value of $\mathrm{f}(-1.4)$. [1]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-2 < x \le 2$. [2]

(iii) Does f^{-1} exist? Justify your answer. [1]

(iv) Find the range of f. [1]

The function g is defined as $\mathrm{g} : x \mapsto \dfrac{ax-3}{x-a}$, $x \in \mathbb{R}, x \ne a$, where $a > 0, a \ne 3$.

(v) Find $\mathrm{g}^2(x)$. Hence, or otherwise, evaluate $\mathrm{g}^{2019}(5)$, leaving your answer in a if necessary. [4]

(vi) Given that $a = 3$, find the range of gf. [1]

5 **(a)** The r^{th} term of a sequence is given by $u_r = \dfrac{4}{M^{3r-1}}$, where $M > 1$.

(i) Write down the first three terms of u_r in terms of M. [1]

(ii) Show that $\displaystyle\sum_{r=1}^{n} u_r = \frac{4M}{M^3 - 1}\left(1 - \frac{1}{M^{3n}}\right)$. [2]

(iii) Give a reason why the series in **(ii)** is convergent and state the sum to infinity. [2]

(b) **(i)** Show that $\cos\left(\dfrac{2r+1}{2}\right) - \cos\left(\dfrac{2r-1}{2}\right) = -2\sin\left(\dfrac{1}{2}\right)\sin(r)$. [2]

(ii) Hence show that $\displaystyle\sum_{r=1}^{n} \sin r = \operatorname{cosec}\left(\frac{1}{2}\right)\sin\frac{n+1}{2}\sin\frac{n}{2}$. [4]

Section B: Probability and Statistics [60 marks]

6 A biased tetrahedral die has four faces, marked with the numbers 1, 2, 3 and 4. On any throw, the probability of the die landing on each face is shown in the table below, where c and d are real numbers.

Number on face	1	2	3	4
Probability of landing on face	0.3	c	d	0.2

(i) Write down an expression for d in terms of c. [1]

(ii) By writing the variance of the result of one throw of the die in the form $-\alpha(c-h)^2+k$, where α, h and k are positive constants to be determined, find the value of c which maximises this variance. [5]

(iii) If $c=0.2$, find the probability that in 10 throws of the die, at least 7 throws land on an even number. [2]

7 $X_1, X_2, X_3, \ldots$ are independent normally-distributed random variables with common mean μ and **different** variances. For each positive integer n, $\text{Var}(X_n)=2n$.

(i) Find $\text{P}(\mu-1<X_2<\mu+1)$. [3]

(ii) Find $\text{P}(X_3 \geq X_4)$. [1]

(iii) For each n, let $Y_n=\dfrac{X_1+X_2+\ldots+X_n}{n}$. By finding the distribution of Y_n in terms of n, determine the smallest integer value of n such that $\text{P}(\mu-1<Y_n<\mu+1)>\dfrac{2}{3}$. [4]

8 For events A and B, it is given that $\text{P}(A)=\dfrac{2}{5}$, $\text{P}(A\cup B)=\dfrac{6}{7}$, and $\text{P}(A\cap B')=\dfrac{1}{3}$. Find:

(i) $\text{P}(B)$; [2]

(ii) $\text{P}(A'\mid B)$. [2]

A third event, C, is such that B and C are independent, and $\text{P}(C)=\dfrac{2}{5}$.

(iii) Find $\text{P}(B'\cap C)$. [2]

(iv) Hence, find the greatest and least possible values of $\text{P}(A\cap B'\cap C)$. [4]

9 **(a)** Find the number of ways of arranging the letters of the word JEWELLERY, if:

(i) there are no restrictions. [1]

(ii) the arrangement starts with 'L', and between any two 'E's there must be at least 2 other letters. [3]

A 4-letter 'codeword' is formed by taking an arrangement of 4 letters from the word JEWELLERY.

(iii) Find the number of 4-letter codewords that can be formed. [3]

(b) Mr and Mrs Lee, their three children, and 5 others are seated at a round table during a wedding dinner. Find the number of ways that everyone can be seated, such that Mr and Mrs Lee are seated together, but their children are not all seated together. [3]

10 A company manufactures packets of potato chips with X mg of sodium in each packet. It is known that the mean amount of sodium per packet is 1053 mg. After some alterations to the production workflow, 50 randomly chosen packets of potato chips were selected for analysis. The amount of sodium in each packet was measured, and the results are summarised as follows:

$$\sum(x-1050)=58.0,\ \sum(x-1050)^2=2326$$

(i) Test at 5% level of significance whether the mean amount of sodium in a packet of potato chips has changed, after the alterations to the workflow. [6]

(ii) Explain what '5% level of significance' means in this context. [1]

(iii) A second tester conducted the same test at $\alpha\%$ level of significance, for some integer α. However, he came to a different conclusion from the first tester. What is the range of α for which the second tester could have taken? [1]

(iv) Without performing another hypothesis test, explain whether the conclusion in part **(i)** would be different if the alternative hypothesis was that the mean amount of sodium had decreased after the alterations to the workflow. [1]

It is given instead that the standard deviation of amount of sodium in a packet of potato chips is 6.0 mg. The mean amount of sodium of a second randomly chosen sample of 40 packets of potato chips is $\bar{y}$.

(v) A hypothesis test on this sample, at 5% level of significance, led to a conclusion that the amount of sodium has decreased.

Find the range of values of $\bar{y}$, to 1 decimal place.

Explain if it is necessary to make any assumptions about the distribution of the amount of sodium in each packet of potato chips. [3]

11 **(a)** Comment on the following statement: "The product moment correlation coefficient between the amount of red wine intake and the risk of heart disease is approximately -1. Thus we can conclude that red wine intake decreases the risk of heart disease." [1]

(b) During an experiment, the radiation intensity, I, from a source at time t, in appropriate units, is measured and the results are tabulated below.

t	0.2	0.4	0.6	0.8	1.0
I	2.81	1.64	0.93	0.55	0.30

(i) Identify the independent variable and explain why it is independent. [1]

(ii) Draw a scatter diagram of these data. With the help of your diagram, explain whether the relationship between I and t is likely to be well modelled by an equation of the form $I = at + b$, where a and b are constants. [3]

(iii) Calculate, to 4 decimal places, the product moment correlation coefficient between

(a) I and t,

(b) $\ln I$ and t. [2]

(iv) Using the model $I = a\mathrm{e}^{bt}$, find the equation of a suitable regression line, and calculate the values of a and b. [3]

(v) Use the regression line found in **(iv)** to estimate the radiation intensity when $t = 0.7$. Comment on the reliability of your estimate. [2]

End of Paper

EJC_H2_2019_JC2_Prelim_P1_Solutions

1	Let x, y, z be the tariff in ¢/kWh in Jan-Apr, May-Aug and Sept-Dec respectively. $677x+586y+699z=52953$ --- (1) $1011x+871y+1048z=79063$ --- (2) $1349x+1174y+1417z=106328$ --- (3) Solving with GC, $x=25.81$ $y=29.68$ $z=25.88$

2	Total length is l, thus we have $4s+2\pi r=l$ …(*) Let the combined area be A. $A=s^2+\pi r^2$ …(#) <u>Method 1: implicit differentiation of (#)</u> Use (#) to find $\frac{\mathrm{d}A}{\mathrm{d}s}$: $A=s^2+\pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}s}=2s+2\pi r\frac{\mathrm{d}r}{\mathrm{d}s}$ Use (*) to find $\frac{\mathrm{d}r}{\mathrm{d}s}$: $4s+2\pi r=l \Rightarrow 4+2\pi\frac{\mathrm{d}r}{\mathrm{d}s}=0$ Thus $\frac{\mathrm{d}r}{\mathrm{d}s}=-\frac{2}{\pi}$ Sub into $\frac{\mathrm{d}A}{\mathrm{d}s}$: $\frac{\mathrm{d}A}{\mathrm{d}s}=2s+2\pi r\left(-\frac{2}{\pi}\right)=2s-4r$ For stationary value of A, $\frac{\mathrm{d}A}{\mathrm{d}s}=0 \Rightarrow s=2r$ Check minimum: $\frac{\mathrm{d}^2A}{\mathrm{d}s^2}=2-4\frac{\mathrm{d}r}{\mathrm{d}s}=2-4\left(-\frac{2}{\pi}\right)>0$ Thus A is a minimum when $s=2r$. Required ratio is $\frac{\text{length of first piece}}{\text{length of second piece}}=\frac{4s}{2\pi r}=\frac{4(2r)}{2\pi r}=\frac{4}{\pi}$ <u>Method 2: differentiation in 1 variable</u> <u>2a: expressing A in terms of r</u> From (*), $s=\frac{l-2\pi r}{4}$ Sub into (#): $A=\pi r^2+\left(\frac{l-2\pi r}{4}\right)^2$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r + 2\left(\frac{l-2\pi r}{4}\right)\left(\frac{-2\pi}{4}\right)$$

$$= \frac{\pi}{4}\left[8r - \left(l - 2\pi r\right)\right] = \frac{\pi}{4}\left[r\left(8+2\pi\right) - l\right]$$

For stationary value of A, $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$:

$$\frac{\pi}{4}\left[r\left(8+2\pi\right) - l\right] = 0 \Rightarrow r = \frac{l}{8+2\pi}$$

Check minimum:

EITHER 2nd Derivative Test

$$\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = \frac{\pi}{4}\left(8+2\pi\right) > 0$$

So $r = \frac{l}{8+2\pi}$ gives a minimum value of A.

OR 1st Derivative Test

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{\pi}{4}\left[r\left(8+2\pi\right) - l\right] = \frac{\pi}{4}\left(8+2\pi\right)\left(r - \frac{l}{8+2\pi}\right)$$

r	$\left(\frac{l}{8+2\pi}\right)^-$	$\left(\frac{l}{8+2\pi}\right)$	$\left(\frac{l}{8+2\pi}\right)^+$
$r - \frac{l}{8+2\pi}$	-ve	0	+ve
$\frac{\mathrm{d}A}{\mathrm{d}r}$	-ve	0	+ve

$$\frac{\text{length of first piece}}{\text{length of second piece}} = \frac{4s}{2\pi r} = \frac{l-2\pi r}{2\pi r} = \frac{l}{2\pi r} - 1$$

When A is minimum,

$$\frac{\text{length of first piece}}{\text{length of second piece}} = \frac{l}{2\pi\left(\frac{l}{8+2\pi}\right)} - 1$$

$$= \frac{\left(8+2\pi\right)}{2\pi} - 1$$

$$= \frac{4}{\pi}$$

<u>2b: expressing A in terms of s</u>

From (*), $r = \frac{l-4s}{2\pi}$

Sub into (#): $A = \pi\left(\frac{l-4s}{2\pi}\right)^2 + s^2$

$$\frac{\mathrm{d}A}{\mathrm{d}S} = 2\pi\left(\frac{l-4s}{2\pi}\right)\left(\frac{-4}{2\pi}\right) + 2s$$

$$= \frac{2}{\pi}\left[s\left(4+\pi\right) - l\right]$$

	For stationary value of A, $\frac{dA}{ds}=0$: $\frac{2}{\pi}\left[s(4+\pi)-l\right]=0 \Rightarrow s=\frac{l}{4+\pi}$ *Check minimum:* *EITHER 2nd Derivative Test* $\frac{d^2A}{ds^2}=\frac{2}{\pi}(4+\pi)>0$ So $s=\frac{l}{4+\pi}$ gives a minimum value of A OR 1st Derivative Test $\frac{dA}{dS}=\frac{2}{\pi}\left[s(4+\pi)-l\right]=\frac{2(4+\pi)}{\pi}\left(s-\frac{l}{4+\pi}\right)$

S	$\left(\frac{l}{4+\pi}\right)^-$	$\left(\frac{l}{4+\pi}\right)$	$\left(\frac{l}{4+\pi}\right)^+$
$s-\frac{l}{4+\pi}$	-ve	0	+ve
$\frac{dA}{ds}$	-ve	0	+ve

$$\frac{\text{length of first piece}}{\text{length of second piece}}=\frac{4s}{2\pi r}=\frac{4s}{2\pi\left(\frac{l-4s}{2\pi}\right)}=\frac{4s}{l-4s}$$

When A is minimum,

$$\frac{\text{length of first piece}}{\text{length of second piece}}=\frac{4\left(\frac{l}{4+\pi}\right)}{l-\frac{4l}{4+\pi}}$$
$$=\frac{4l}{4+\pi}\times\frac{4+\pi}{\pi l}$$
$$=\frac{4}{\pi}$$

Other possible methods:
Method 3: Differentiate w.r.t. ratio
Method 4: Complete the square

3	(i)

$$\frac{2}{2}\left[2a+(2-1)d\right]=\frac{a\left(r^2-1\right)}{r-1}$$

$$\Rightarrow 2a+d=a(r+1)--------(1)$$

$$\frac{4}{2}\left[2a+(4-1)d\right]=\frac{a\left(r^4-1\right)}{r-1}$$

$$\Rightarrow 4a+6d=a\left(r^2+1\right)(r+1)--------(2)$$

From (1): Sub $d=a(r+1)-2a$ into (2)

i.e. $d=ar-a$

$$4a+6[ar-a]=a\left(r^2+1\right)(r+1)$$

$$6ar-2a=a\left(r^3+r^2+r+1\right)$$

$r^3+r^2-5r+3=0$ (shown)

$$(r-1)^2(r+3)=0$$

$r=-3$ or $r=1$ (rej)

[If $r=1$, $d=a(r+1)-2a=0$, but $d\neq 0$]

<u>Alternative solution</u>

$$2a+d=\frac{a\left(r^2-1\right)}{r-1} \text{ -- (1)}$$

$$4a+6d=\frac{a\left(r^4-1\right)}{r-1} \text{ -- (2)}$$

6 x (1)-2:

$$8a=\frac{6a\left(r^2-1\right)}{r-1}-\frac{a\left(r^4-1\right)}{r-1}$$

$$8a(r-1)=a\left(6r^2-6-r^4+1\right)$$

$$8r-8=6r^2-r^4-5$$

$$r^4-6r^2+8r-3=0$$

Solving:

$r=-3$ or 1 (rej)

(ii)

$a(-3)^{n-1} > 1000\left[a+(n-1)d\right]$

Note: $d = a(-3+1) - 2a = -4a$

$a(-3)^{n-1} > 1000\left[a+(n-1)(-4a)\right]$

$a(-3)^{n-1} > 1000a(5-4n)$

Since $a < 0$,

$(-3)^{n-1} < 1000(5-4n)$

Since n^{th} term of GP is positive, i.e. $a(-3)^{n-1} > 0$

$(-3)^{n-1}$ is negative $\Rightarrow n$ is even

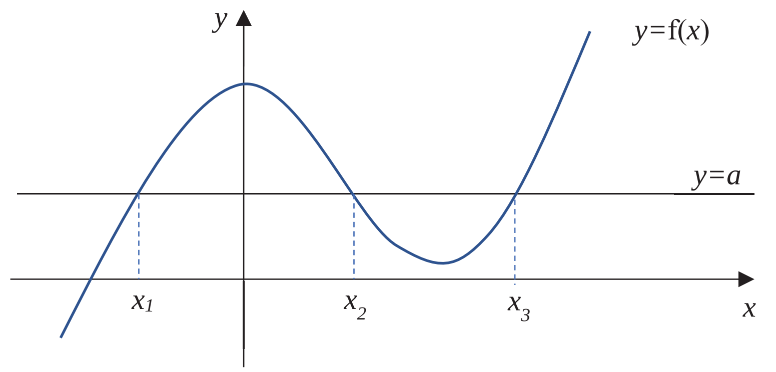
From GC (table)

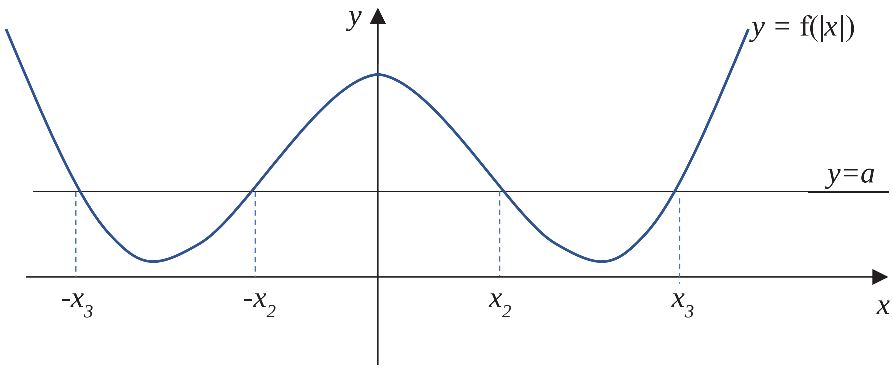
n	$(-3)^{n-1}$	$1000(5-4n)$
10	$-19683\ >$	-35000
12	$-177147\ <$	-43000

Smallest $n = 12$

4	(i) $(1+ax)^n = 1 + n(ax) + \frac{n(n-1)}{2!}(ax)^2 + \frac{n(n-1)(n-2)}{3!}(ax)^3 + \ldots$
	(ii) Since the three coefficients form a GP, we have $\dfrac{\frac{n(n-1)}{2}a^2}{na} = \dfrac{\frac{n(n-1)(n-2)}{6}a^3}{\frac{n(n-1)}{2}a^2}$ $\frac{(n-1)a}{2} = \frac{(n-2)a}{3}$ $3(n-1) = 2(n-2)$ $n = -1$
	(iii) To prove GP, we need to show that $\frac{u_r}{u_{r-1}} = \text{constant}$ $u_r = \frac{n(n-1)\ldots(n-r+1)}{r!}a^r = \frac{(-1)(-2)\ldots(-1-r+1)}{r!}a^r$ $u_{r-1} = \frac{n(n-1)\ldots(n-r+2)}{(r-1)!}a^{r-1} = \frac{(-1)(-2)\ldots(-1-r+2)}{(r-1)!}a^{r-1}$ $\frac{u_r}{u_{r-1}} = \frac{(-1-r+1)}{r}a$ Since $n=-1$, $\frac{u_r}{u_{r-1}} = \frac{(-1-r+1)}{r}a = -a$ (constant) Alternatively, $(1+ax)^{-1} = 1 - ax + a^2x^2 - a^3x^3 + \ldots + (-a)^{r-1} + \ldots$ $\frac{u_r}{u_{r-1}} = \frac{(-a)^{r-1}}{(-a)^{r-2}} = \frac{(-1)^{r-1}(a)^{r-1}}{(-1)^{r-2}(a)^{r-2}} = -a$ (constant)
	(iv) The cofficients of the terms in x of odd powers form a GP with first term $-a$, and common ratio a^2. Sum to infinity $= \frac{-a}{1-a^2} = \frac{a}{a^2-1}$

5 (a)(i)

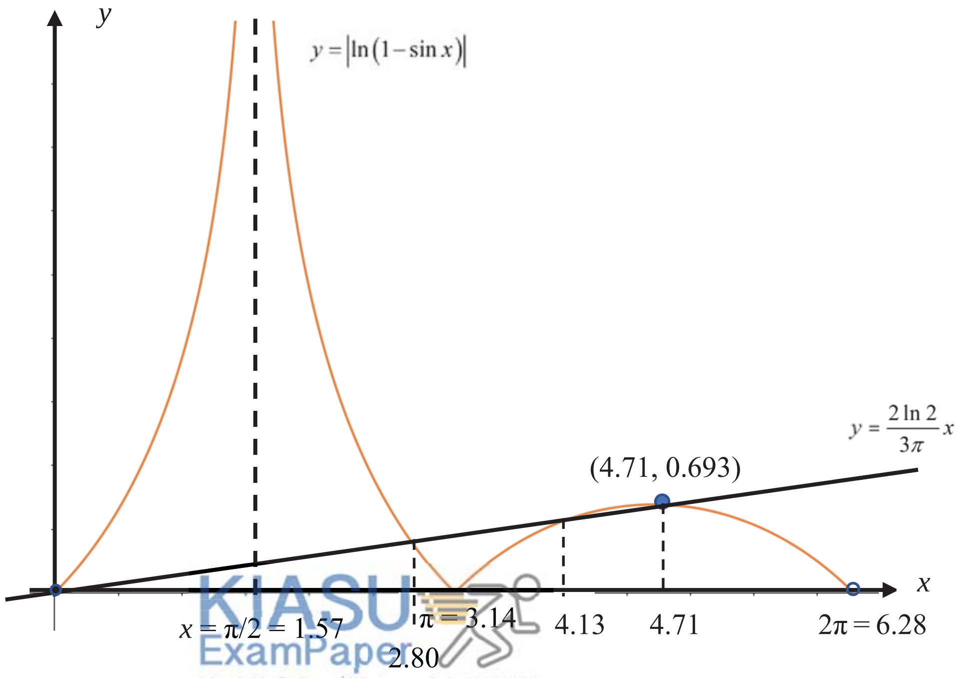




Since the graph of $y=\mathrm{f}\left(|x|\right)$ retains the part with positive x-values, $\mathrm{f}\left(|-x_2|\right)=\mathrm{f}\left(x_2\right)=a$. Similarly for $-x_3$.

Thus there will be 4 roots, i.e. $x_2, x_3, -x_2, -x_3$

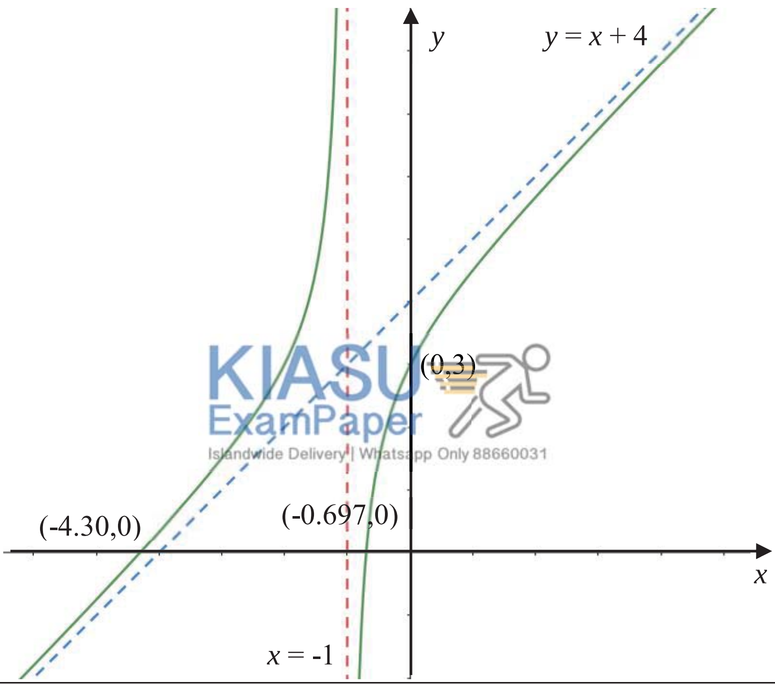
(b)



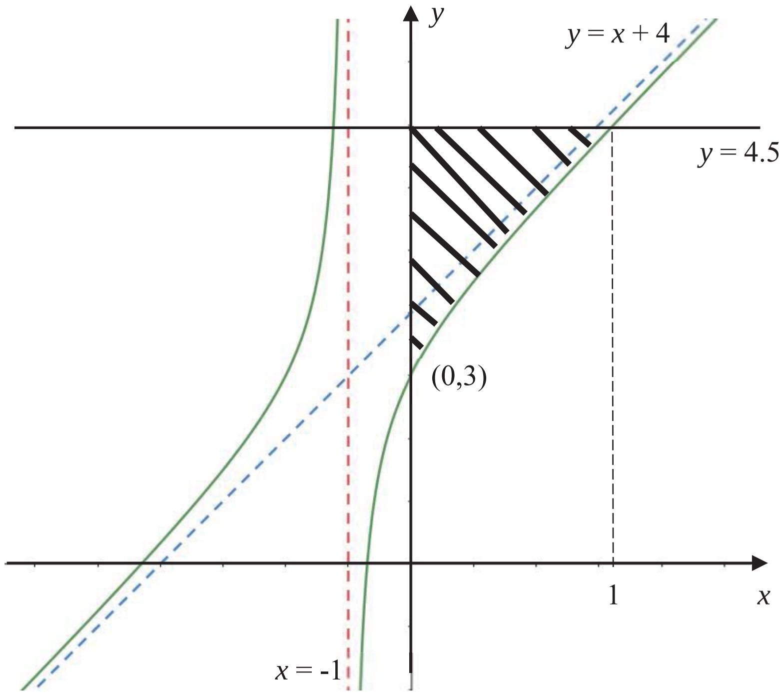
$$\frac{2\ln 2}{3\pi}x \le \left|\ln\left(1-\sin x\right)\right|$$

From the graph,

	$\frac{2\ln 2}{3\pi}x \le \lvert \ln(1-\sin x) \rvert$ $0 \le x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \le 2.80$ or $4.13 \le x \le \frac{3\pi}{2}$ OR $0 \le x < 1.57$ or $1.57 < x \le 2.80$ or $4.13 \le x \le 4.71$ (3 s.f)

6	(i) $y = \frac{x^2+5x+3}{x+1} = x+4-\frac{1}{x+1}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 1+(x+1)^{-2} = 1+\frac{1}{(x+1)^2}$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+1)(2x+5)-(x^2+5x+3)(1)}{(x+1)^2}$ $= \frac{x^2+2x+2}{(x+1)^2} = \frac{1+(x+1)^2}{(x+1)^2} = 1+\frac{1}{(x+1)^2}$ Since $(x+1)^2 \ge 0$, $\frac{1}{(x+1)^2} > 0$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 1+\frac{1}{(x+1)^2} > 1$. Since $\frac{\mathrm{d}y}{\mathrm{d}x} \ne 0$ for any real value of x, C has no stationary points.
	(ii) $y = \frac{x^2+5x+3}{x+1} = x+4-\frac{1}{x+1}$ Asymptotes: $y = x+4$, $x = -1$ y-intercept: when $x = 0$, $y = 3$ 

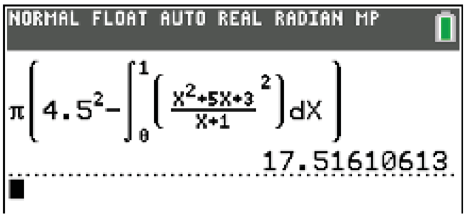
(iii)

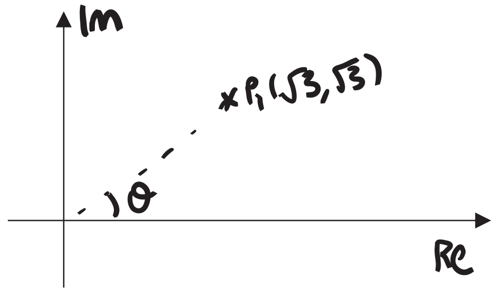


Required volume $= \pi\left(4.5\right)^2(1) - \pi\int_0^1 \left(\frac{x^2+5x+3}{x+1}\right)^2 \mathrm{d}x$

$= 17.516$

$= 17.5$ units^3 (3 s.f.) (by G.C.)

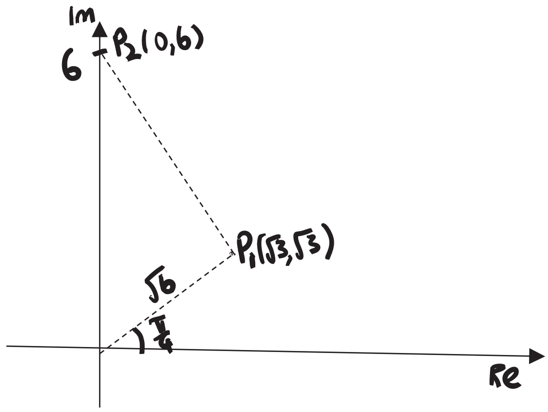


7	(i) $\lvert z \rvert = \sqrt{3+3} = \sqrt{6}$ $\arg(z) = \frac{\pi}{4}$ (refer to Argand diagram)	

(ii)

$z = \sqrt{6}e^{i\left(\frac{\pi}{4}\right)}$

$\Rightarrow z^2 = 6e^{i\left(\frac{\pi}{2}\right)}$



Area of triangle $OP_1P_2 = \frac{1}{2}(6)\sqrt{3} = 3\sqrt{3}$

(iii)

$$\arg\left(w^n z^3\right) = \arg\left(w^n\right) + \arg\left(z^3\right)$$
$$= n\arg w + 3\arg z$$
$$= n\left(-\frac{\pi}{3}\right) + 3\left(\frac{\pi}{4}\right)$$

$$\arg\left(w^n z^3\right) = -\frac{\pi}{4}$$

$$\Rightarrow n\left(-\frac{\pi}{3}\right) + \frac{3\pi}{4} = -\frac{\pi}{4} + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

$$-\frac{n\pi}{3} = -\pi + 2k\pi$$

$$n = 3 - 6k$$

$\{n \in \mathbb{Z} : n = 3 - 6k, k \in \mathbb{Z}\}$

i.e. $n \in \{\dots, -9, -3, 3, 9, \dots\}$

8	(a) $2^y = 2+\sin 2x$ Differentiate w.r.t x: $2^y \ln 2 \dfrac{dy}{dx} = 2\cos 2x \quad \ldots(1)$ Differentiate w.r.t x: $2^y \ln 2 \dfrac{d^2y}{dx^2} + 2^y(\ln 2)^2\left(\dfrac{dy}{dx}\right)^2 = -4\sin 2x \ldots(2)$ When $x=0, y=1, \dfrac{dy}{dx}=\dfrac{1}{\ln 2}, \dfrac{d^2y}{dx^2}=-\dfrac{1}{\ln 2}$ $y = 1+\dfrac{1}{\ln 2}x - \dfrac{1}{2\ln 2}x^2 + \ldots$	<u>Alternative method</u> $y\ln 2 = \ln(2+\sin 2x)$ $(\ln 2)\dfrac{dy}{dx} = \dfrac{2\cos 2x}{2+\sin 2x}$ $(\ln 2)\dfrac{d^2y}{dx^2} = \dfrac{-4\sin 2x(2+\sin 2x)-(2\cos 2x)^2}{(2+\sin 2x)^2}$ When $x=0, y=1, \dfrac{dy}{dx}=\dfrac{1}{\ln 2}, \dfrac{d^2y}{dx^2}=-\dfrac{1}{\ln 2}$ $y = 1+\dfrac{1}{\ln 2}x - \dfrac{1}{2\ln 2}x^2 + \ldots$
	(b)(i) $BC = AC\cos\left(\dfrac{\pi}{3}\right),\ DC = AC\sin\left(\dfrac{\pi}{6}-\theta\right)$ $\dfrac{BC}{DC} = \dfrac{AC\cos\left(\dfrac{\pi}{3}\right)}{AC\sin\left(\dfrac{\pi}{6}-\theta\right)} = \dfrac{1}{2\left(\sin\dfrac{\pi}{6}\cos\theta - \cos\dfrac{\pi}{6}\sin\theta\right)}$ $= \dfrac{1}{2\left(\dfrac{1}{2}\cos\theta - \dfrac{\sqrt{3}}{2}\sin\theta\right)}$ $= \dfrac{1}{\cos\theta - \sqrt{3}\sin\theta}$ (shown)	
	(b)(ii) Since θ is sufficiently small, $\dfrac{BC}{DC} \approx \dfrac{1}{1-\dfrac{\theta^2}{2}-\sqrt{3}\theta} = \left(1-\left(\sqrt{3}\theta+\dfrac{\theta^2}{2}\right)\right)^{-1}$ $= 1+\sqrt{3}\theta+\dfrac{\theta^2}{2}+\left(\sqrt{3}\theta+\dfrac{\theta^2}{2}\right)^2+\ldots$ $= 1+\sqrt{3}\theta+\dfrac{\theta^2}{2}+3\theta^2+\ldots$ $\approx 1+\sqrt{3}\theta+\dfrac{7\theta^2}{2}$ $\therefore a=\sqrt{3}, b=\dfrac{7}{2}$	

9

(a)(i)

Using factor formula (MF26),

$2\sin x\cos 3x=\sin 4x-\sin 2x$.

Hence

$$\int 2\sin x\cos 3x\,\mathrm{d}x=\int\left(\sin 4x-\sin 2x\right)\mathrm{d}x$$

$$=-\frac{\cos 4x}{4}+\frac{\cos 2x}{2}+C$$

(a)(ii)

Let

$$u=x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=1$$

$$\frac{dv}{dx}=2\sin x\cos x\Rightarrow v=-\frac{\cos 4x}{4}+\frac{\cos 2x}{2}$$

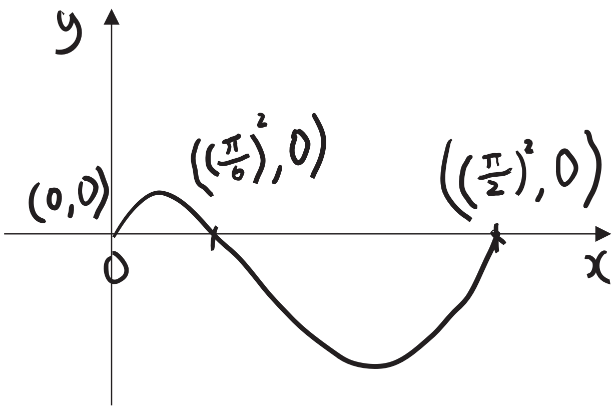
$$\int 2x\sin x\cos 3x\,\mathrm{d}x$$

$$=x\left(-\frac{\cos 4x}{4}+\frac{\cos 2x}{2}\right)-\int-\frac{\cos 4x}{4}+\frac{\cos 2x}{2}\,\mathrm{d}x$$

$$=-\frac{x\cos 4x}{4}+\frac{x\cos 2x}{2}+\frac{\sin 4x}{16}-\frac{\sin 2x}{4}+C$$

$$=\frac{1}{16}\left[-4x\cos 4x+8x\cos 2x+\sin 4x-4\sin 2x\right]+C$$

(b)(i)



To find x-intercepts, $y=\sin\theta\cos 3\theta=0$

$\sin\theta=0$ or $\cos 3\theta=0$

$\theta=0$ or $3\theta=\frac{\pi}{2},\frac{3\pi}{2}$ $\left(0\le\theta\le\frac{\pi}{2}\Rightarrow 0\le 3\theta\le\frac{3\pi}{2}\right)$

$\theta=\frac{\pi}{6},\frac{\pi}{2}$

Thus intercepts are when $\theta=0,\frac{\pi}{6},\frac{\pi}{2}$.

(b)(ii)

Area of S

$$=\int_0^{\left(\frac{\pi}{6}\right)^2} y\ \mathrm{d}x-\int_{\left(\frac{\pi}{6}\right)^2}^{\left(\frac{\pi}{2}\right)^2} y\ \mathrm{d}x$$

$$=\int_0^{\frac{\pi}{6}} \sin\theta\cos 3\theta(2\theta)\ \mathrm{d}\theta-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin\theta\cos 3\theta(2\theta)\ \mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{6}} 2\theta\sin\theta\cos 3\theta\ \mathrm{d}\theta-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\theta\sin\theta\cos 3\theta\ \mathrm{d}\theta$$

$$=\frac{1}{16}\left[-4\theta\cos 4\theta+8\theta\cos 2\theta+\sin 4\theta-4\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$-\frac{1}{16}\left[-4\theta\cos 4\theta+8\theta\cos 2\theta+\sin 4\theta-4\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$=\frac{1}{16}\left(-4\left(\frac{\pi}{6}\right)\left(-\frac{1}{2}\right)+8\left(\frac{\pi}{6}\right)\left(\frac{1}{2}\right)+\frac{\sqrt{3}}{2}-4\left(\frac{\sqrt{3}}{2}\right)\right)-\frac{1}{16}(0)$$

$$-\frac{1}{16}\left(-2\pi+4\pi(-1)\right)+\frac{1}{16}\left(-4\left(\frac{\pi}{6}\right)\left(-\frac{1}{2}\right)+8\left(\frac{\pi}{6}\right)\left(\frac{1}{2}\right)+\frac{\sqrt{3}}{2}-4\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$=2\times\frac{1}{16}\left(\frac{\pi}{3}+\frac{2\pi}{3}+\frac{\sqrt{3}}{2}-2\sqrt{3}\right)-\frac{1}{16}[-6\pi]$$

$$=\frac{\pi}{2}-\frac{3}{16}\sqrt{3}$$

10

(i)

Since $\frac{\mathrm{d}T}{\mathrm{d}t}>0$ as the object is being heated up, and $T_H-T>0$ as hotplate temperature is higher than that of the object, it follows that k is positive.

(ii)

$$\frac{\mathrm{d}T}{\mathrm{d}t}=k(275-T)$$

$$\int\frac{1}{275-T}\mathrm{d}T=\int k\ \mathrm{d}t$$

$$-\ln(275-T)=kt+C$$

$$275-T=A\mathrm{e}^{-kt} \quad \text{where } A=\mathrm{e}^{-C}$$

Substituting $t=0$, $T=25$,

$250=A\mathrm{e}^0$ thus $A=250$

$T=275-250\mathrm{e}^{-kt}$

Substituting $t=100$, $T=75$,

$75=275-250\mathrm{e}^{-100k}$

$k\approx 0.0022314$

So $T=275-250\mathrm{e}^{-0.0022314t}$.

	(iii) Curve B is a possible graph. Curve A does not fit because: • Temperature does not exceed equilibrium as object is being heated continuously; OR • The curve cannot have different gradients for same value of T *(note that the* $\frac{\mathrm{d}T}{\mathrm{d}t}$ *is linear in* T); OR • Gradient cannot be negative at any point because the object is being heated continuously. OR • Observe that $\frac{\mathrm{d}T}{\mathrm{d}t}=k\left(T_H-T\right)-m\left(T-T_S\right)$ $=\left(k+m\right)\left(\frac{kT_H+mT_S}{k+m}-T\right)$ So $\frac{\mathrm{d}T}{\mathrm{d}t}$ is always > 0.
	(iv) As $T\to 125$, $\frac{\mathrm{d}T}{\mathrm{d}t}\to k(275-125)-m(125-25)$. From graph, as $T\to 125$, $\frac{\mathrm{d}T}{\mathrm{d}t}\to 0$. So, $0=k(275-125)-m(125-25)$ $\Rightarrow m=\frac{3k}{2}\approx 0.00335$ (3s.f.)

11	(i) $\overrightarrow{PQ}=\begin{pmatrix}9\\8\\3\end{pmatrix}-\begin{pmatrix}9\\2\\9\end{pmatrix}=\begin{pmatrix}0\\6\\-6\end{pmatrix}$, $\overrightarrow{PR}=\begin{pmatrix}3\\2\\3\end{pmatrix}-\begin{pmatrix}9\\2\\9\end{pmatrix}=\begin{pmatrix}-6\\0\\-6\end{pmatrix}$ A vector normal to Π_1 is $\begin{pmatrix}0\\1\\-1\end{pmatrix}\times\begin{pmatrix}-1\\0\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$ $\mathbf{r}.\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\2\\3\end{pmatrix}.\begin{pmatrix}-1\\1\\1\end{pmatrix}=2$ So a Cartesian equation of Π_1 is $-x+y+z=2$
	(ii) Position vector of midpoint of PQ is $\frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OQ}\right)=\begin{pmatrix}9\\5\\6\end{pmatrix}$ Π_2 is perpendicular to $\overrightarrow{PQ}$, so $\overrightarrow{PQ}$ is normal to Π_2

So $\mathbf{r}.\begin{pmatrix}0\\6\\-6\end{pmatrix}=\begin{pmatrix}9\\5\\6\end{pmatrix}.\begin{pmatrix}0\\6\\-6\end{pmatrix}=-6$

So a Cartesian equation of Π_2 is $6y-6z=-6 \Rightarrow y-z=-1$.

(iii)

Eqn of line passing through S and F is

$$\mathbf{r}=\begin{pmatrix}3\\8\\9\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}, \quad \lambda\in\mathbb{R}$$

So $\overrightarrow{OF}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}$ for some λ

F lies on Π_1

So $\overrightarrow{OF}.\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}.\begin{pmatrix}-1\\1\\1\end{pmatrix}=2$

$\Rightarrow \lambda=-4$

So coordinates of F are $(7,4,5)$.

(iv)

Note that T lies on the line SF,

So $\overrightarrow{OT}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}$ for some λ from (iii)

$\overrightarrow{PT}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}-\begin{pmatrix}9\\2\\9\end{pmatrix}=\begin{pmatrix}-6-\lambda\\6+\lambda\\\lambda\end{pmatrix}$ and

$\overrightarrow{ST}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}-\begin{pmatrix}3\\8\\9\end{pmatrix}=\begin{pmatrix}-\lambda\\\lambda\\\lambda\end{pmatrix}$

Since $\left|\overrightarrow{PT}\right|=\left|\overrightarrow{ST}\right|$,

$(6+\lambda)^2+(6+\lambda)^2+\lambda^2=(-\lambda)^2+\lambda^2+\lambda^2$

$\Rightarrow (6+\lambda)^2=\lambda^2$

$\Rightarrow (6+\lambda)^2-\lambda^2=0$

$\Rightarrow (6+\lambda+\lambda)(6+\lambda-\lambda)=0$

$\Rightarrow \lambda=-3$

Hence coordinates of $T(6,5,6)$.

End of Paper

EJC_H2_2019_JC2_Prelim_P2_Solutions

Section A: Pure Mathematics [40 marks]

1	(a) Sub $z=(2+2\text{i})w$ into the other equation $\Rightarrow (1-2\text{i})(2+2\text{i})w = 39-11w\text{i}$ $\Rightarrow w = \dfrac{39}{(1-2\text{i})(2+2\text{i})+11\text{i}} = 2-3\text{i}$ (using GC) Thus, $z=(2+2\text{i})(2-3\text{i})=10-2\text{i}$	OR Sub $w=\dfrac{z}{2+2\text{i}}$ into the other equation $\Rightarrow (1-2\text{i})z = 39-11\text{i}\left(\dfrac{z}{2+2\text{i}}\right)$ $\Rightarrow z = \dfrac{39}{\dfrac{11\text{i}}{2+2\text{i}}+(1-2\text{i})} = 10-2\text{i}$ (using GC) Thus, $w=\dfrac{10-2\text{i}}{2+2\text{i}}=2-3\text{i}$.
	(b) $(1+\text{i}c)^3 = 1+3\text{i}c+3(\text{i}c)^2+(\text{i}c)^3$ $= 1+3\text{i}c-3c^2-\text{i}c^3$ $= 1-3c^2+\text{i}(3c-c^3)$ Since $(1+\text{i}c)^3$ is real, $3c-c^3=0$ $c(3-c^2)=0$ $c=0, \pm\sqrt{3}$	

2 (i)

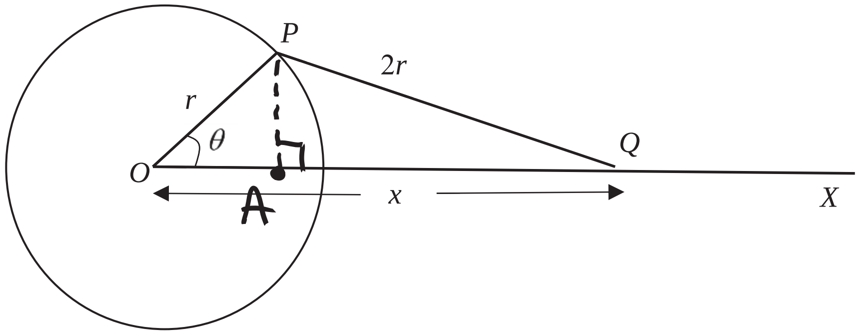
Max $x=3r$ when $\theta=0$

Min $x=r$ when $\theta=\pi$

(ii)

Method 1

Consider triangle OPA.



$$\cos\theta=\frac{OA}{r}\Rightarrow OA=r\cos\theta$$

Consider triangle PAQ. By pythagoras theorem,

$$\begin{aligned}AQ&=\sqrt{(2r)^2-(PA)^2}\\&=\sqrt{(2r)^2-(r\sin\theta)^2}\\&=r\sqrt{4-\sin^2\theta}\end{aligned}$$

$$x=OA+AQ=r\cos\theta+r\sqrt{4-\sin^2\theta}=r\left[\cos\theta+\sqrt{4-\sin^2\theta}\right]\text{ (shown)}$$

Method 2: Cosine Rule

$$\begin{aligned}(2r)^2&=r^2+x^2-2rx\cos\theta\\4r^2&=r^2+x^2-2rx\cos\theta\\&=(x-r\cos\theta)^2+r^2\sin^2\theta\end{aligned}$$

$$x-r\cos\theta=r\sqrt{4-\sin^2\theta}\quad\left(\text{reject }-r\sqrt{4-\sin^2\theta}\because x\ge r\ge r\cos\theta\right)$$

$$x=r\left(\cos\theta+\sqrt{4-\sin^2\theta}\right)\quad\text{(shown)}$$

(iii)

Method 1:

$$\begin{aligned}\frac{dx}{dt}&=\frac{dx}{d\theta}\times\frac{d\theta}{dt}\\&=r\left[-\sin\theta+\frac{(-2\sin\theta\cos\theta)}{2\sqrt{4-\sin^2\theta}}\right]\times\frac{d\theta}{dt}\end{aligned}$$

When $\theta=\frac{\pi}{6}$ and $\frac{d\theta}{dt}=0.3$,

$$\begin{aligned}\frac{dx}{dt}&=r\left[-\sin\frac{\pi}{6}-\frac{\sin\frac{\pi}{6}\cos\frac{\pi}{6}}{\sqrt{4-\sin^2\left(\frac{\pi}{6}\right)}}\right](0.3)\\&=-0.217r\end{aligned}$$

Method 2:
Differentiate implicitly w.r.t *t*,

$$\frac{\mathrm{d}x}{\mathrm{d}t}=r\left(\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}+\frac{(-2\sin\theta\cos\theta)}{2\sqrt{4-\sin^2\theta}}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)$$

When $\theta=\frac{\pi}{6}$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t}=0.3$,

$$\frac{\mathrm{d}x}{\mathrm{d}t}=r\left[-\left(\sin\frac{\pi}{6}\right)(0.3)-\frac{\sin\frac{\pi}{6}\cos\frac{\pi}{6}}{\sqrt{4-\sin^2\left(\frac{\pi}{6}\right)}}(0.3)\right]$$

$$=-0.217r$$

3 (i)

Length of projection of q onto p $=|\mathbf{q}.\hat{\mathbf{p}}|=\left|\frac{\mathbf{q}.\mathbf{p}}{|\mathbf{p}|}\right|$

Method 1

$$3\overrightarrow{PR}=5\overrightarrow{PQ}\Rightarrow 3(\mathbf{r}-\mathbf{p})=5(\mathbf{q}-\mathbf{p})\Rightarrow \mathbf{q}=\frac{1}{5}(2\mathbf{p}+3\mathbf{r})$$

Sub into $|\mathbf{q}.\hat{\mathbf{p}}|$:

$$|\mathbf{q}.\hat{\mathbf{p}}|=\left|\frac{\frac{1}{5}(2\mathbf{p}+3\mathbf{r}).\mathbf{p}}{|\mathbf{p}|}\right|$$

$$=\left|\frac{\frac{2}{5}\mathbf{p}.\mathbf{p}+\frac{3}{5}\mathbf{p}.\mathbf{r}}{|\mathbf{p}|}\right|$$

$$=\frac{\frac{2}{5}(29)+\frac{3}{5}(11)}{\sqrt{29}}=\frac{91}{5\sqrt{29}}\ \text{(or 3.38)}$$

Method 2

$$3\overrightarrow{PR}=5\overrightarrow{PQ}\Rightarrow 3(\mathbf{r}-\mathbf{p})=5(\mathbf{q}-\mathbf{p})\Rightarrow \mathbf{r}=\frac{1}{3}(5\mathbf{q}-2\mathbf{p})$$

Sub into $\mathbf{p}.\mathbf{r}=11$:

$$\Rightarrow\frac{1}{3}(5\mathbf{q}-2\mathbf{p}).\mathbf{p}=11$$

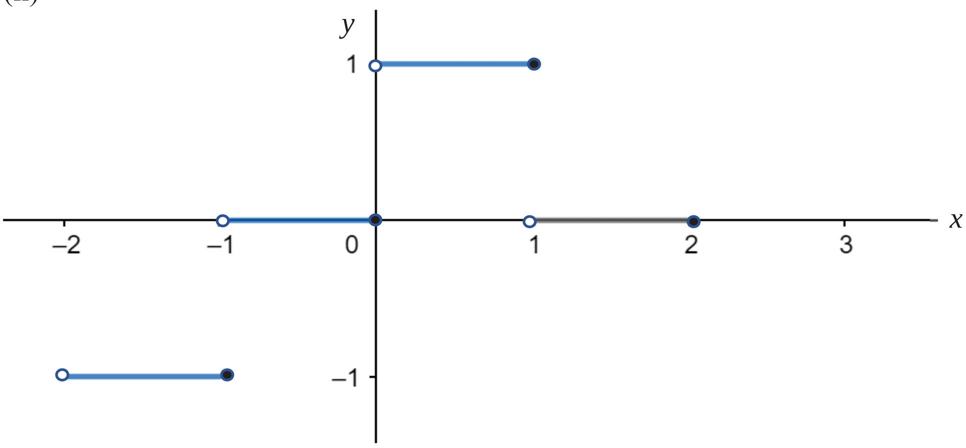
$$\Rightarrow 5\mathbf{q}.\mathbf{p}-2\mathbf{p}.\mathbf{p}=33$$

$$\Rightarrow \mathbf{q}.\mathbf{p}=\frac{91}{5}\Rightarrow|\mathbf{q}.\hat{\mathbf{p}}|=\frac{91}{5\sqrt{29}}\ \text{(or 3.38)}$$

Method 3
Dot p to both sides,

$$3(\mathbf{r}-\mathbf{p}).\mathbf{p}=5(\mathbf{q}-\mathbf{p}).\mathbf{p}$$

	$\Rightarrow 3(\mathbf{r}-\mathbf{p}).\mathbf{p} = 5(\mathbf{q}-\mathbf{p}).\mathbf{p}$ $\Rightarrow 3\mathbf{r}.\mathbf{p} - 3\mathbf{p}.\mathbf{p} = 5\mathbf{q}.\mathbf{p} - 5\mathbf{p}.\mathbf{p}$ $\Rightarrow \mathbf{p}.\mathbf{q} = \frac{1}{5}(3\mathbf{p}.\mathbf{r} + 2\mathbf{p}.\mathbf{p}) = \frac{1}{5}\left(3(11)+2\left(\sqrt{29}\right)^2\right) = \frac{91}{5}$ So $\lvert \mathbf{q}.\hat{\mathbf{p}} \rvert = \frac{91}{5\sqrt{29}}$ (or 3.38)
	(ii) $\overrightarrow{PS} = \mathbf{r}$ so $OPSR$ is a parallelogram spanned by OP and OR. So area of $OPSR = \lvert \mathbf{p} \times \mathbf{r} \rvert$ $= \left\lVert \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \right\rVert = \left\lVert \begin{pmatrix} -6-8 \\ -(-9+4) \\ -6-2 \end{pmatrix} \right\rVert = \left\lVert \begin{pmatrix} -14 \\ 5 \\ -8 \end{pmatrix} \right\rVert = \sqrt{285}$

4	(i) $\mathrm{f}(-1.4) = \lceil -1.4 \rceil = -1$
	(ii) 
	(iii) Method 1 No, because the horizontal line $y=1$ (for example) cuts the graph more than once from $(0,1]$. So f is not 1-1 so f^{-1} does not exist. Method 2 No, because for example, $\mathrm{f}(1.1) = \mathrm{f}(1.2) = 0$. So f is not 1-1 so f^{-1} does not exist.
	(iv) $R_{\mathrm{f}} = \{-1, 0, 1\}$
	(v)

	$\mathrm{g}^2(x)=\dfrac{a\left(\dfrac{ax-3}{x-a}\right)-3}{\dfrac{ax-3}{x-a}-a}$ $=\dfrac{a^2x-3a-3x+3a}{ax-3-ax+a^2}$ $=x$ Then $\mathrm{g}^3(x)=\mathrm{g}^2(\mathrm{g}(x))$ $=\dfrac{ax-3}{x-a}$ Observe that even compositions give x, odd compositions give $\mathrm{g}(x)$. So $\mathrm{g}^{2019}(x)=\dfrac{ax-3}{x-a}\Rightarrow \mathrm{g}^{2019}(5)=\dfrac{5a-3}{5-a}$.
	(vi) $\mathrm{g}(x)=\dfrac{3x-3}{x-3}$ $D_{\mathrm{f}}=(-2,2]\xrightarrow{\mathrm{f}}\{-1,0,1\}\xrightarrow{\mathrm{g}}\left\{\dfrac{3}{2},1,0\right\}$

5	(a)(i) $u_1=\dfrac{4}{M^2},u_2=\dfrac{4}{M^5},u_3=\dfrac{4}{M^8}$
	(a) (ii) $\displaystyle\sum_{r=1}^{n}\frac{4}{M^{3r-1}}=\frac{\frac{4}{M^2}\left(1-\frac{1}{M^{3n}}\right)}{1-\frac{1}{M^3}}$ $=\dfrac{4}{M^2}\left(1-\dfrac{1}{M^{3n}}\right)\times\dfrac{M^3}{M^3-1}$ $=\dfrac{4M}{M^3-1}\left(1-\dfrac{1}{M^{3n}}\right)$ (shown)
	(a) (iii) Method 1 (consider expression) Since as $n\to\infty$, $\dfrac{1}{M^{3n}}\to 0$, $\therefore \dfrac{4M}{M^3-1}\left(1-\dfrac{1}{M^{3n}}\right)\to\dfrac{4M}{M^3-1}$ The sum to infinity is $\dfrac{4M}{M^3-1}$. Method 2 (consider GP) This is a GP with common ratio $=\dfrac{1}{M^3}$

$M > 1 \Rightarrow 0 < \frac{1}{M} < 1 \Rightarrow 0 < \frac{1}{M^3} < 1$, so the series is convergent.

$$S = \frac{\frac{4}{M^2}}{1 - \frac{1}{M^3}} = \frac{4M}{M^3 - 1}$$

(b)(i)

$$\cos\left(\frac{2r+1}{2}\right) - \cos\left(\frac{2r-1}{2}\right)$$

$$= -2\sin\left(\frac{1}{2}\left(\frac{2r+1}{2} + \frac{2r-1}{2}\right)\right)\sin\left(\frac{1}{2}\left(\frac{2r+1}{2} - \frac{2r-1}{2}\right)\right)$$

$$= -2\sin(r)\sin\left(\frac{1}{2}\right) \quad \text{(shown)}$$

(b)(ii)

$$\sum_{r=1}^{n} \sin r = -\frac{1}{2\sin\left(\frac{1}{2}\right)} \sum_{r=1}^{n} \left(\cos\left(\frac{2r+1}{2}\right) - \cos\left(\frac{2r-1}{2}\right)\right)$$

$$= -\frac{1}{2\sin\left(\frac{1}{2}\right)} \left[\begin{array}{l} \cos\left(\frac{3}{2}\right) - \cos\left(\frac{1}{2}\right) \\ + \cos\left(\frac{5}{2}\right) - \cos\left(\frac{3}{2}\right) \\ + \cos\left(\frac{7}{2}\right) - \cos\left(\frac{5}{2}\right) \\ + \qquad \vdots \\ + \cos\left(\frac{2n-1}{2}\right) - \cos\left(\frac{2n-3}{2}\right) \\ + \cos\left(\frac{2n+1}{2}\right) - \cos\left(\frac{2n-1}{2}\right) \end{array}\right]$$

$$= -\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left[\cos\left(n + \frac{1}{2}\right) - \cos\left(\frac{1}{2}\right)\right]$$

$$= \operatorname{cosec}\left(\frac{1}{2}\right) \sin\frac{n+1}{2} \sin\frac{n}{2} \qquad \text{shown}$$

Section B: Probability and Statistics [60 marks]

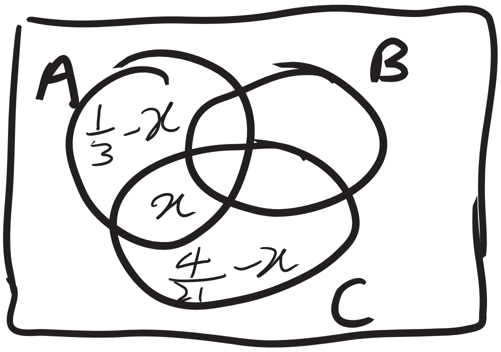
6	(i) $d = 0.5 - c$
	(ii) Let X be the result of one throw of the die. $\mathrm{E}(X) = (1)(0.3) + (2)(c) + (3)(0.5 - c) + (4)(0.2) = 2.6 - c$ $\mathrm{E}(X^2) = (1)(0.3) + (4)(c) + (9)(0.5 - c) + (16)(0.2) = 8 - 5c$ $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ $= (8 - 5c) - (2.6 - c)^2$ $= 8 - 5c - c^2 + 5.2c - 6.76$ $= -c^2 + 0.2c + 1.24$ $= -(c - 0.1)^2 + 1.25$ (completing the square) Thus, the variance is maximum when $c = 0.1$
	(iii) Let Y be the number of throws, out of 10, that land on an even number. $Y \sim \mathrm{B}(10, 0.4)$ Required probability $= \mathrm{P}(Y \geq 7)$ $= 1 - \mathrm{P}(Y \leq 6)$ $= 0.054762...$ $= 0.0548$ (to 3 s.f.)

7	(i) $X_2 \sim \mathrm{N}(\mu, 4)$ $\mathrm{P}(\mu - 1 < X_2 < \mu + 1)$ $= \mathrm{P}\left(\frac{\mu - 1 - \mu}{2} < \frac{X_2 - \mu}{2} < \frac{\mu + 1 - \mu}{2}\right)$ $= \mathrm{P}\left(-\frac{1}{2} < Z < \frac{1}{2}\right)$ where $Z \sim \mathrm{N}(0,1)$ $= 0.38292\ldots$ ≈ 0.383 (3 s.f.)
	(ii) $X_3 - X_4 \sim \mathrm{N}(0, 14)$ $\mathrm{P}(X_3 \geq X_4) = \mathrm{P}(X_3 - X_4 \geq 0) = \frac{1}{2}$ (by symmetry)
	(iii) $\mathrm{Var}(Y_n) = \frac{1}{n^2}\mathrm{Var}(X_1 + X_2 + \ldots + X_n)$ $= \frac{1}{n^2}\left(\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n)\right)$ $= \frac{1}{n^2}\left(2 + 4 + 6 + \ldots + 2n\right)$

	$= \frac{1}{n^2} \times n(n+1)$ (Sum of A.P.) $= 1 + \frac{1}{n}$ Since the X_n's are independent Normal distributions with common mean, $Y_n \sim \mathrm{N}\left(\mu, 1+\frac{1}{n}\right)$ (NB: The variance of Y_n decreases as n increases.) Either $\mathrm{P}(\mu - 1 < Y_n < \mu + 1) > \frac{2}{3}$ $\mathrm{P}\left(\frac{\mu-1-\mu}{\sqrt{1+\frac{1}{n}}} < \frac{Y_n-\mu}{\sqrt{1+\frac{1}{n}}} < \frac{\mu+1-\mu}{\sqrt{1+\frac{1}{n}}}\right) > \frac{2}{3}$ $\mathrm{P}\left(\frac{-1}{\sqrt{1+\frac{1}{n}}} < Z < \frac{1}{\sqrt{1+\frac{1}{n}}}\right) > \frac{2}{3}$ $\frac{1}{\sqrt{1+\frac{1}{n}}} > 0.96742$ Solving this inequality, $n > 14.6017\ldots$ Hence, the smallest possible value of n is 15. Alternatively $Y_n - \mu \sim \mathrm{N}\left(0, 1+\frac{1}{n}\right)$ From GC, <table><tr><td>n</td><td>$\mathrm{P}(-1 < Y_n - \mu < 1)$</td></tr><tr><td>14</td><td>0.6660</td></tr><tr><td>15</td><td>0.6671</td></tr></table>$\therefore$ smallest value of n is 15.

8	(i) $\mathrm{P}(B) = \mathrm{P}(A \cup B) - \mathrm{P}(A \cap B') = \frac{6}{7} - \frac{1}{3} = \frac{11}{21}$
	(ii) $\mathrm{P}(A' \mid B) = \frac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A \cup B) - \mathrm{P}(A)}{\mathrm{P}(B)} = \frac{\frac{6}{7} - \frac{2}{5}}{\frac{11}{21}} = \frac{\frac{16}{35}}{\frac{11}{21}} = \frac{48}{55}$
	(iii) $\mathrm{P}(B' \cap C) = P(C) - P(B \cap C)$ $= \frac{2}{5} - P(B)P(C)$ ($\because B, C$ independent) $= \frac{2}{5} - \frac{11}{21} \times \frac{2}{5}$ $= \frac{4}{21}$

(iv)



Let $\mathrm{P}(A\cap B'\cap C)=x$

Since $\frac{4}{21}-x\geq 0$, $x\leq\frac{4}{21}$

Furthermore, since $\mathrm{P}(A\cup B)=\frac{6}{7}$, $\frac{4}{21}-x\leq\frac{1}{7}$, so $x\geq\frac{1}{21}$

<u>Alternative:</u>

$A\cap B'\cap C\subseteq B'\cap C\Rightarrow P(A\cap B'\cap C)\leq P(B'\cap C)$

So greatest possible value is $\frac{4}{21}$.

Furthermore, $P(A\cap B'\cap C)=P(B'\cap C)-P(A'\cap B'\cap C)$

And $P(A'\cap B'\cap C)\leq P(A'\cap B')=1-P(A\cup B)=\frac{1}{7}$

So $P(A\cap B'\cap C)\geq P(B'\cap C)-\frac{1}{7}=\frac{1}{21}$

So least possible value is $\frac{1}{21}$.

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9 (a)(i)

9 letters with 3 'E' and 2 'L'

No. of ways $=\frac{9!}{3!2!}=30240$

(a)(ii)

L is fixed

_J_W_L_R_Y_: $5!=120$

Case 1: separated by 2 and 2 – 2 ways

Case 2: separated by 2 and 3 / 3 and 2 – 2 ways

Total number of ways: $120\times(2+2)=480$ ways

(a)(iii)

All distinct: ${}^{6}C_{4}\times 4!=360$

EE or LL (but not both): ${}^{2}C_{1}\times{}^{5}C_{2}\times\frac{4!}{2!}=240$

EE and LL: $\frac{4!}{2!2!}=6$

EEE: ${}^{5}C_{1}\times\frac{4!}{3!}=20$

Total: $360+240+6+20=626$

(b)

Mr and Mrs Lee together: $(9-1)! \times 2! = 80640$

Mr and Mrs Lee together and 3 children together: $(7-1)! \times 2! \times 3! = 8640$

Number of ways: $80640 - 8640 = 72000$

OR

Let A be the event that Mr and Mrs Lee are seated together and
B be the event that the 3 children are all seated together.

Then no. of ways $= n(A) - n(A \cap B)$

$= (9-1)! \times 2! - (7-1)! \times 2! \times 3!$

$= 80640 - 8640 = 72000$

10 (i)

$$\bar{x} = 1050 + \frac{58.0}{50} = 1051.16$$

$$s^2 = \frac{1}{n-1}\left(\sum(x-1050)^2 - \frac{\left[\sum(x-1050)\right]^2}{n}\right)$$

$$= \frac{1}{49}\left(2326 - \frac{58.0^2}{50}\right)$$

$$= 46.096 \text{ (5 s.f.)}$$

To test H_0: $\mu = 1053$ against

H_1: $\mu \neq 1053$ at 5% level of significance

Since $n = 50$ is large, by Central Limit Theorem,

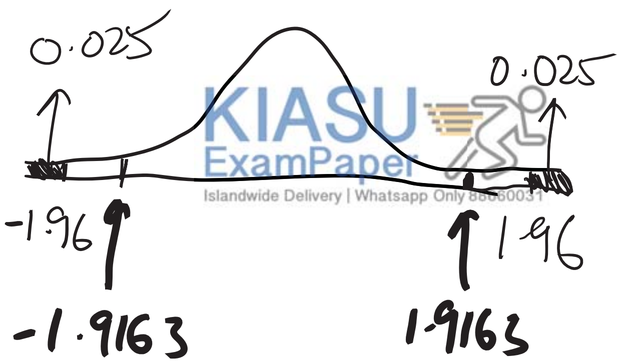
under H_0, $\bar{X} \sim N\left(1053, \frac{46.096}{50}\right)$ approximately

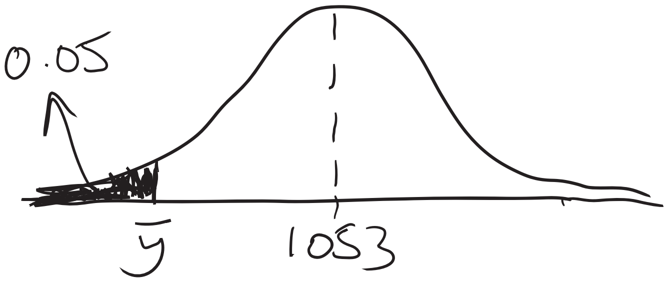
either

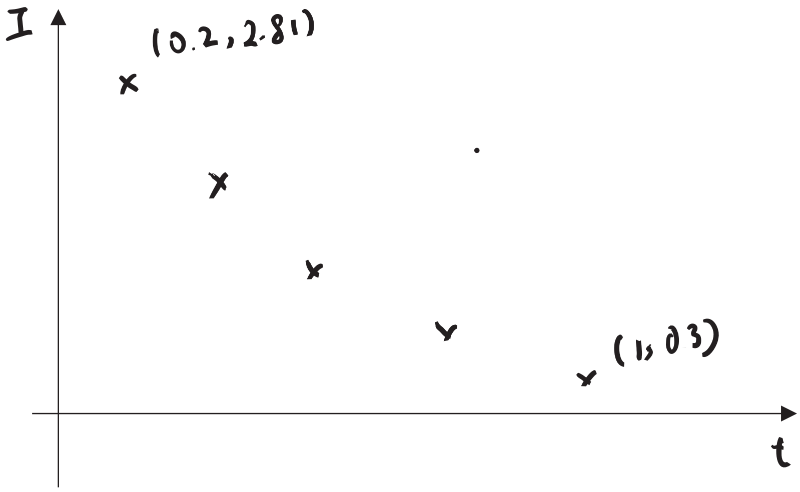
p-value: 0.055322

or

z-value: -1.9163 and critical region: $|z_{0.025}| = 1.96$



	Since p-value > 0.05 (or |z-value| < 1.96), we do not reject H_0 and conclude at 5% level of significance that there is insufficient evidence that the population mean amount of sodium per packet has changed after alterations to the workflow.
	(ii) The probability of wrongly concluding that the mean amount of sodium is not 1053mg, when it is in fact 1053mg, is 0.05.
	(iii) Since p-value is 0.055322, $\alpha \geq 6$
	(iv) If we tested H_1: $\mu < 1053$, Either $p\text{-value} = 0.027661 < 0.05$ or $z\text{-value} = -1.9163 < -1.645$ So we may reject H_0 and conclude at 5% level of significance that the population mean amount of sodium had decreased.
	(v) To test H_0: $\mu = 1053$ against H_1: $\mu < 1053$ at 5% level of significance Since $n = 50$ is large, by Central Limit Theorem, under H_0, $\bar{X} \sim N\left(1053, \frac{6.0^2}{40}\right)$ approximately To reject H_0, p-value < 0.05  $\Rightarrow \bar{y} < 1051.4$ (to 1 d.p.) It is not necessary to assume anything about the population distribution, as sample size (= 40) is large enough, so the Central Limit Theorem says the sample mean amount of sodium approximately follows a normal distribution.

11	(a) There may be a strong negative linear correlation between the amount of red wine intake and the risk of heart disease, but we cannot conclude that amount of red wine intake causes risk of heart disease to decrease, as causality cannot be inferred from correlation.
	(b)(i) The variable t is the independent variable, as we are able to control, or determine, the intervals at which we measure the corresponding radiation.
	(b)(ii)  From the scatter diagram, we can see that the points lie along a curve, rather than a straight line. Hence $I = at + b$ is not a likely model.
	(b)(iii) r between I and t = -0.9565 r between ln I and t = -0.9998
	(b)(iv) $I = ae^{bt} \Rightarrow \ln I = bt + \ln a$ Equation of regression line: $\ln I = -2.7834239t + 1.6007544 \Rightarrow \ln I = -2.78t + 1.60$ $\ln a = 1.600754 \Rightarrow a = 4.96$ (3 s.f.) $b = -2.78$ (3 s.f.)
	(b)(v) $t = 0.7$, $I = 0.706$ (to 3 sig fig) The answer is reliable as r is close to -1, and $t = 0.7$ is within the data range (0.2 to 1.0) and thus the estimate is obtained via interpolation.

End of Paper

1 The number of units, $D(x)$ of a particular product that people are willing to purchase per week in city A at a price \$$x$ is given by the function $D(x)=\dfrac{40320}{\mathrm{g}(x)}$, where $\mathrm{g}(x)$ is a quadratic polynomial in x. The following table shows the number of units people are willing to purchase at different prices.

x	5	8	10
$D(x)$	384	224	168

Find the number of units of the product that people are willing to purchase at a price of \$18. [4]

2 It is given that $p_n = \ln\dfrac{1+x^n}{1+x^{n+1}}$, where $-1<x<1$ and n is a positive integer.

(i) Find $\displaystyle\sum_{n=1}^{N} p_n$, giving your answer in terms of N and x. [3]

(ii) Hence find the sum to infinity of the series in part **(i)** in terms of x. [2]

3 In the triangle ABC, $AB=1$, $AC=2$ and angle $ABC=\left(\dfrac{\pi}{2}-x\right)$ radians. Given that x is sufficiently small for x^3 and higher powers of x to be ignored, show that $BC \approx p+qx+rx^2$, where p, q, r are constants to be determined in exact form. [5]

4 A part of a hyperbola has equation given by $\mathrm{f}(x)=\sqrt{\dfrac{(x+5)^2}{36}-1}$, $x\in D$, where $D\subseteq\mathbb{R}$.

(i) State the largest possible set D. [1]

(ii) State the equations of the asymptotes of $y=\mathrm{f}(x)$. [1]

(iii) Sketch the graph of $y=\mathrm{f}(x)$ for D in part **(i)**, showing clearly all the features of the curve. [2]

(iv) On separate diagrams, sketch the graphs of $y=\dfrac{1}{\mathrm{f}(x)}$ and $y=\mathrm{f}'(x)$, showing clearly all the features of the curves. [4]

5 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OB produced such that $\overrightarrow{OC} = \lambda\overrightarrow{OB}$ where $\lambda > 1$. Point D is such that $OCDA$ is a parallelogram. Point M lies on AD, between A and D, such that $AM : MD = 1 : 2$. Point N lies on OC, between O and C, such that $ON : NC = 4 : 3$.

(i) Find the position vectors of M and N, in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. [2]

(ii) Show that the area of triangle OMD is $k\lambda\left|\mathbf{a}\times\mathbf{b}\right|$, where k is a constant to be determined. [4]

(iii) The vector $\mathbf{p}$ is a unit vector in the direction of $\overrightarrow{OD}$. Give a geometrical meaning of $\left|\mathbf{p}.\mathbf{a}\right|$. [1]

6 A curve C is defined by the parametric equations

$$x = 25\sin^2 t, \quad y = 2\cos t$$

where $0 \le t \le \pi$.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(ii) The tangent to C at the point where $t = \dfrac{\pi}{4}$ cuts the x-axis at P and the y-axis at Q, find the exact area of the triangle OPQ, where O is the origin. [4]

(iii) State the equation of the normal to C where the normal is parallel to the x-axis. [1]

7 **(a)** **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(2\mathrm{e}^{\cos\frac{x}{2}}\right)$. [1]

(ii) Hence find $\displaystyle\int \frac{1}{2}\sin x\,\mathrm{e}^{\cos\frac{x}{2}}\,\mathrm{d}x$. [4]

(b) Using the substitution $u = 1 - \mathrm{e}^x$, find $\displaystyle\int \frac{1}{1-\mathrm{e}^x}\,\mathrm{d}x$. [4]

8 The function f is defined by $\mathrm{f}: x \mapsto \sqrt{a^2 - \frac{(x-a)^2}{4}}$ for $x \in \mathbb{R}$, $-a \le x \le 3a$, where a is a positive constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$, giving the coordinates of any stationary points and the points where the graph meets the axes. [2]

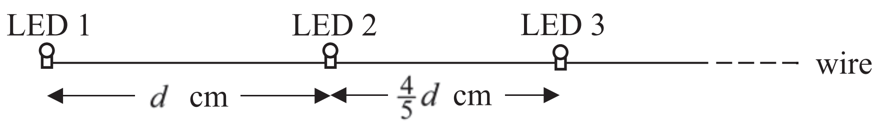
(ii) If the domain of f is further restricted to $-a \le x \le k$, state the greatest value of k for which the function f^{-1} exists. [1]

(iii) Using the restricted domain found in part **(ii)**, find f^{-1} in similar form. [3]

The function g is defined by $\mathrm{g}(x) = \mathrm{f}\left(\frac{3}{2}x\right)$ for $x \in \mathbb{R}$, $-\frac{2}{3}a \le x \le 2a$.

(iv) Explain why the composite function gf exists and find the range of gf. [3]

9 A company produces festive decorative Light Emitting Diode (LED) string lights, where micro LEDs are placed at intervals along a thin wire. In a particular design, the first LED (LED 1) is placed on one end of a wire with the second LED (LED 2) placed at a distance of d cm from LED 1, and each subsequent LED is placed at a distance $\frac{4}{5}$ times the preceding distance as shown.



(i) If the distance between LED 8 and LED 9 is 56.2 cm, find the value of d correct to 1 decimal place. [2]

(ii) Find the theoretical maximum length of the wire, giving your answer in centimetres correct to 1 decimal place. [2]

The LEDs consisting of three colours red, orange and yellow, are arranged in that order in a repeated manner, that is, LED 1 is red, LED 2 is orange, LED 3 is yellow, LED 4 is red, LED 5 is orange, LED 6 is yellow, and so on.

(iii) Find the colour of the LED nearest to a point on the wire 12.9 m from LED 1. [3]

(iv) If the minimum distance between any two consecutive LEDs is 1 cm so that they can be mounted on the wire, find the colour of the last LED. [3]

10 **(a)** Solve the simultaneous equations $v + \mathrm{i}u = 2$ and $av - 2u = 3\mathrm{i}$, where a is a real constant. Simplify your answers to cartesian form $x + \mathrm{i}y$, where x and y are in terms of a. [4]

(b) It is given that $(x + k)$ is a factor of the equation,

$$bx^3 + (12b + \mathrm{i})x^2 + (b + 12\mathrm{i})x + 12b = 0,$$

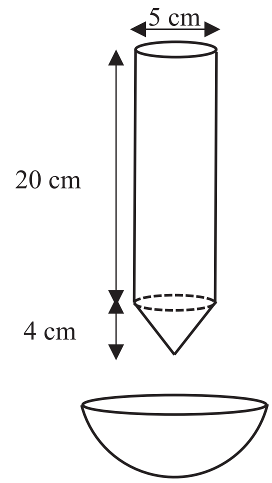
where k and b are non-zero real constants.

(i) Find the value of k. [2]

(ii) Show that the roots of the equation $bx^2 + \mathrm{i}x + b = 0$ are purely imaginary. [2]

(iii) Hence express $\mathrm{f}(x) = bx^3 + (12b + \mathrm{i})x^2 + (b + 12\mathrm{i})x + 12b$ as a product of three linear factors, leaving your answers in terms of b. [2]

11 A container is made up of a cylinder and an inverted right circular cone as shown in the diagram below. The height and the diameter of the cylinder are 20 cm and 5 cm respectively. The height of the cone is 4 cm. An external device ensures liquid flows out through a small hole at the vertex of the cone into a bowl below at a constant rate of 18 cm^3 per minute. The depth of the liquid and the radius of the liquid surface area in the container at the time t minutes are x cm and r cm respectively. The container is full of liquid initially.

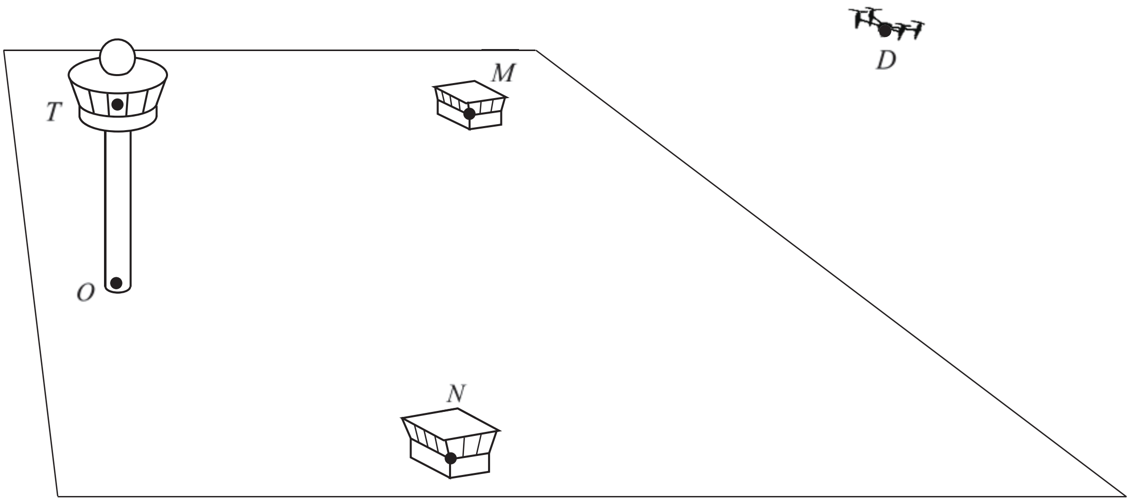


(a) When $x > 4$, find the rate of change of the depth of the liquid in the container. [4]

(b) Find the rate of change of r when $x = 2$. [5]

(c) The bowl as shown in the diagram in part **(a)** is generated by rotating part of the curve $\frac{x^2}{225} + \frac{y^2}{100} = 1$ which is below the x-axis through π radians about the y-axis. Assuming the bowl has negligible thickness, find the volume of the empty space in the bowl when the liquid has completely flowed from the container into the bowl, giving your answer correct to 3 decimal places. [3]

12 At an airport, an air traffic control room T is located in a vertical air traffic control tower, 70 m above ground level. Let $O(0,0,0)$ be the foot of the air traffic control tower and all points (x, y, z) are defined relative to O where the units are in kilometres. Two observation posts at the points $M(0.8, 0.6, 0)$ and $N(0.4, -0.9, 0)$ are located within the perimeters of the airport as shown.



An air traffic controller on duty at T spots an errant drone in the vicinity of the airport. The two observation posts at M and N are alerted immediately. A laser rangefinder at M directs a laser beam in the direction $\begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$ at the errant drone to determine D, the position of the errant drone. The position D is confirmed using another laser beam from N, which passes through the point $(0.8, 0.75, 0.3)$, directed at the errant drone.

(i) Show that D has coordinates $(0.56, -0.24, 0.12)$. [4]

A Drone Catcher, an anti-drone drone which uses a net to trap and capture errant drones, is deployed instantly from O and flies in a straight line directly to intercept the errant drone.

(ii) Find the acute angle between the flight path of the Drone Catcher and the horizontal ground. [2]

At the same time, a Jammer Gun, which emits a signal to jam the control signals of the errant drone, is fired at the errant drone. The Jammer Gun is located at a point G on the plane p containing the points T, M and N.

(iii) Show that the equation of p is $\mathbf{r} \cdot \begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix} = -6.72$. [3]

It is also known that the Jammer Gun is at the foot of the perpendicular from the errant drone to plane p.

(iv) Find the coordinates of G. [3]

(v) Hence, or otherwise, find the distance GD in metres. [2]

Section A: Pure Mathematics [40 marks]

1 On the same axes, sketch the graphs of $y = 2(x-a)^2$ and $y = 3a|x-a|$, where a is a positive constant, showing clearly all axial intercepts. [2]

(i) Solve the inequality $2(x-a)^2 \geq 3a|x-a|$. [4]

(ii) Hence solve $2\left(x-\frac{a}{2}\right)^2 \geq 3a\left|x-\frac{a}{2}\right|$. [2]

2 It is given that $y = \frac{e^{\sin x}}{\sqrt{1+2x}}$.

(i) Show that $\frac{1}{y}\frac{dy}{dx} + \frac{1}{1+2x} = \cos x$. [2]

(ii) By further differentiation of the result in part **(i)**, find the Maclaurin series for y in ascending powers of x, up to and including the term in x^3. [5]

(iii) Use your result from part **(ii)** to approximate the value of $\int_0^1 \frac{e^{\sin x}}{\sqrt{1+2x}} dx$. Explain why this approximation obtained is not good. [2]

(iv) Deduce the Maclaurin series for $\frac{1}{e^{\sin x}\sqrt{1-2x}}$ in ascending powers of x, up to and including the term in x^3. [1]

3 The complex numbers p and q are given by $\frac{a}{1+\sqrt{3}i}$ and $-\frac{a}{2}i$ respectively, where a is a positive real constant.

(i) Find the modulus and argument of p . [2]

(ii) Illustrate on an Argand diagram, the points P, Q and R representing the complex numbers p, q and $p+q$ respectively. State the shape of $OPRQ$.
Hence, find the argument of $p+q$ in terms of π and the modulus of $p+q$ in exact trigonometrical form. [6]

(iii) Find the smallest positive integer n such that $(p+q)^n$ is purely imaginary. [2]

4 **(a)** When studying a colony of bugs, a scientist found that the birth rate of the bugs is inversely proportional to its population and the death rate is proportional to its population. The population of the bugs (in thousands) at time t days after they were first observed is denoted by P. It was found that when the population is 2000, it remains constant.

(i) Assuming that P and t are continuous variables, show that $\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(\frac{4}{P} - P\right)$, where k is a constant. [3]

(ii) Given that the initial population of the bugs was 4000, and that the population was decreasing at the rate of 3000 per day at that instant, find P in terms of t. [4]

(iii) Sketch the graph of P against t, giving the equation of any asymptote(s). State what happens to the population of the bugs in the long run. [2]

(b) Another population of bugs, N (in thousands) in time t days can be modelled by the differential equation $\frac{\mathrm{d}N}{\mathrm{d}t} = 4 + \frac{N}{t}$ for $t \geq 1$. Using the substitution $u = \frac{N}{t}$, solve this equation, given that the population was 1000 when $t = 1$. [3]

Section B: Statistics [60 marks]

5 The daily rainfall in a town follows a normal distribution with mean μ mm and standard deviation σ mm. Assume that the rainfall each day is independent of the rainfall on other days. It is given that there is a 10% chance that the rainfall on a randomly chosen day exceeds 9.8 mm, and there is a 10% chance that the mean daily rainfall in a randomly chosen 7-day week exceeds 8.2 mm.

(i) Show that $\sigma = 2.01$, correct to 2 decimal places. [4]

(ii) Find the maximum value of k such that there is a chance of at least 10% that the mean daily rainfall in a randomly chosen 30-day month exceeds k mm. Give your answer correct to 1 decimal place. [2]

6 Miss Tan carried out an investigation on whether there is a correlation between the amount of time spent on social media and exam scores. The average amount of time spent per month on social media, x hours, and the final exam score, y marks, of 6 randomly selected students from HCI were recorded. The data is shown below.

x	80	84	70	74	58	48
y	44	40	49	45	58	82

(i) Draw a scatter diagram to illustrate the data. [2]

(ii) It is found that the inclusion of a 7th point (x_7, y_7) will not affect the product moment correlation coefficient for the data. Find a possible point (x_7, y_7). [1]

Omit the 7th point (x_7, y_7) for the rest of this question.

(iii) State, with reason, which of the following equations, where a and b are constants, provides the most appropriate model for the relationship between x and y.

(A) $y = a + bx^2$,

(B) $e^y = ax^b$,

(C) $y = a + b\sqrt{x}$. [3]

(iv) Using the model chosen in part **(iii)**, estimate the score of a student who spent an average of 60 hours per month on social media, giving your answer correct to the nearest whole number. [2]

(v) Sam spends an average of 4 hours a day on social media. Assuming a 30-day month, suggest whether it is still reasonable to use the model in part **(iii)** to estimate his score. [1]

7 A cafe sells sandwiches in 2 sizes, "footlong" and "6-inch". The lengths in inches of "footlong" loaves have the distribution $\mathrm{N}(12.2, 0.04)$ and the lengths in inches of "6-inch" loaves have the distribution $\mathrm{N}(6.1, 0.02)$.

(i) Is a randomly chosen "footlong" loaf more likely to be less than 12 inches in length or a randomly chosen "6-inch" loaf more likely to be less than 6 inches in length? [2]

(ii) Find the probability that two randomly chosen "6-inch" loaves have total length more than one randomly chosen "footlong" loaf. [2]

Sue buys a "6-inch" sandwich 3 times a week.

(iii) Find the probability that Sue gets at most one sandwich that is less than 6 inches in length in a randomly chosen week. [2]

(iv) Given that Sue gets more than four sandwiches that are less than 6 inches in length in a randomly chosen 4-week period, find the probability that she gets exactly one such sandwich in the first week. [3]

8 The individual letters of the word PARALLEL are printed on identical cards and arranged in a straight line.

(a) Find the number of arrangements such that

(i) there are no restrictions, [1]

(ii) no L is next to any other L, [2]

(iii) the arrangements start and end with a consonant and all the vowels are together. [3]

(b) The cards are now placed in a bag and Tom draws the cards randomly from the bag one at a time.

(i) 4 cards are drawn without replacement. Find the probability that there is at least one vowel drawn. [2]

(ii) Tom decides to record the letter of the card drawn, on a piece of paper. If the letter on the card drawn is a vowel, Tom will put the drawn card back into the bag and continue with the next draw.

If the letter on the card drawn is a consonant, Tom will remove the card from subsequent draws. Find the probability that Tom records more consonants than vowels at the end of 3 draws. [3]

9 A company purchased a machine to pack shower gel into its bottles. The expected mean volume of shower gel in a bottle is 950 ml.

(a) The floor supervisor believes that the machine is packing less amount of shower gel than expected. A random sample of 80 bottles is taken and the data is as follows:

Volume of shower gel in a bottle (correct to nearest ml)	948	949	950	951	952	953	955
Number of bottles	9	22	36	6	4	1	2

(i) Find unbiased estimates of the population mean and variance, giving your answers correct to 2 decimal places. [2]

(ii) Write down the appropriate hypotheses to test the floor supervisor's belief. You should define any symbols used. [2]

(iii) Using the given data, find the p-value of the test. State what is meant by this p-value in the context of this question. [2]

(iv) It was concluded at $\alpha\%$ level of significance that the machine is indeed packing less amount of shower gel than expected. State the set of values of α. [1]

(b) Due to a change in marketing policy, the machine is being recalibrated to pack smaller bottles of shower gel with mean volume of 250 ml. The volume of a recalibrated bottle of shower gel is denoted by Y ml. A random sample of 50 bottles of y ml each is taken and the data obtained is summarised by:

$$\sum(y-250)=-25, \qquad \sum(y-250)^2=k.$$

Another test was conducted at the 1% significance level. The test concluded that the machine had been calibrated incorrectly. Find the range of values of k, correct to 1 decimal place. [4]

(c) Explain why there is no need for the floor supervisor to know anything about the population distribution of the volume of shower gel in a bottle for both parts **(a)** and **(b)**. [1]

10 In a game with a 4-sided fair die numbered 1 to 4 on each face, the score for a throw is the number on the bottom face of the die. A player gets to choose either option A or option B.

Option A: The player rolls the die once. The score x is the amount of money $\$x$ that the player wins.

Option B: The player rolls the die twice. The first score is x and the second score is y. If $y > x$, the player wins $\$2xy$, but if $y < x$, the player loses $\$(x - y)$. Otherwise, he neither wins nor loses any money.

(i) Find the expected amounts won by a player in one game when playing option A and when playing option B. Show that option B is a better option. [5]

(ii) Suggest why a risk averse player would still choose option A. [1]

(iii) Show that the variance of the amount won by a player in one game when playing option A is 1.25. [2]

In a competition, Abel and Benson each play the game 50 times. Abel chooses option A and Benson chooses option B.

It is given that the variance of the amount won by a player in one game when playing option B is $\frac{887}{16}$.

(iv) Find the distributions of the total amounts won by Abel and Benson respectively in the competition. [2]

(v) Show that the probability of the total amount won by Abel exceeding the total amount won by Benson in the competition is approximately 0.120. [3]

ANNEX

H2 MA 2019 JC2 Prelim (Paper 1 and Paper 2)

***Filename**: Change* SCHOOL *to your school name, e.g. NYJC*

Paper 1

Select topic from the dropdown list. If the question consists of multiple topics, choose 1 topic.

QN	TOPIC (H2) Paper 1	ANSWERS (Exclude graphs and text answers)
1	Equations & Inequalities	72 units
2	Sigma Notation & MOD	(i) $\ln\dfrac{1+x}{1+x^{N+1}}$ (ii) $\ln(1+x)$
3	Maclaurin & Binomial Series	$p=\sqrt{3}, q=1, r=\dfrac{1}{2\sqrt{3}}$
4	Graphs & Transformations	(i) $D=(-\infty,-11]\cup[1,\infty)$ (ii) $y=\pm\dfrac{x+5}{6}$
5	Vectors	(i) $\overrightarrow{OM}=\dfrac{1}{3}(3\mathbf{a}+\lambda\mathbf{b}), \overrightarrow{ON}=\dfrac{4}{7}\lambda\mathbf{b}$ (ii) $k=\dfrac{1}{3}$
6	Differentiation & Applications	(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-1}{25\cos t}$ (ii) $\dfrac{225}{4\sqrt{2}}$ (iii) $y=0$
7	Integration & Applications	(a)(i) $-\sin\dfrac{x}{2}\left(\mathrm{e}^{\cos\frac{x}{2}}\right)$ (ii) $-2\cos\dfrac{x}{2}\left(\mathrm{e}^{\cos\frac{x}{2}}\right)+2\mathrm{e}^{\cos\frac{x}{2}}+C$ (b) $-\ln\left\|1-\mathrm{e}^{x}\right\|+x+C$
8	Functions	(ii) greatest value of k is a. (iii) $\mathrm{f}^{-1}: x\mapsto a-2\sqrt{a^2-x^2},\ x\in\mathbb{R},\ 0\le x\le a$ (iv) $\mathrm{R}_{\mathrm{gf}}=\left[\dfrac{\sqrt{3}}{2}a, a\right]$
9	APGP	(i) $d=268.0$ (ii) 1339.9 cm (iii) red

		(iv) yellow
10	Complex Numbers	(a) $u=\frac{a}{4+a^2}-\frac{6+2a^2}{4+a^2}\text{i}, v=\frac{2}{4+a^2}-\frac{a}{4+a^2}\text{i}$ (b)(i) $k=12$ (iii) $b(x+12)(x+\frac{1+\sqrt{1+4b^2}}{2b}\text{i})(x+\frac{1-\sqrt{1+4b^2}}{2b}\text{i})$
11	Differentiation & Applications	(a) -0.916 cm/min (b) -2.29 cm/min (c) 4293.510 cm^3
12	Vectors	(ii) 11.1° (iv) $(0.547,-0.237,0.00325)$ (v) 117 m
13	H2 Prelim P1 Q13 Topic	
14	H2 Prelim P1 Q14 Topic	

Paper 2

Select topic from the dropdown list. If the question consists of multiple topics, choose 1 topic.

QN	TOPIC (H2) Paper 2	ANSWERS (Exclude graphs and text answers)
1	Equations & Inequalities	(i) $x\le -\frac{a}{2}$ or $x=a$ or $x\ge\frac{5a}{2}$ (ii) $x\le -a$ or $x=\frac{a}{2}$ or $x\ge 2a$
2	Maclaurin & Binomial Series	(ii) $y=1+x^2-\frac{3}{2}x^3+\ldots$ (iii) 0.985 (iv) $\frac{1}{e^{\sin x}\sqrt{1-2x}}\approx 1+x^2+\frac{3}{2}x^3$
3	Complex Numbers	(i) $\frac{a}{2}$; $-\frac{\pi}{3}$ (ii) $-\frac{5\pi}{12}$; $a\cos\frac{\pi}{12}$ (iii) 6
4	Differential Equations	(a) (ii) $P=2\sqrt{1+3e^{-2t}}$ (iii) The population of the bugs will decrease and approach 2000 in the long run. (b) $N=4t\ln t+t$
5	Normal Distribution	(ii) 7.6
6	Correlation & Regression	(ii) $(69,53)$ (iii) Since $\lvert r\rvert=0.96785$ for Model B is nearest to 1, Model B is the most accurate model (iv) 62 marks (v) unreasonable as 120 is not within data range

		where $48 \le x \le 84$
7	Normal Distribution	(i) a "6-inch" loaf more likely to be less than 6 inches (ii) $\frac{1}{2}$ (iii) 0.855 (iv) 0.446
8	PnC & Probability	(a) (i) 3360 (ii) 1200 (iii) 240 (b) (i) $\frac{13}{14}$ (ii) 0.644
9	Hypothesis Testing	(a) (i) 949.84 ; 1,73 (ii) H_0: $\mu = 950$ H_1: $\mu < 950$ (iii) 0.135 (iv) $\{\alpha \in \mathbb{R} : 13.5 \le \alpha \le 100\}$ (b) $12.5 \le k \le 104.8$ (c) There is no need for the floor supervisor to assume the volume of shower gel follow a normal distribution as the sample sizes in both part (a) and (b) are large, the sample mean volume of shower gel can be approximated to follow a normal distribution by Central Limit Theorem.
10	DRV	(i) \$2.50; \$3.75 (ii) Option A is a "sure win" option where the player would definitely gain a positive amount in all cases, whereas option B has a risk of losing money in some cases. (iv) $A \sim \mathrm{N}\left(125, \frac{125}{2}\right)$; $B \sim \mathrm{N}\left(\frac{375}{2}, \frac{22175}{8}\right)$
11	H2 Prelim P2 Q11 Topic	
12	H2 Prelim P2 Q12 Topic	
13	H2 Prelim P2 Q13 Topic	
14	H2 Prelim P2 Q14 Topic	

Qn	Solutions	C
1	$D(x)=\dfrac{40320}{g(x)}$ $\Rightarrow g(x)=\dfrac{40320}{D(x)}$ $ax^2+bx+c=\dfrac{40320}{D(x)}$ Given $5^2a+5b+c=\dfrac{40320}{384}=105$ -- (1) $8^2a+8b+c=\dfrac{40320}{224}=180$ -- (2) $10^2a+10b+c=\dfrac{40320}{168}=240$ -- (3) Using GC, $a=1,\ b=12,\ c=20$ When $x=18$, $D(18)=\dfrac{40320}{18^2+12(18)+20}=72$ units	G w A g(5² of 5² A de in pc
2i	<u>Method 1</u>: (method of differences) $\displaystyle\sum_{n=1}^{N}p_n=\sum_{n=1}^{N}\ln\frac{1+x^n}{1+x^{n+1}}$ $\displaystyle=\sum_{n=1}^{N}\left(\ln(1+x^n)-\ln(1+x^{n+1})\right)$ $=\ln(1+x)-\cancel{\ln(1+x^2)}$ $+\cancel{\ln(1+x^2)}-\cancel{\ln(1+x^3)}$ $+\cancel{\ln(1+x^3)}-\cancel{\ln(1+x^4)}$ $\vdots$ $+\cancel{\ln(1+x^N)}-\ln(1+x^{N+1})$ $=\ln(1+x)-\ln(1+x^{N+1})$ $=\ln\dfrac{1+x}{1+x^{N+1}}$	Th dc dc su di cl th Sc en lin co se A fin ln of

Method 2: (using property of logarithm)

$$\sum_{n=1}^{N} p_n = \sum_{n=1}^{N} \ln \frac{1+x^n}{1+x^{n+1}}$$

$$= \ln \frac{1+x}{1+x^2} + \ln \frac{1+x^2}{1+x^3} + \ln \frac{1+x^3}{1+x^4} + \ldots + \ln \frac{1+x^N}{1+x^{N+1}}$$

$$= \ln \frac{1+x}{\cancel{1+x^2}} \cdot \frac{\cancel{1+x^2}}{\cancel{1+x^3}} \cdot \frac{\cancel{1+x^3}}{\cancel{1+x^4}} \cdot \ldots \cdot \frac{\cancel{1+x^N}}{1+x^{N+1}}$$

$$= \ln \frac{1+x}{1+x^{N+1}}$$

2ii

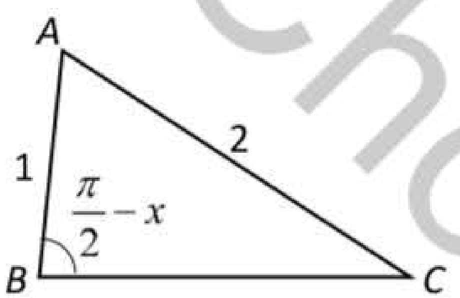
Since $-1 < x < 1$, as $N \to \infty$, $x^{N+1} \to 0$.

$$\therefore \sum_{n=1}^{\infty} p_n = \ln \frac{1+x}{1+0}$$

$$= \ln(1+x)$$

So
as

3



By Cosine Rule,

$$\cos\left(\frac{\pi}{2} - x\right) \quad 2^2 = 1^2 + (BC)^2 - 2(BC)\cos\left(\frac{\pi}{2} - x\right)$$

$$4 = 1 + (BC)^2 - 2(BC)\sin x$$

$$(BC)^2 - 2(BC)\sin x - 3 = 0$$

$$BC = \frac{2\sin x \pm \sqrt{4\sin^2 x + 12}}{2}$$

$$= \sin x \pm \sqrt{\sin^2 x + 3}$$

$$\approx x \pm \sqrt{x^2 + 3}$$

$$= x + \sqrt{x^2 + 3} \text{ , since } BC > 0$$

$$= x + \sqrt{3}\left(1 + \frac{x^2}{3}\right)^{1/2}$$

$$= x + \sqrt{3}\left[1 + \frac{1}{2}\frac{x^2}{3} + \ldots\right]$$

$$\approx \sqrt{3} + x + \frac{x^2}{2\sqrt{3}}$$

$$p = \sqrt{3}, q = 1, r = \frac{1}{2\sqrt{3}}$$

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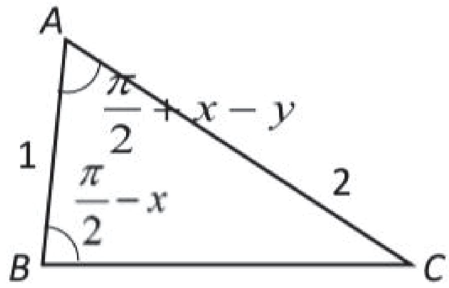
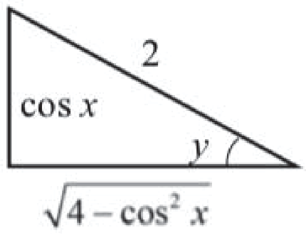
Alternative Solution



By sine rule,

$$\frac{BC}{\sin\left(\frac{\pi}{2}+x-y\right)}=\frac{2}{\sin\left(\frac{\pi}{2}-x\right)}=\frac{1}{\sin y}$$



$$\Rightarrow \frac{BC}{\cos(y-x)}=\frac{2}{\cos x}=\frac{1}{\sin y}$$

$$\Rightarrow BC=\frac{2\cos(y-x)}{\cos x}$$

$$\Rightarrow BC=\frac{2(\cos y\cos x+\sin y\sin x)}{\cos x}$$

$$\Rightarrow BC=\frac{2(\cos y\cos x+\sin y\sin x)}{\cos x}$$

$$\Rightarrow BC=\frac{2\left(\frac{\sqrt{4-\cos^2 x}}{2}\cos x+\frac{\cos x}{2}\sin x\right)}{\cos x}$$

$$\Rightarrow BC=\frac{\sqrt{4-\cos^2 x}\,\cos x+\cos x\sin x}{\cos x}$$

$$\Rightarrow BC=\sqrt{4-\cos^2 x}+\sin x$$

$$\Rightarrow BC=\sqrt{3+\sin^2 x}+\sin x$$

4i

Sketch the graph using GC, remember to set window appropriately.

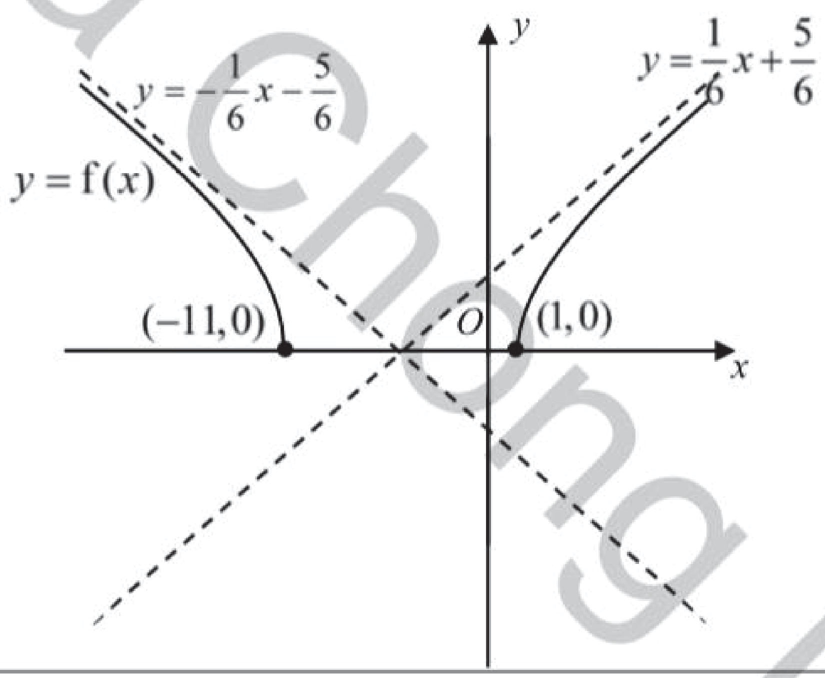
OR

Notice that it is the upper half of a hyperbola.

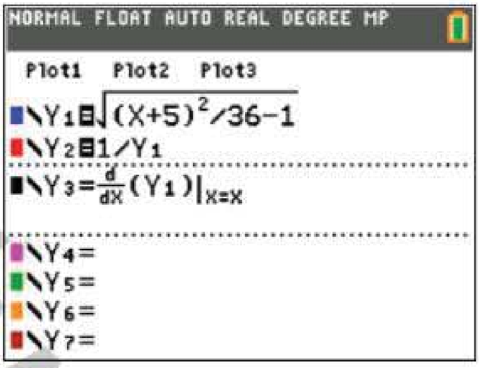
$$y=\sqrt{\frac{(x+5)^2}{36}-1}$$

$$\frac{(x+5)^2}{6^2}-y^2=1$$

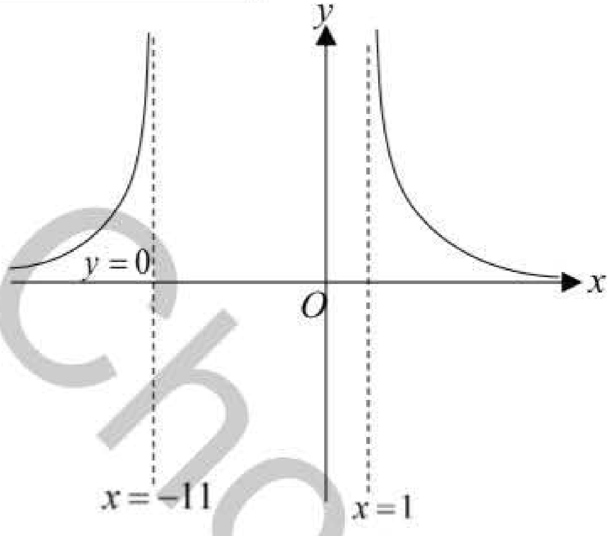
We have $D=(-\infty,-1]\cup[1,\infty)$.

ii	Let $y=\sqrt{\frac{(x+5)^2}{36}-1}$ $\frac{(x+5)^2}{6^2}-y^2=1$ To find asymptotes: $y^2=\frac{(x+5)^2}{6^2}$ $y=\pm\frac{x+5}{6}$
iii	

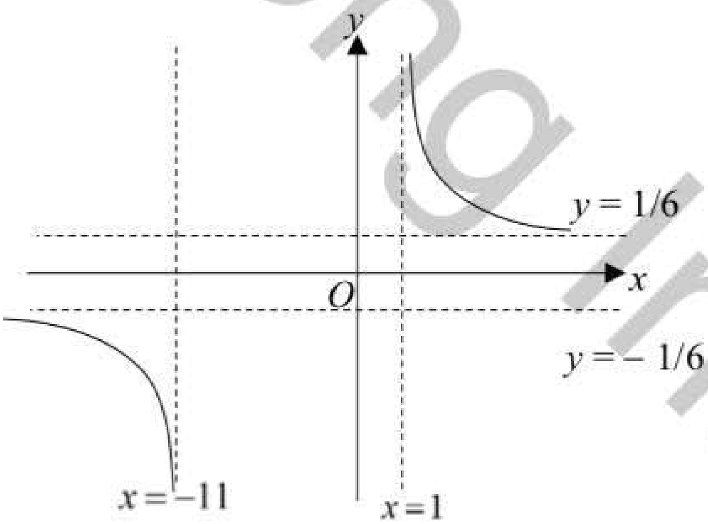
iv Using GC to help if students is not sure



$y=\dfrac{1}{\mathrm{f}(x)}$



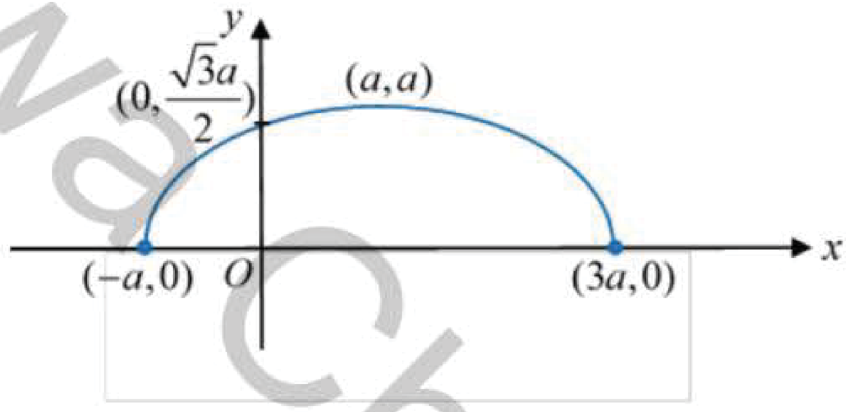
$y=\mathrm{f}'(x)$

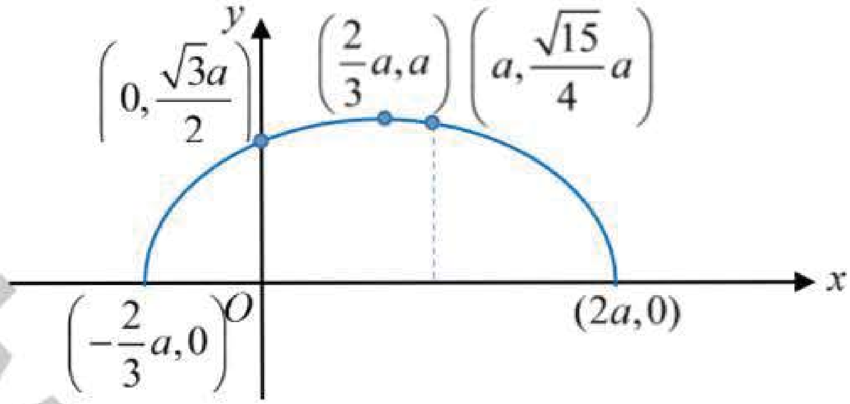


5i	$\overrightarrow{OC} = \lambda\mathbf{b}$ $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OC} = a + \lambda\mathbf{b}$ $\overrightarrow{ON} = \frac{4}{7}\lambda\mathbf{b}$ $\overrightarrow{OM} = \frac{\overrightarrow{OD} + 2\overrightarrow{OA}}{3}$ $= \frac{\mathbf{a} + \lambda\mathbf{b} + 2\mathbf{a}}{3}$ $= \frac{1}{3}(3\mathbf{a} + \lambda\mathbf{b})$	
ii	Area of triangle $OMD = \frac{1}{2}\left\|\overrightarrow{OM} \times \overrightarrow{OD}\right\|$ $= \frac{1}{2}\left\|\frac{1}{3}(\lambda\mathbf{b} + 3\mathbf{a}) \times (\lambda\mathbf{b} + \mathbf{a})\right\|$ $= \frac{1}{6}\left\|\lambda^2\mathbf{b} \times \mathbf{b} + \lambda\mathbf{b} \times \mathbf{a} + 3\lambda\mathbf{a} \times \mathbf{b} + 3\mathbf{a} \times \mathbf{a}\right\|$ $= \frac{1}{6}\left\|-\lambda\mathbf{a} \times \mathbf{b} + 3\lambda\mathbf{a} \times \mathbf{b}\right\| \quad (\because \mathbf{b} \times \mathbf{b} = \mathbf{0} = \mathbf{a} \times \mathbf{a})$ $= \frac{1}{3}\lambda\left\|\mathbf{a} \times \mathbf{b}\right\|$	
iii	$\left\|\mathbf{p}.\mathbf{a}\right\|$ is the length of projection of $\overrightarrow{OA}$ on $\overrightarrow{OD}$	

6i	$\frac{\mathrm{d}x}{\mathrm{d}t} = 50\sin t\cos t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = -2\sin t$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{25\cos t}$	
ii	When $t = \frac{\pi}{4}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sqrt{2}}{25}$, $x = \frac{25}{2}$, $y = \sqrt{2}$ $y = -\frac{\sqrt{2}}{25}x + c$ $c = \sqrt{2} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$ Equation of tangent: $y = -\frac{\sqrt{2}}{25}x + \frac{3}{\sqrt{2}}$ When $x = 0$, $y = \frac{3}{\sqrt{2}}$ When $y = 0$, $x = \frac{75}{2}$ Area of triangle $OPQ = \frac{1}{2}\left(\frac{75}{2}\right)\left(\frac{3}{\sqrt{2}}\right) = \frac{75}{4}\left(\frac{3}{\sqrt{2}}\right) = \frac{225}{4\sqrt{2}}$	

iii	$y=0$	
7ai	Let $y=2e^{\cos\frac{x}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}\sin\frac{x}{2}\left(2e^{\cos\frac{x}{2}}\right)$ $=-\sin\frac{x}{2}\left(e^{\cos\frac{x}{2}}\right)$	
aii	$\int\frac{1}{2}\sin x\left(e^{\cos\frac{x}{2}}\right)dx$ $=\int\frac{1}{2}\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)\left(e^{\cos\frac{x}{2}}\right)dx$ $=-\int\left(\cos\frac{x}{2}\right)\left(-\sin\frac{x}{2}e^{\cos\frac{x}{2}}\right)dx$ $=\left(-\cos\frac{x}{2}\right)\left(2e^{\cos\frac{x}{2}}\right)-\int\left(\frac{1}{2}\sin\frac{x}{2}\right)\left(2e^{\cos\frac{x}{2}}\right)dx$ $=-2\cos\frac{x}{2}\left(e^{\cos\frac{x}{2}}\right)+2e^{\cos\frac{x}{2}}+C$	
b	$\int\frac{1}{1-e^x}dx$ Let $u=1-e^x\Rightarrow\frac{du}{dx}=-e^x=u-1\Rightarrow\frac{dx}{du}=\frac{1}{u-1}$ $=\int\frac{1}{u(u-1)}du$ $=\int-\frac{1}{u}+\frac{1}{u-1}du$ $=-\ln\lvert u\rvert+\ln\lvert u-1\rvert+C$	

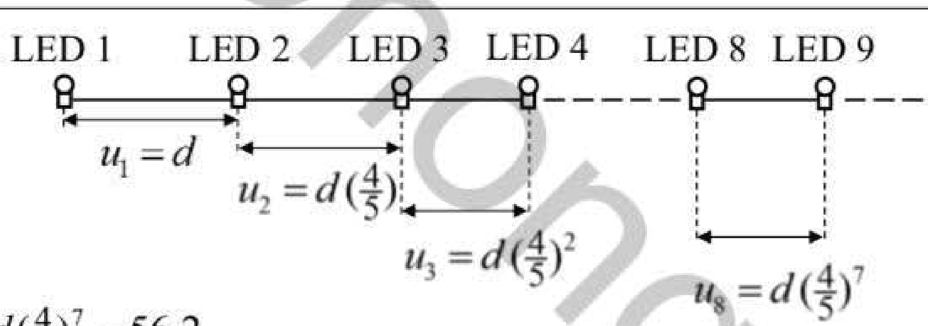
	$= -\ln\lvert 1-\mathrm{e}^x\rvert + \ln\lvert -\mathrm{e}^x\rvert + C$ $= -\ln\lvert 1-\mathrm{e}^x\rvert + x + C$
8i	
ii	For f^{-1} to exist, f must be one-one. i.e. every horizontal line $y=h$, $h\in\mathbb{R}$ can only cut the graph of $y=\mathrm{f}(x)$ at most once. Hence, the greatest value of k is a.
iii	Let $y=\sqrt{a^2-\frac{(x-a)^2}{4}}$ $\Rightarrow y^2=a^2-\frac{(x-a)^2}{4}$ $\Rightarrow (x-a)^2=4(a^2-y^2)$ $\Rightarrow x=a\pm 2\sqrt{a^2-y^2}$ $\Rightarrow x=a-2\sqrt{a^2-y^2}\ (\because -a\le x\le a)$ $\mathrm{f}^{-1}: x\mapsto a-2\sqrt{a^2-x^2},\ x\in\mathbb{R},\ 0\le x\le a$
iv	$\mathrm{g}(x)=\mathrm{f}\left(\frac{3}{2}x\right)$ for $x\in\mathbb{R},\ -\frac{2}{3}a\le x\le 2a.$ Since $\mathrm{R_f}=[0,a]\subseteq\left[-\frac{2}{3}a,2a\right]=\mathrm{D_g}$, the composite function gf exists.



$$\mathrm{D_{gf}} = \mathrm{D_f} \xrightarrow{\mathrm{f}} \mathrm{R_f} = [0, a] \xrightarrow{\mathrm{g}} \left[\frac{\sqrt{3}}{2}a, a\right] = \mathrm{R_{gf}}$$

$$\mathrm{R_{gf}} = \left[\frac{\sqrt{3}}{2}a, a\right]$$

9i



$d(\frac{4}{5})^7 = 56.2$

$\therefore\ d = 267.9824829 = 268.0$ (1 d.p.)

ii

Since $|r| = \left|\frac{4}{5}\right| < 1$, sum to infinity exists.

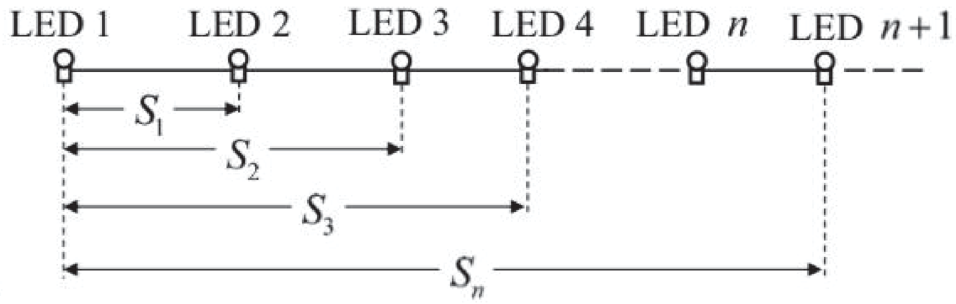
Hence maximum theoretical length of wire

$= \dfrac{d}{1-r}$

$= \dfrac{267.9824829}{1-\frac{4}{5}}$

$= 1339.912415$

$= 1339.9$ cm (1 d.p.)

iii	

Let S_n be distance from LED 1 to LED $n+1$.

<u>Method 1</u>: (using GC table)

Using sum of GP, $S_n = \dfrac{267.9824829[1-(\frac{4}{5})^n]}{1-\frac{4}{5}}$

Using GC,

n	S_n	$\lvert S_n - 1290 \rvert$
14	1281.0	9.0
15	1292.8	2.8

$\therefore$ closest LED to 1290 is when $n=15$.

$\therefore$ required LED is LED $(15+1)$ = LED 16.

Hence colour of LED 16 is red.

<u>Method 2</u>: (using algebraic manipulation)

Let $S_n = \dfrac{267.9824829[1-(\frac{4}{5})^n]}{1-\frac{4}{5}} = 1290$

$1-(\frac{4}{5})^n = 0.9627494943$

$n\ln(\frac{4}{5}) = \ln 0.0372505057$

$\therefore\ n = 14.74427443$

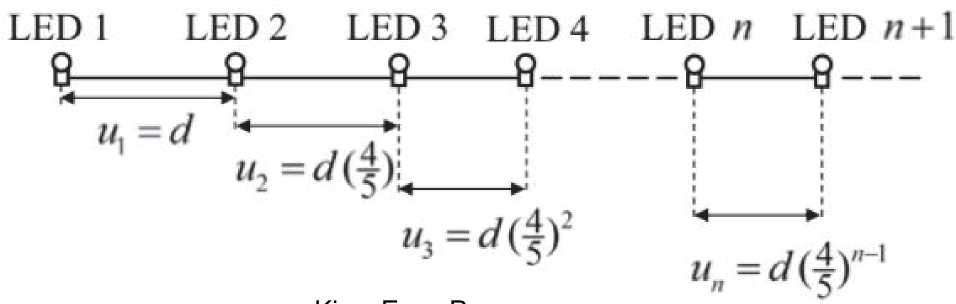
Using GC,

n	S_n	$\lvert S_n - 1290 \rvert$
14	1281.0	9.0
15	1292.8	2.8

$\therefore$ closest LED to 1290 is when $n=15$.

$\therefore$ required LED is LED $(15+1)$ = LED 16.

Hence colour of LED 16 is red.

iv	

	Let $u_n = d\,(\frac{4}{5})^{n-1} = 267.9824829(\frac{4}{5})^{n-1} \geq 1$ $(\frac{4}{5})^{n-1} \geq \frac{1}{267.9824829}$ $\ln(\frac{4}{5})^{n-1} \geq \ln \frac{1}{267.9824829}$ $(n-1)\ln(\frac{4}{5}) \geq \ln \frac{1}{267.9824829}$ $n-1 \leq 25.05526859$ $\therefore \quad n \leq 26.05526859$ Since largest integer $n = 26$, last LED is LED $(26+1) =$ LED 27 Hence colour of last LED 27 is yellow.	
10a	$v + \mathrm{i}u = 2$ ----(1) $av - 2u = 3\mathrm{i}$ ----(2) $(1)\times a - (2)$: $\mathrm{i}au + 2u = 2a - 3\mathrm{i}$ $u = \frac{2a-3\mathrm{i}}{2+\mathrm{i}a} \times \frac{2-\mathrm{i}a}{2-\mathrm{i}a}$ $u = \frac{4a - 6\mathrm{i} - 2a^2\mathrm{i} - 3a}{4+a^2}$ $u = \frac{a-(6+2a^2)\mathrm{i}}{4+a^2}$ $= \frac{a}{4+a^2} - \frac{6+2a^2}{4+a^2}\mathrm{i}$ Sub u into (1) $v = 2 - \mathrm{i}u$ $= 2 - \mathrm{i}(\frac{a}{4+a^2} - \frac{6+2a^2}{4+a^2}\mathrm{i})$ $= 2 - \frac{6+2a^2}{4+a^2} - \frac{a}{4+a^2}\mathrm{i}$ $= \frac{2}{4+a^2} - \frac{a}{4+a^2}\mathrm{i}$	

bi	Sub $x=-k$ into $bx^3+(12b+\mathrm{i})x^2+(12\mathrm{i}+b)x+12b=0$ We have: $-bk^3+(12b+\mathrm{i})k^2-(12\mathrm{i}+b)k+12b=0$ $-bk^3+12bk^2-bk+12b=0\ ---(1)$ and $k^2-12k=0----(2)$ (2) implies $k(k-12)=0$ $k=0,12$ Reject $k=0$ as $b\neq 0$.	• • • •
bii	$bx^2+\mathrm{i}x+b=0$ $x=\dfrac{-\mathrm{i}\pm\sqrt{-1-4b^2}}{2b}$ $x=\dfrac{-1\pm\sqrt{1+4b^2}}{2b}\mathrm{i}$ Hence roots are purely imaginary	• • •
biii	Hence $bx^3+(12b+\mathrm{i})x^2+(12\mathrm{i}+b)x+12b=(x+12)(bx^2+cx+b)$ By compare coeff of x, $c=\mathrm{i}$, using (i):	•

	$bx^3+(12b+\mathrm{i})x^2+(12\mathrm{i}+b)x+12b=(x+12)(bx^2+\mathrm{i}x+b)$ $=b(x+12)(x+\frac{1+\sqrt{1+4b^2}}{2b}\mathrm{i})(x+\frac{1-\sqrt{1+4b^2}}{2b}\mathrm{i})$	• •
11 a	Height of liquid in the cylinder $= x-4$ $V=\pi(2.5)^2(x-4)+\frac{1}{3}\pi(2.5)^2(4)$ $=6.25\pi(x-4)+\frac{25}{3}\pi$ $\frac{\mathrm{d}V}{\mathrm{d}x}=6.25\pi$ and $\frac{\mathrm{d}V}{\mathrm{d}t}=-18$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ $=\frac{-18}{6.25\pi}$ $=-0.916$ cm/min	• •
b	$\frac{r}{x}=\frac{2.5}{4}\Rightarrow x=1.6r$ When $x=2$, $r=\frac{5}{4}$ $V_c=\frac{1}{3}\pi r^2 x$ $=\frac{1}{3}\pi r^2(1.6r)$ $=\frac{1.6}{3}\pi r^3$ $\frac{\mathrm{d}V}{\mathrm{d}r}=1.6\pi r^2$	• •

$$\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$$
$$=\frac{1}{1.6\pi\left(\frac{5}{4}\right)^2}\times(-18)$$
$$=-2.29 \text{ cm/min}$$

c

$$\text{Vol of bowl } = \pi\int_{-10}^{0} 225\left(1-\frac{y^2}{100}\right)\mathrm{d}y$$

Vol. of empty space

$$=\pi\int_{-10}^{0} 225\left(1-\frac{y^2}{100}\right)\mathrm{d}y-\underbrace{\pi(2.5)^2(20)}_{\boxed{\text{A}}}-\underbrace{\frac{1}{3}\pi(2.5)^2(4)}_{\boxed{\text{B}}}$$
$$=4293.510 \text{ cm}^3 \text{ (3 d.p.)}$$

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12i

Equation of line through MD : $\mathbf{r}=\begin{pmatrix}0.8\\0.6\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\7\\-1\end{pmatrix},\ \lambda\in\mathbb{R}$

$$\begin{pmatrix}0.8\\0.75\\0.3\end{pmatrix}-\begin{pmatrix}0.4\\-0.9\\0\end{pmatrix}=\begin{pmatrix}0.4\\1.65\\0.3\end{pmatrix}$$

Equation of line through ND: $\mathbf{r}=\begin{pmatrix}0.4\\-0.9\\0\end{pmatrix}+\mu\begin{pmatrix}0.4\\1.65\\0.3\end{pmatrix},\ \lambda\in\mathbb{R}$

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Since lines through MD and ND intersect at D,

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$$\begin{cases} 0.8+2\lambda = 0.4+0.4\mu \\ 0.6+7\lambda = -0.9+1.65\mu \\ -\lambda = 0.3\mu \end{cases}$$

Solving using GC,

$\lambda = -\dfrac{3}{25}$, $\mu = \dfrac{2}{5}$.

Substitute $\lambda = -1.2$ into equation of line through MD,

$$\begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix} - \frac{3}{25}\begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.56 \\ -0.24 \\ 0.12 \end{pmatrix}$$

$\therefore$ coordinates of D is $(0.56, -0.24, 0.12)$. (shown)

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Equation of horizontal ground: $\mathbf{r}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$

Required angle

$$= 90° - \cos^{-1}\left|\frac{\begin{pmatrix} 0.56 \\ -0.24 \\ 0.12 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{0.56^2+(-0.24)^2+0.12^2}\sqrt{1}}\right|$$

$$= 11.14233567°$$

$$= 11.1° \quad (1 \text{ d.p.})$$

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iii

Given $T(0,0,0.07)$.

$$\overrightarrow{NM} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.4 \\ -0.9 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 1.5 \\ 0 \end{pmatrix}$$

$$\overrightarrow{TM} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0.07 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ -0.07 \end{pmatrix}$$

$$\begin{pmatrix} 0.4 \\ 1.5 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0.8 \\ 0.6 \\ -0.07 \end{pmatrix} = \begin{pmatrix} -0.105 \\ 0.028 \\ -0.96 \end{pmatrix} = 0.01\begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix}$$

$\therefore$ equation of p:

$$\mathbf{r}\cdot\begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix} = -6.72 \text{ (Shown)}$$

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Equation of line perpendicular to p passing through D is

$$\mathbf{r}=\begin{pmatrix}0.56\\-0.24\\0.12\end{pmatrix}+t\begin{pmatrix}-10.5\\2.8\\-96\end{pmatrix},\ t\in\mathbb{R}.$$

Substitute equation of line into p,

$$\begin{pmatrix}0.56-10.5t\\-0.24+2.8t\\0.12-96t\end{pmatrix}\cdot\begin{pmatrix}-10.5\\2.8\\-96\end{pmatrix}=-6.72$$

$\therefore\ t=0.00121618712$

Hence

$$\begin{pmatrix}0.56\\-0.24\\0.12\end{pmatrix}+0.00121618712\begin{pmatrix}-10.5\\2.8\\-96\end{pmatrix}=\begin{pmatrix}0.5472300352\\-0.2365946761\\0.00324603648\end{pmatrix}$$

$$=\begin{pmatrix}0.547\\-0.237\\0.00325\end{pmatrix}$$

$\therefore$ coordinates of G is $(0.547,-0.237,0.00325)$.
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<u>Method 1</u>: [Hence using $(0.547,-0.237,0.00325)$]

Required distance DG

$=\sqrt{(0.56-0.54723)^2+(-0.24-(-0.23659))^2+(0.12-0.003246)^2}$

$=0.1174998$

$=0.117\quad(3\text{ s.f.})$

$\therefore$ required distance $=117$ m

<u>Method 2</u>: [Otherwise using dot product]

$$\overrightarrow{MD}=\begin{pmatrix}0.56\\-0.24\\0.12\end{pmatrix}-\begin{pmatrix}0.8\\0.6\\0\end{pmatrix}=\begin{pmatrix}-0.24\\-0.84\\0.12\end{pmatrix}$$

Required distance

$$=\left|\begin{pmatrix}-0.24\\-0.84\\0.12\end{pmatrix}\cdot\frac{\begin{pmatrix}-10.5\\2.8\\-96\end{pmatrix}}{\sqrt{(-10.5)^2+2.8^2+(-96)^2}}\right|$$

$=0.1174996006$

$=0.117\quad(3\text{ s.f.})$

$\therefore$ required distance $=117$ m
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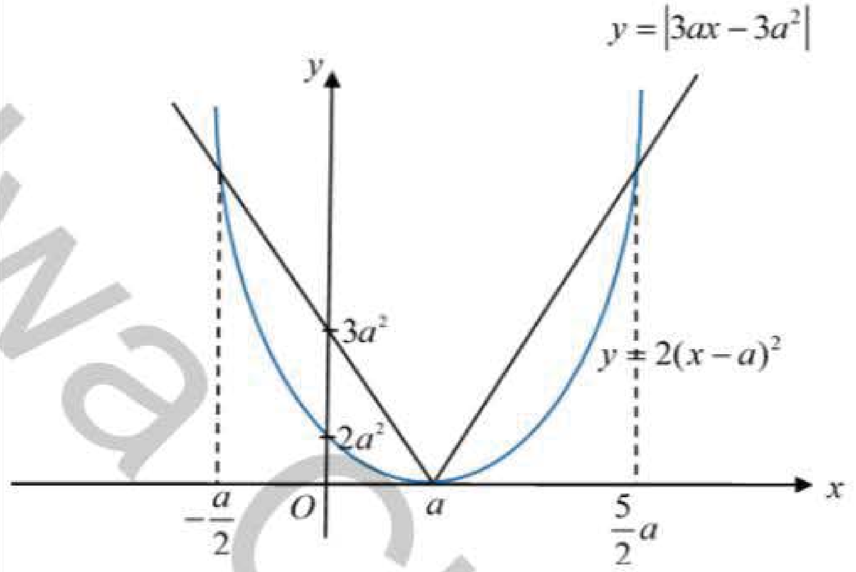
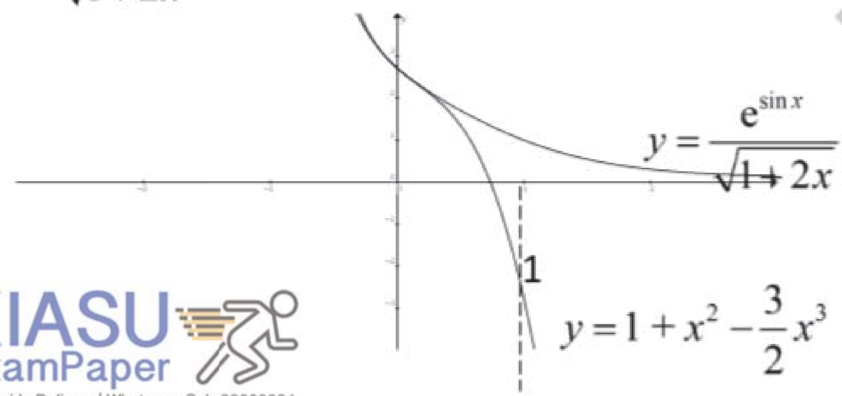
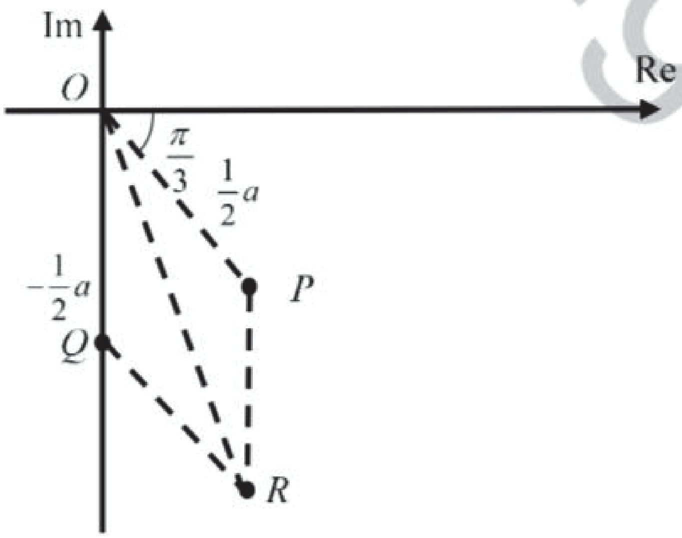
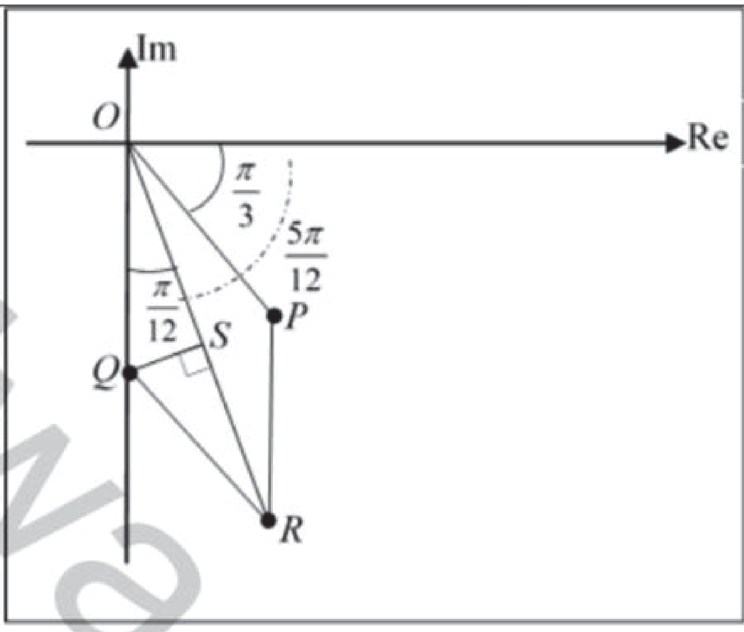
	<u>Method 3</u> $\left\|\overrightarrow{DG}\right\| = \beta\left\|\begin{pmatrix}-10.5\\2.8\\-96\end{pmatrix}\right\|$ $= 0.00121618712\sqrt{9334.09}$ $= 0.1174996006$ $= 0.117 \quad (3 \text{ s.f.})$ $\therefore$ required distance $= 117$ m	

Qn	Solutions	Commen
1		In genera part. Common 1. Fc in 2. Tl stu of 3. Sc nu gr
1i	To find points of intersection, we solve $y=\|3ax-3a^2\|$ $y=2(x-a)^2$ i.e. $2(x-a)^2=\|3ax-3a^2\|$ $2(x-a)^2=3ax-3a^2$ ------(1) $\Rightarrow 2x^2-7ax+5a^2=0$ $\Rightarrow (2x-5a)(x-a)=0$ $\Rightarrow x=\frac{5}{2}a$ or $x=a$ Or $2(x-a)^2=-(3ax-3a^2)$ -------(2) $\Rightarrow 2x^2-ax-a^2=0$ $\Rightarrow (2x+a)(x-a)=0$ $\Rightarrow x=-\frac{1}{2}a$ or $x=a$ For $2(x-a)^2\geq\|3ax-3a^2\|$, $x\leq-\frac{a}{2}$ or $x=a$ or $x\geq\frac{5a}{2}$	A lot of st previous g Majority that $x=a$ The easie intercepts based on final conc A lot of st method to directly w manipulat Common 1. $\|a$ m M lo 2. Sc wa in fa Th (

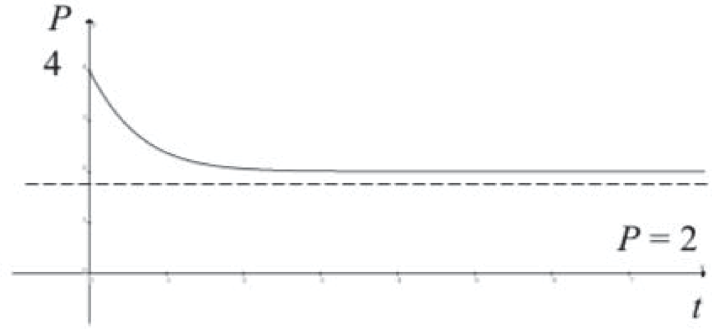
1ii	Replace x with $x+\frac{a}{2}$, $2(x+\frac{a}{2}-a)^2 \geq \left\|3a\left(x+\frac{a}{2}\right)-3a^2\right\|$ $2\left(x-\frac{a}{2}\right)^2 \geq \left\|3ax-\frac{3a^2}{2}\right\|$ $\therefore\ x+\frac{a}{2} \leq -\frac{a}{2}$ or $x+\frac{a}{2}=a$ or $x+\frac{a}{2} \geq \frac{5a}{2}$ $\Rightarrow x \leq -a$ or $x=\frac{a}{2}$ or $x \geq 2a$	A lot replac
2i	$y=\frac{e^{\sin x}}{\sqrt{1+2x}}$ $y\sqrt{1+2x}=e^{\sin x}$ $\sqrt{1+2x}\frac{dy}{dx}+\frac{1}{2}\frac{2}{\sqrt{1+2x}}y=\cos x\, e^{\sin x}$ $\frac{dy}{dx}+\frac{y}{1+2x}=\frac{e^{\sin x}}{\sqrt{1+2x}}\cos x$ $=y\cos x$ $\frac{1}{y}\frac{dy}{dx}+\frac{1}{1+2x}=\cos x$ Alternatively, $\ln\left[y\sqrt{1+2x}\right]=\sin x$ $\frac{\sqrt{1+2x}\left(\frac{dy}{dx}\right)+(1+2x)^{-\frac{1}{2}}}{y\sqrt{1+2x}}=\cos x$ $\frac{1}{y}\frac{dy}{dx}+\frac{1}{1+2x}=\cos x$ (Shown)	This c Stude detail questi credit

ii	Differentiating again w.r.t. x, $\frac{1}{y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{1}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-\frac{2}{(1+2x)^2}=-\sin x$ $\frac{1}{y}\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{2}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{2}{y^3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+\frac{8}{(1+2x)^3}=-\cos x$ $\frac{1}{y}\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-\frac{3}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{2}{y^3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+\frac{8}{(1+2x)^3}=-\cos x$ When $x=0, y=1, \frac{\mathrm{d}y}{\mathrm{d}x}=0, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2, \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-9$. The Maclaurin's series for y is $y=1+\frac{2}{2!}x^2-\frac{9}{3!}x^3+\ldots$ $=1+x^2-\frac{3}{2}x^3+\ldots$	The n questi differ $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}$ a expre differ $\frac{\mathrm{d}}{\mathrm{d}x}\Big($
iii	$\int_0^1 \frac{e^{\sin x}}{\sqrt{1+2x}}\,\mathrm{d}x \approx \int_0^1 1+x^2-\frac{3}{2}x^3\,\mathrm{d}x$ $=\frac{23}{24}$ $=0.985$ (3 s.f.)	A lot they c integr evalu
iii	$\int_0^1 \frac{e^{\sin x}}{\sqrt{1+2x}}\,\mathrm{d}x = 0.058$  The graphs of $y=\frac{e^{\sin x}}{\sqrt{1+2x}}$ and $y=1+x^2-\frac{3}{2}x^3$ differ significantly near to $x=1$. Hence the approximation is not good.	1. Ma the M powe powe appro answe expla good. 2. Ma that x witho affect 3. So range range for ap valid

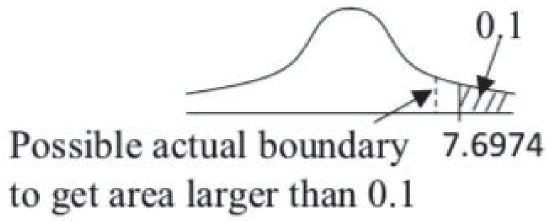
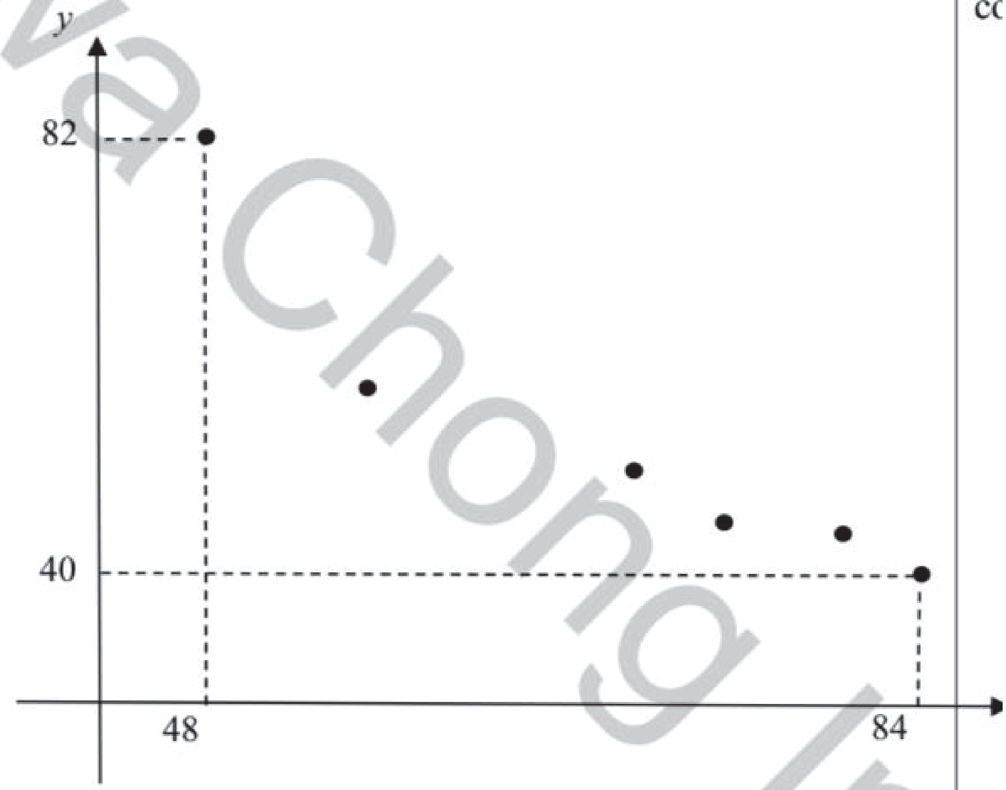
		not sı less tl good
iv	Replace x with $-x$ $y=\dfrac{\mathrm{e}^{\sin(-x)}}{\sqrt{1-2x}}=\dfrac{\mathrm{e}^{-\sin x}}{\sqrt{1-2x}}=\dfrac{1}{\mathrm{e}^{\sin x}\sqrt{1-2x}}$ $\dfrac{1}{\mathrm{e}^{\sin x}\sqrt{1-2x}}\approx 1+x^2+\dfrac{3}{2}x^3$	Many series conne (ii).
3i	$\lvert p\rvert=\left\lvert\dfrac{a}{1+\sqrt{3}\mathrm{i}}\right\rvert=\dfrac{a}{2}$ $\arg(p)=\arg\left(\dfrac{a}{1+\sqrt{3}\mathrm{i}}\right)$ $=\arg(a)-\arg\left(1+\sqrt{3}\mathrm{i}\right)$ $=0-\dfrac{\pi}{3}$ $=-\dfrac{\pi}{3}$	• Ins arg mu p. mi • A 4th • A
ii	 The shape is a rhombus.	• Al fo ve p sh • Sc di • A qu tri
ii	$\angle QOR=\dfrac{\frac{\pi}{2}-\frac{\pi}{3}}{2}=\dfrac{\pi}{12}$ $\arg(p+q)=-(\dfrac{\pi}{3}+\dfrac{\pi}{12})=-\dfrac{5\pi}{12}$	• Sc ar on ro • Sc ar an

	 $\lvert p+q\rvert = 2\times OS = 2\times\frac{a}{2}\cos\frac{\pi}{12} = a\cos\frac{\pi}{12}$	• M fo • Sc ar
iii	For $(p+q)^n$ to be purely imaginary, $(p+q)^n = (a\cos\frac{\pi}{12})^n[\cos(-\frac{5n\pi}{12}) + \mathrm{i}\sin(-\frac{5n\pi}{12})]$ $\cos(-\frac{5n\pi}{12}) = 0$ Hence the smallest positive integer n is 6, as $\cos(-\frac{5\pi}{2}) = 0$.	• Sc su • a f

4ai	$\frac{dP}{dt} = \frac{a}{P} - bP$, where a, b are constants When $P = 2$, $\frac{dP}{dt} = \frac{a}{2} - 2b = 0 \Rightarrow a = 4b$ $\frac{dP}{dt} = \frac{4b}{P} - bP$ $= b\left(\frac{4}{P} - P\right)$ $= k\left(\frac{4}{P} - P\right)$, where $k = b$
4aii	When $P = 4$, $\frac{dP}{dt} = -3$. $\frac{dP}{dt} = k\left(\frac{4}{4} - 4\right) = -3$ $\Rightarrow k = 1$ $\frac{dP}{dt} = \frac{4}{P} - P$ $= \frac{4 - P^2}{P}$ $\frac{dt}{dP} = \frac{P}{4 - P^2}$ $= -\frac{1}{2}\frac{-2P}{4 - P^2}$ $t = -\frac{1}{2}\ln\lvert 4 - P^2 \rvert + C$ $-2t = \ln\lvert 4 - P^2 \rvert - 2C$ $\ln\lvert 4 - P^2 \rvert = 2C - 2t$ $4 - P^2 = \pm e^{2C} e^{-2t}$ $= Ae^{-2t}$, where $A = \pm e^{2C}$ When $t = 0$, $P = 4$. $A = -12$ $4 - P^2 = -12e^{-2t}$ $P^2 = 4 + 12e^{-2t}$ $P = 2\sqrt{1 + 3e^{-2t}}$

4a iii	 The population of the bugs will decrease and approach 2000 in the long run.	St in M in lo cc ap Tl as St
4b	**Method 1:** $u=\frac{N}{t}$ $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{t\frac{\mathrm{d}N}{\mathrm{d}t}-N}{t^2}$ $t\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\mathrm{d}N}{\mathrm{d}t}-\frac{N}{t}$ Substitute into $\frac{\mathrm{d}N}{\mathrm{d}t}=4+\frac{N}{t}$: $t\frac{\mathrm{d}u}{\mathrm{d}t}=4$ $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{4}{t}$ $u=4\ln t+C$ $\frac{N}{t}=4\ln t+C$ When $t=1$, $N=1 \Rightarrow C=1$. $N=4t\ln t+t$ **Method 2:** $u=\frac{N}{t}$ $N=ut$ $\frac{\mathrm{d}N}{\mathrm{d}t}=\frac{\mathrm{d}u}{\mathrm{d}t}t+u$ Substitute into $\frac{\mathrm{d}N}{\mathrm{d}t}=4+\frac{N}{t}$:	Tl at ei di N A m $\frac{N}{t}$ = Tl of D

	$\frac{\mathrm{d}u}{\mathrm{d}t}t+u=4+u$ $\frac{\mathrm{d}u}{\mathrm{d}t}t=4$ $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{4}{t}$ $u=4\ln t+C$ $\frac{N}{t}=4\ln t+C$ When $t=1$, $N=1 \Rightarrow C=1$. $N=4t\ln t+t$	
5 **(i)**	Let X denote the daily rainfall in mm. $X\sim \mathrm{N}\left(\mu,\sigma^2\right)$ In 7 days, $\bar{X}\sim \mathrm{N}\left(\mu,\frac{\sigma^2}{7}\right)$ $\mathrm{P}\left(X>9.8\right)=0.1$ $\mathrm{P}\left(Z>\frac{9.8-\mu}{\sigma}\right)=0.1$ $\frac{9.8-\mu}{\sigma}=1.2815516$ $9.8=\mu+1.2815516\sigma$ --- (1) $\mathrm{P}\left(\bar{X}>8.2\right)=0.1$ $\mathrm{P}\left(Z>\frac{8.2-\mu}{\sigma/\sqrt{7}}\right)=0.1$ $\frac{8.2-\mu}{\sigma/\sqrt{7}}=1.2815516$ $8.2=\mu+0.4843810\sigma$ --- (2) Solving (1) and (2), $\mu=7.22780,\sigma=2.00710$ $\therefore \sigma=2.01\left(2\text{dp}\right)\left(\text{shown}\right)$	S μ pa
(ii)	In 30 days, $\bar{X}\sim \mathrm{N}\left(7.22780,\frac{2.01^2}{30}\right)$ $\mathrm{P}\left(\bar{X}>k\right)\geq 0.1$ $k\leq 7.6981\left(5\text{sf}\right)$ $\therefore$ max $k=7.6\left(1\text{dp}\right)$	N al A th S ar

<table>
<tr><td></td><td>Note: change in inequality sign because for area to be greater than or equal to 0.1, the value of k can be at 7.6974 or to its left, i.e. less than 7.6974 (see diagram below).

</td><td></td></tr>
<tr><td>6
(i)</td><td>
</td><td>A
cc</td></tr>
<tr><td>(ii)</td><td>(x_7, y_7) is $(\bar{x}, \bar{y})$
Using G.C,
$(\bar{x}, \bar{y}) = (69, 53)$
OR
$(x_7, y_7) =$
$\left(\dfrac{80+84+70+74+58+48}{6}, \dfrac{44+40+49+45+58+82}{6}\right)$
$= (69, 53)$</td><td>M</td></tr>
<tr><td>(iii)</td><td>Model A: $y = a + bx^2$ $\quad r$-value $= -0.92386$
Model B: $y = \ln a + b\ln x$ $\quad r$-value $= -0.96785$
Model C: $y = a + b\sqrt{x}$ $\quad r$-value $= -0.95855$

Since $|r|$ for Model B is nearest to 1, Model B is the most accurate model</td><td>S
di
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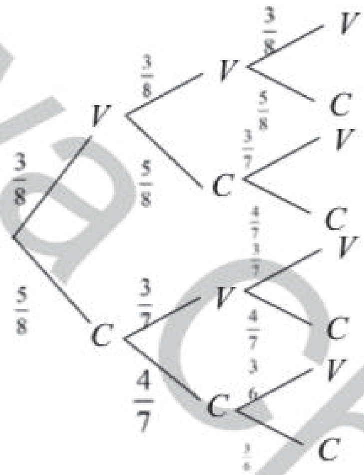
</table>

(iv)	$y = \ln a + b \ln x$ By GC, $\ln a = 350.11$ $b = -70.469$ $y = 350.11 - 70.469 \ln x$ When $x = 60$, $y = 62$ marks. The student is estimated to score 62 marks.	N **us** of m A w A ec
(v)	$4 \times 30 = 120$ unreasonable as 120 is not within data range where $48 \le x \le 84$.	S h T th ar th m
7 **(i)**	Let X and Y denote the length (in inches) of a "footlong" and a "6-inch" loaf respectively. $X \sim \mathrm{N}\left(12.2, 0.2^2\right)$, $Y \sim \mathrm{N}\left(6.1, \left(0.1\sqrt{2}\right)^2\right)$ $\mathrm{P}(Y < 6) = 0.239750 = 0.240(3\mathrm{sf})$ $\mathrm{P}(X < 12) = 0.158655 = 0.159(3\mathrm{sf}) < \mathrm{P}(Y < 6)$ $\therefore$ a "6-inch" loaf more likely to be less than 6 inches.	A co va th 6, ga It pr St fu as qu
(ii)	$\mathrm{E}(Y_1 + Y_2 - X) = 2(6.1) - 12.2 = 0$ $\mathrm{Var}(Y_1 + Y_2 - X) = 2\left(0.1\sqrt{2}\right)^2 + 0.2^2 = 0.08$ $Y_1 + Y_2 - X \sim \mathrm{N}(0, 0.08)$ $\mathrm{P}(Y_1 + Y_2 > X) = \mathrm{P}(Y_1 + Y_2 - X > 0) = \dfrac{1}{2}$	It in M th th S w as ar

(iii)	Let A denote the number of “6-inch” sandwiches less than 6-inches in length in a week. $A \sim \mathrm{B}(3, 0.239750)$ $\mathrm{P}(A \le 1) = 0.855122 = 0.855(3\text{sf})$ <u>Alternative</u> $\text{required probability} = \mathrm{P}(Y<6)\left[\mathrm{P}(Y>6)\right]^2 \times 3 + \left[\mathrm{P}(Y>6)\right]^3$ $= 0.23975(1-0.23975)^2 \times 3 + (1-0.23975)^3$ $= 0.855(3\text{sf})$	St va w or di of sa “l S w m “x sa in
(iv)	Let B and C denote the number of “6-inch” sandwiches less than 6-inches in length in a 4-week and 3-week period respectively. $B \sim \mathrm{B}(12, 0.239750)$ and $C \sim \mathrm{B}(9, 0.239750)$ $\mathrm{P}(A=1 \mid B>4) = \dfrac{\mathrm{P}(A=1)\mathrm{P}(C>3)}{\mathrm{P}(B>4)}$ $= \dfrac{\mathrm{P}(A=1)\left[1-\mathrm{P}(C \le 3)\right]}{1-\mathrm{P}(B \le 4)}$ $= \dfrac{0.41571 \times 0.14710}{0.13721}$ $= 0.445674$ $= 0.446(3\text{sf})$	Tl ch fr S co S de nu 4t P = S 2 up P — =
8 (a)(i)	Number of arrangements$= \dfrac{8!}{3!2!} = 3360$	S re
8 (a)(ii)	Number of arrangements$= \dfrac{5!}{2!} \times {}^6\mathrm{C}_3 = 1200$	M co 2
8 (a)(iii)	Arrange 3 vowels (into a group) $\dfrac{3!}{2!} = 3$	M es er

	Arrange 5 consonants (for slotting) $\frac{5!}{3!}=20$ Since the ends must be consonants, only the middle 4 slots can be used for the group to slot. (ie. ${}^4C_1$) Number of arrangements= $3\times 20\times 4=240$ OR Case 1: First and Last 'L' $4!\times\frac{3!}{2!}=72$ Case 2: First and last 'L' and non-L $2\times 2\times\frac{4!}{2!}\times\frac{3!}{2!}=144$ Case 3: First and last non- L $2\times\frac{4!}{3!}\times\frac{3!}{2!}=24$ Total= 240 ways	
(b)(i)	P(1V) + P(2V) + P(3V) $=\frac{{}^5C_3\,{}^3C_1+{}^5C_2\,{}^3C_2+{}^5C_1\,{}^3C_3}{{}^8C_4}$ $=\frac{65}{70}=\frac{13}{14}$ Probability = 1 – P(all consonants) $=1-\left(\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\right)=\frac{13}{14}$ OR 1 – P(no vowels) $=1-\frac{{}^5C_4}{{}^8C_4}$ $=1-\frac{5}{70}$ $=\frac{13}{14}$	G

	OR P(1 vowel) + P(2 vowels) + P(3 vowels) $=\frac{3\times5\times4\times3}{8\times7\times6\times5}\times\frac{4!}{3!}+\frac{3\times2\times5\times4}{8\times7\times6\times5}\times\frac{4!}{2!2!}+\frac{3\times2\times1\times5}{8\times7\times6\times5}\times\frac{4!}{3!}$ $=\frac{13}{14}$	
8 (b)(ii)	 Probability = P(VCC)+P(CVC)+P(CCV)+P(CCC) $=\frac{3}{8}\times\frac{5}{8}\times\frac{4}{7}+\frac{5}{8}\times\frac{3}{7}\times\frac{4}{7}+\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}+\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}$ $=\frac{505}{784}$ $=0.644$	M di
9(i)	Using GC, Unbiased estimate of population mean is $\bar{x}=949.84\ (2\text{ d.p.})$ Unbiased estimate of population variance is $s^2=1.31634^2=1.73\ (2\text{ d.p.})$ Note: If student write $\mu=949.84$ or $\sigma^2=1.31634^2=1.73$, annotate but do not deduct.	G St *ai* ar gi T m st 1. ge 2.

		N 1. σ sa U va (1 (1 fo N μ
9 (a) (ii)	Let X be the volume of shower gel dispensed by the machine. Let μ denote the population mean volume of shower gel dispensed by the machine. $\bar{X} \sim \mathrm{N}\left(\mu, \frac{s^2}{n}\right)$ approx. by Central Limit Theorem. $\bar{X} \sim \mathrm{N}\left(950, \frac{s^2}{n}\right)$ $\mathrm{H_0}: \mu = 950$ $\mathrm{H_1}: \mu < 950$	Pa ar M th de 's de of ex th as er
9 (a)(iii)	Under $\mathrm{H_0}$, using GC, $p-\text{value} = 0.13474 \approx 0.135$	R m fa st cc

	The p - value is the probability of the sample mean volume of shower gel in a bottle is as extreme as 949.84 when the mean is actually 950.	Tl w 9 m M th cc S th w of hy st p- cc
9 (a)(iv)	Since we reject H_0 , $p-\text{value} < \alpha/100$ $\alpha \geq 13.476$ $\{\alpha \in \mathbb{R} : 13.5 \leq \alpha \leq 100\}$	B di nc up St us in rc th
9(b)	Let Y be the volume of shower gel dispensed by the machine after recalibration. Let μ denote the population mean volume of refill dispensed by the machine. $\bar{Y} \sim \mathrm{N}\left(\mu, \frac{s^2}{n}\right)$ approx. by Central Limit Theorem H_0: $\mu = 250$ H_1: $\mu \neq 250$ Test statistic: $Z = \frac{\bar{Y}-\mu}{s/\sqrt{n}}$ Level of significance: 1% Reject H_0 if $p-\text{value} \leq 0.01$	C th 1. 2. to 3. as cc 4.

$$\frac{249.5-250}{s/\sqrt{50}} \le -2.5758293 \quad \text{or} \quad \frac{249.5-250}{s/\sqrt{50}} \ge 2.5758293 \text{ (NA)}$$

$$0 < \frac{s}{\sqrt{50}} \le 0.19411$$

$$0 < s \le 1.3726$$

$$0 < \frac{1}{49}\left(k - \frac{(-25)^2}{50}\right) \le 1.3726^2$$

$$12.5 < k \le 104.8$$

M lo w in M th 2 re th

9(c)

There is no need for the floor supervisor to assume the volume of shower gel follow a normal distribution as the sample sizes in both part (a) and (b) are large. The sample mean volume of shower gel can be approximated to follow a normal distribution by Central Limit Theorem.

A Tl sa ca nc Tl w 's St ar fc

10 (i)

Let $\$W$ denote the amount won by a player in one game.
Under option A,

$$\mathrm{E}(W_A) = 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right) + 4\left(\frac{1}{4}\right)$$

$$= (1+2+3+4)\left(\frac{1}{4}\right)$$

$$= \frac{5}{2} = \$2.50$$

Under option B, the winning amount for each combination is listed below:

x \ y	1	2	3	4
1	0	4	6	8
2	-1	0	12	16
3	-2	-1	0	24
4	-3	-2	-1	0

St ra di nc ar pı sc G ex A H nι kı w re as pc di cc

	$\mathrm{E}(W_B)=\left(\frac{1}{16}\right)[4+6+12+8+16+24$ $-1-2-3-1-2-1]$ $=\left(\frac{1}{16}\right)[70-10]$ $=\frac{15}{4}=\$3.75$ Since $\mathrm{E}(W_B)>\mathrm{E}(W_A)$, option B is better. (shown)	ca ar It ex ar O
(ii)	Option A is a "sure win" option where the player would definitely gain a positive amount in all cases, whereas option B has a risk of losing money in some cases.	G
(iii)	$\mathrm{E}(W_A{}^2)=(1^2+2^2+3^2+4^2)\left(\frac{1}{4}\right)=\frac{15}{2}$ $\mathrm{Var}(W_A)=\mathrm{E}(W_A{}^2)-[\mathrm{E}(W_A)]^2=\frac{15}{2}-\left(\frac{5}{2}\right)^2=\frac{5}{4}=1.25(\text{shown})$	G Tl st cc
(iv)	Let A and B denote the total amount won by Abel and Benson respectively. $A=W_{A1}+W_{A2}+\cdots+W_{A50}$ Since $n=50$ is large, by Central Limit Theorem, $A\sim \mathrm{N}\left(50\left(\frac{5}{2}\right),50\left(\frac{5}{4}\right)\right)$ approximately i.e. $A\sim \mathrm{N}\left(125,\frac{125}{2}\right)$ $B=W_{B1}+W_{B2}+\cdots+W_{B50}$ $B\sim \mathrm{N}\left(50\left(\frac{15}{4}\right),50\left(\frac{887}{16}\right)\right)$ approximately by Central Limit Theorem since $n=50$ is large $B\sim \mathrm{N}\left(\frac{375}{2},\frac{22175}{8}\right)$	A w va ar w th Tl ap of ar O - - - - -

		-
(v)	$A-B \sim \mathrm{N}\left(125-\frac{375}{2}, \frac{125}{2}+\frac{22175}{8}\right)$ $A-B \sim \mathrm{N}\left(-\frac{125}{2}, \frac{22675}{8}\right)$ $\mathrm{P}(A>B)=\mathrm{P}(A-B>0)$ $=0.120207$ $=0.120(3\mathrm{sf})$	M at re pı A cc aı fo - -

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2019

MATHEMATICS **9758/01**

Higher 2 **18 September 2019**

Paper 1 **3 hours**

1 **(i)** The first four terms of a sequence are given by $u_1 = -13$, $u_2 = -12.8$, $u_3 = 1.8$ and $u_4 = 38$. Given that u_m is a cubic polynomial in m, find u_m in terms of m. [3]

(ii) Find the range of values of m for which u_m is greater than 2000. [2]

2 **(a)** Express $y = \dfrac{3x-1}{x-2}$ in the form $y = A + \dfrac{B}{x-2}$, where A and B are constants to be found. Hence, state a sequence of transformations that transforms the graph of $y = \dfrac{1}{x}$ to the graph of $y = \dfrac{3x-1}{x-2}$. [4]

(b) It is given that $\mathrm{g}(x) = x^2 - 2x + 2$. Sketch the graph of $y = \mathrm{g}(|x|)$, stating clearly the coordinates of any turning points and axial intercepts. Find numerically, the volume of revolution when the region bounded by the curve $y = \mathrm{g}(|x|)$ and the line $y = 5$ is rotated completely about the x-axis. [5]

3 Referred to an origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point C is such that $OACB$ is a parallelogram. The point D is on BC such that $\overrightarrow{BD} = \lambda\overrightarrow{BC}$ and the point E is on AC such that $\overrightarrow{AE} = \mu\overrightarrow{AC}$, where λ and μ are positive constants. The area of triangle ODE is k times the area of triangle OCE.

(i) By finding the area of triangle ODE and OCE in terms of **a** and **b**, find k in terms of μ and λ. [6]

(ii) The point F is on OC and ED such that $OF : FC = 6 : 1$ and $DF : FE = 3 : 4$. By finding the values of λ and μ, calculate the value of k. [3]

4 The curve C has equation $y = \dfrac{x^2+5}{x-2}$.

(i) Prove, using an algebraic method, that C cannot lie between two values to be determined. [4]

(ii) Sketch C, showing clearly the equations of any asymptotes and coordinates of any turning points and axial intercepts. [4]

(iii) By adding a suitable graph to your sketch in **(ii)**, deduce the range of values of h for which the equation

$$\left(x^2+5\right)^2 + (x+1)^2(x-2)^2 = h^2(x-2)^2$$

has at least one positive real root. [3]

[Turn over

5 **Do not use a graphing calculator in answering this question.**

(a) **(i)** It is given that $w_1 = -3 + \sqrt{5}\,\mathrm{i}$. Find the value of w_1^3, showing clearly how you obtain your answer. [2]

(ii) Given that $-3 + \sqrt{5}\,\mathrm{i}$ is a root of the equation

$$4w^3 + pw^2 + qw - 14 = 0,$$

using your result in **(i)** , find the values of the real numbers p and q. [3]

(iii) For these values of p and q, find the other two roots of the equation in part **(ii)**. [3]

(b) It is given that $z = -1 - \sqrt{3}\,\mathrm{i}$.

Find the set of values of n for which $\dfrac{z^*}{z^n}$ is purely imaginary. [4]

6 The function f is defined for all real x by

$$\mathrm{f}(x) = \mathrm{e}^{2x} - 9\mathrm{e}^{-2x}.$$

(i) Show that $\mathrm{f}'(x) > 0$ for all x. [2]

(ii) Show that the set of values of x for which the graph $y = \mathrm{f}(x)$ is concave upward is the same as the set of values of x for which $\mathrm{f}(x) > 0$, and find this set of values of x, in the form of $k\ln 3$, where k is a constant to be found. [3]

(iii) Sketch the graph of $y = \mathrm{f}(x)$, showing clearly any points of intersections with the axes. [2]

(iv) Hence, find the exact value of $\int_0^2 \left|\mathrm{e}^{2x} - 9\mathrm{e}^{-2x}\right| \mathrm{d}x$. [4]

7 It is given that

$$\mathrm{f}(x) = \begin{cases} 4a^2 - x^2, & \text{for } 0 < x \le 2a, \\ 2a(x - 2a), & \text{for } 2a < x \le 4a, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x + 4a)$ for all real values of x, where a is a positive real constant.

(i) Evaluate $\mathrm{f}(2019a)$ in terms of a. [1]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-3a \le x \le 5a$. [3]

The function g is defined by

$$\mathrm{g} : x \mapsto \sqrt{4a^2 - (x - 2a)^2}, \qquad 2a < x < 4a.$$

(iii) Determine whether the composite function gf exists, justifying your answer. [1]

(iv) Give, in terms of a, a definition of fg. [2]

(v) Given that $(\mathrm{fg})^{-1}(27) = \dfrac{7}{2}a$, find the exact value of a. [2]

8

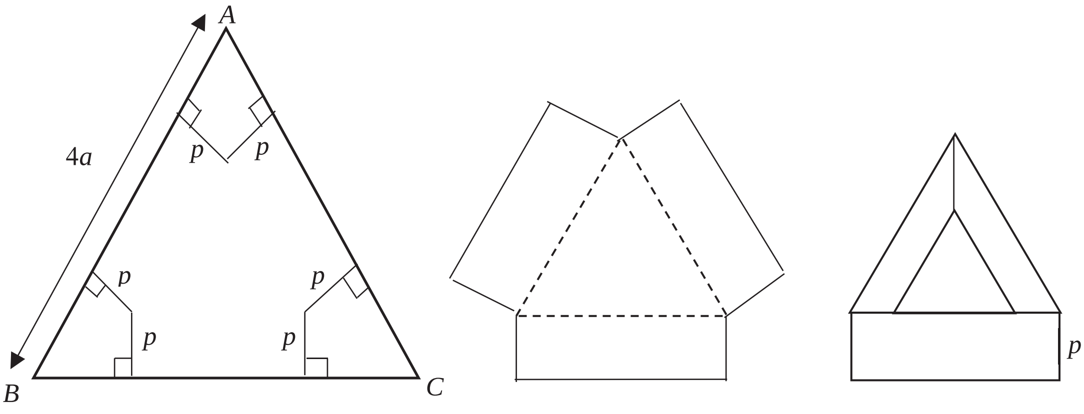


Fig.1 **Fig.2** **Fig.3**

Fig. 1 shows a metal sheet, *ABC*, in the form of an equilateral triangle of side 4*a* cm. A kite shape is cut from each corner, to give the shape as shown in Fig. 2. The remaining metal sheet shown in Fig. 2 is bent along the dotted lines, to form an open triangular prism of height *p* cm shown in Fig. 3.

(i) Show that the volume of the prism is given by $V = \sqrt{3}p\left(2a - \sqrt{3}p\right)^2 \text{ cm}^3$. [3]

(ii) Without using a calculator, find in terms of *a*, the exact value of *p* that gives a stationary value of *V*, and explain why there is only one answer. [6]

(iii) The prism is used by a housewife as a mould for making a dessert. To make the dessert, The housewife has to fill up $\frac{3}{4}$ of the mould with coconut milk. The cost of coconut milk is 0.4 cents per cm^3. What is the exact maximum cost in terms of *a* she needs to pay for the coconut milk? [3]

9 Find

(a) $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$, [2]

(b) $\int \cos kx \cos(k+2)x\,dx$, where *k* is a positive constant, [2]

(c) $\int x\tan^{-1}(3x)\,dx$. [6]

10 **(a)** The Deep Space spacecraft launched in October 1998 used an ion engine to travel from Earth to the Comet Borrelly. The average speed of the spacecraft in October 1998 was 44 000 km/hr. The monthly average speed, v_n of the spacecraft in month *n* based on its first 5 months of operation was given by:

Month, n	1	2	3	4	5
Average speed, v_n	44 000	44335	44 670	45 005	45 340

Assume that v_n follows the same increment for the rest of its flight.

(i) State a general formula for v_n in terms of *n*. [1]

(ii) In which month and year did the average speed of the spacecraft first exceed 53 500 km/hr ? [3]

[Turn over

(iii) Assume that there are 30 days per month. It is known that the total distance travelled by the spacecraft from Earth is given by $\sum_{r=1}^{n}\left(v_r T\right)$ where T is the time taken, in hours, by the spacecraft to travel in one month. Given that the spacecraft travelled from Earth continuously for 3 years to reach Comet Borrelly, find the total distance that it travelled. [3]

(b) Dermontt's Law is an empirical formula for the orbital period of major satellites orbiting planets in the solar system. It is represented by the equation $T_n = T_0 C^n$, where T_n is the orbital period, in days, of the $(n + 1)^{\text{th}}$ satellite and C is a constant associated with the satellite system in question. It is known that the planet Jupiter has 67 satellites. The orbital period of its first satellite is 0.44 days and $C = 2.03$.

(i) Find the longest orbital period of a satellite of Jupiter. [2]

(ii) Find the largest value of n for which the total orbital periods of the first n satellites of Jupiter is within 5×10^6 days of the orbital period of the 20^{th} satellite of Jupiter. [3]

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2019

MATHEMATICS 9758/02

Section A : Pure Mathematics [40 Marks]

1 A sequence a_0, a_1, a_2, … is given by $a_0 = \frac{3}{5}$ and $a_{n+1} = a_n + 3^n - n$ for $n \geq 0$. By considering $\sum_{r=0}^{n-1}\left(a_{r+1} - a_r\right)$, find a formula for a_n in terms of *n*. [5]

2 **In this question, you may use expansions from the List of Formula (MF26).**

(a) **(i)** Find the Maclaurin expansion of ln(cos 3*x*) in ascending powers of *x*, up to and including the term in x^6. [5]

(ii) Hence, state the Maclaurin expansion of tan 3*x*, up to and including the term in x^5. [2]

(b) Given that *x* is sufficiently small, find the series expansion of $\frac{e^{\tan x}}{\left(2+x\right)^2}$ in ascending powers of *x*, up to and including the term in x^2. [3]

3 A curve *C* has parametric equations $x = 4\sin 2t$, $y = 4\cos 2t$, where $0 \leq t \leq \frac{\pi}{4}$.

(i) Sketch *C*. [2]

(ii) State the exact value(s) of *t* at the point(s) where the tangent(s) to *C* are parallel to the
(a) *y*-axis,
(b) *x*-axis. [2]

(iii) *P* is a point on *C* where the normal at *P* is parallel to the line $y = \sqrt{3}x - 2$. Find the equation of the tangent at *P*, in the form of $y = \frac{a}{3}\left(b - x\right)$, where *a* and *b* are the constants to be determined. [3]

(iv) Find the exact area bounded by *C*, the tangent at *P* and the axes. [2]

4 A differential equation is given by $2u^2\frac{\mathrm{d}^2x}{\mathrm{d}u^2} + 4u\frac{\mathrm{d}x}{\mathrm{d}u} = 15u + 12$ where $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$ when $u = 1$. By differentiating $u^2\frac{\mathrm{d}x}{\mathrm{d}u}$ with respect to u, show that the solution of the differential equation is given by $x = au + \frac{b}{u} + c\mathrm{f}(u) + d$, where *a*, *b*, *c* and *d* are constants to be determined and f(*u*) is a function in *u* to be found. [6]

5 The plane p has equation $\mathbf{r}=\begin{pmatrix}1\\0\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-4\end{pmatrix}$, and the line l has equation $\mathbf{r}=\begin{pmatrix}-10\\0\\4\end{pmatrix}+t\begin{pmatrix}8\\-4\\1\end{pmatrix}$, where λ, μ and t are parameters.

(i) Show that l is perpendicular to p and find the values of λ, μ and t which give the coordinates of the point at which l and p intersect. [5]

(ii) Find the cartesian equations of the planes such that the perpendicular distance from each plane to p is 2. [5]

Section B : Statistics [60 Marks]

6 On average, 35% of the boxes of Brand *A* cereal contain a voucher. Brandon buys one box of cereal each week. The number of vouchers he obtains is denoted by X.

(i) State, in context, two conditions needed for X to be well modelled by a binomial distribution. [2]

Assume now that X has a binomial distribution. In order to claim a free gift, 8 vouchers are required. Find the probability that Brandon

(ii) obtains at most 3 vouchers in 9 weeks, [1]

(iii) will be able to claim a free gift only in the 10^{th} week. [2]

$100p$ % of the boxes of Brand *B* cereal contain a voucher. Brandon also buys a box of Brand *B* cereal each week for 10 weeks. Given that the probability that Brandon obtains at most 1 voucher in ten weeks is 0.4845, write down an equation in terms of p and hence, find the value of p. [2]

7 A company sells peanut butter in jars. Each jar is labelled as containing m grams of peanut butter on average. A consumer group suspects that the average mass of peanut butter in a jar is overstated. To test this suspicion, the consumer group checks a random sample of 50 jars and the mass of peanut butter per jar, x grams, are summarized by

$$\sum(x-390)=120 \qquad \text{and} \qquad \sum(x-390)^2=3100.$$

The consumer group uses the above data to carry out a test at the 2% level of significance. The result leads the consumer group to conclude that the company has overstated the average mass of peanut butter in a jar.

(i) Explain why the consumer group is able to carry out a hypothesis test without knowing anything about the distribution of the mass of the peanut butter of the jars. [1]

(ii) Find the least possible value of m, to the nearest gram that leads to the result of the hypothesis test as stated above. [7]

8 **(a)** Find the number of different 6-digit numbers that can be formed from the digits 0, 2, 4, 5, 6 and 8, if no digit is repeated and the numbers formed are divisible by 5. [2]

(b) Find how many 4-letter code words can be formed from the letters of the word ***DIFFERENT***. [4]

9 In this question you should state the parameters of any distributions that you use.

The masses of Grade *A* and Grade *B* strawberries are normally distributed with mean 18 grams and 12 grams respectively and standard deviation 3 grams and 2 grams respectively.

(i) Find the probability that a randomly chosen Grade *A* strawberry has mass between 17 and 20 grams. [1]

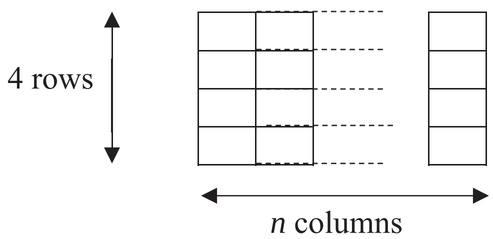
(ii) Any Grade *B* strawberry that weighs less than *m* grams will be downgraded to Grade *C*. Given that there is a probability of at least 0.955 that a Grade *B* strawberry will not be downgraded to Grade *C*, find the greatest value of *m*. [2]

Grade *A* strawberries are packed into bags of 12 while Grade *B* strawberries are packed into bags of 15.

(iii) Find the probability that a bag of Grade *A* strawberries weighs more than a bag of Grade *B* strawberries. [3]

(iv) State an assumption needed for your calculation in part **(iii)**. [1]

10 At a particular booth in a funfair, Kathryn is given some boxes which are arranged in the layout as shown below.



Each box contains a numbered card. One card is numbered '4', three cards are numbered '2', and the rest of the cards are numbered '1'. All the boxes are closed initially and Kathryn is required to open 2 different boxes. Her score is the sum of the numbers obtained.

If she scores more than 4, she wins \$10. If she scores less than 4, she loses \$2. If she scores 4, she does not win anything. The random variable *W* is her winnings after one game.

(i) Show that $\mathrm{P}(W=10)=\dfrac{1}{2n}$. [2]

(ii) Given that $\mathrm{P}(W=10)=\dfrac{1}{8}$, find the value of *n*. Hence, find the probability distribution of *W*. [4]

(iii) Find E(*W*) and Var(*W*). Explain whether Kathryn should play the game. [4]

50 other participants play the game. Find the probability that the mean winnings is at most \$1. [2]

11 **[Leave your answers in fraction]**

The events A and B are such that $\mathrm{P}(A|B) = \dfrac{7}{10}$, $\mathrm{P}(B|A) = \dfrac{4}{15}$ and $\mathrm{P}(A \cup B) = \dfrac{3}{5}$.

Find the exact values of

(i) $\mathrm{P}(A \cap B)$, [3]

(ii) $\mathrm{P}(A' \cap B)$. [2]

For a third event C, it is given that $\mathrm{P}(C) = \dfrac{3}{10}$ and that A and C are independent.

(iii) Find $\mathrm{P}(A' \cap C)$. [2]

(iv) Hence state an inequality satisfied by $\mathrm{P}(A' \cap B \cap C)$. [1]

12 As part of a medical research on diabetes, a team of researchers conducts a study to investigate the amount of glucose y, measured to the nearest 0.5 mg/dl, present in human bodies at different age x, measured in years. The results are given in the table.

x	20	28	36	44	52	60	68	76
y	86.0	90.5	94.5	97.5	100.0	105.0	103.5	104.0

(i) Calculate the product moment correlation coefficient between x and y and explain whether your answer suggests that a linear model is appropriate. [2]

(ii) Draw a scatter diagram for the data, labelling the axes clearly. [1

One of the values of y appears to be incorrect.

(iii) Circle the corresponding point on your diagram and label it P. [1]

For part **(iv)** and **(v)** of this question, you should omit P.

(iv) Explain from your scatter diagram why the relationship between x and y should not be modelled by an equation of the form $y = ax + b$. [1]

(v) Suppose that the relationship between x and y is modelled by an equation of the form $y = c + d\ln x$, where c and d are constants. Find the product moment correlation coefficient between y and $\ln x$ and the constants c and d. [2]

Assume that the value of x at P is correct.

(vi) Use the model $y = c + d\ln x$, with the values of c and d found in **(v)** to estimate the correct value of y at P, giving your answer to the nearest 0.5 mg/dl. Explain why you would expect this estimate to be reliable. [2]

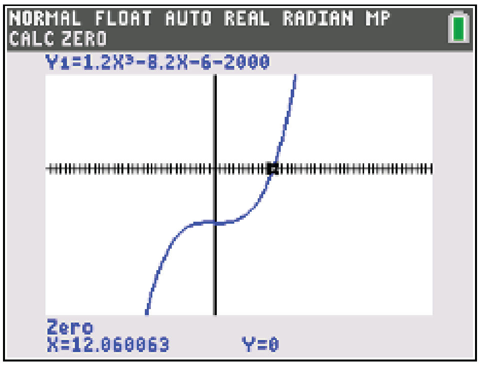
(vii) If the correct value of y at P is used, the 8 data points may be fitted by the model $y = 43.942 + 14.079\ln x$. Find the correct value of y at P, giving your answer to the nearest 0.5 mg/dl. [3]

Jurong Pioneer Junior College
H2 Mathematics Preliminary Exam P1 Solution

Q1

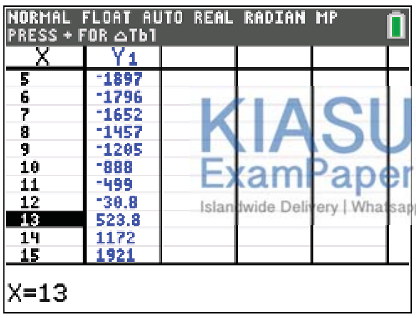
(i)
Let $u_m = am^3 + bm^2 + cm + d$
$u_1 = a + b + c + d = -13$
$u_2 = 8a + 4b + 2c + d = -12.8$
$u_3 = 27a + 9b + 3c + d = 1.8$
$u_4 = 64a + 16b + 4c + d = 38$
Using GC, $a = 1.2$, $b = 0$, $c = -8.2$, $d = -6$
$u_m = 1.2m^3 - 8.2m - 6$

(ii)
$u_m > 2000$
Let $y = 1.2m^3 - 8.2m - 6 - 2000$



From GC graphing, $m > 12.06$
$m \geq 13$ where m is an integer

Or : From GC table,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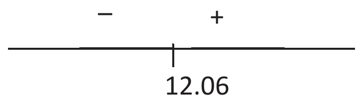
X	Y1
5	-1897
6	-1796
7	-1652
8	-1457
9	-1205
10	-888
11	-499
12	-30.8
13	523.8
14	1172
15	1921

X=13

$m \geq 13$ where m is an integer

Or : GC poly root finder



$m > 12.06$
$m \geq 13$ where m is an integer

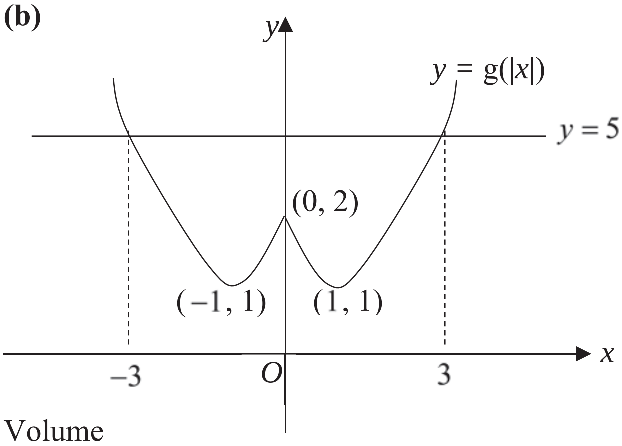
Q2

(a)

$$y = \frac{3x-1}{x-2} = 3 + \frac{5}{x-2}$$

- Translation of 2 units in the positive x direction.
- Scaling parallel to the y-axis by a scale factor of 5.
- Translation of 3 units in the positive y direction.

(b)



Volume

$$= 2\left[\pi(5)^2(3) - \pi\int_0^3 \left(x^2 - 2x + 2\right)^2 \mathrm{d}x\right]$$

$$= 373(3\text{sf})$$

or

Volume

$$= \pi(5)^2(6) - \pi\int_{-3}^{3} \left(|x|^2 - 2|x| + 2\right)^2 \mathrm{d}x$$

$$= 373(3\text{sf})$$

Note :

Cannot use

$$\pi(5)^2(6) - \pi\int_{-3}^{3} \left(x^2 - 2x + 2\right)^2 \mathrm{d}x$$

Cannot use

$$\pi(5)^2(6) - \pi\int_{-3}^{3} \left(|x|^2 - 2|x| + 2\right)^2 \mathrm{d}x$$ if question ask for exact.

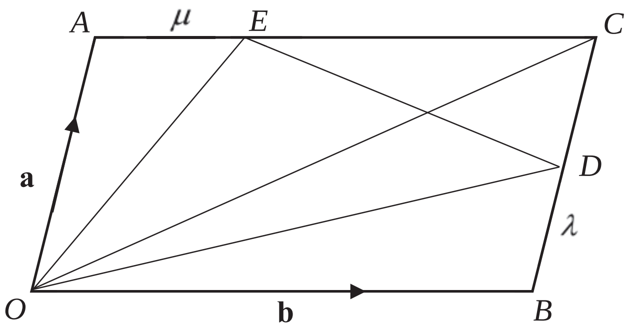
Q3
(i)

$\overrightarrow{OD} = \mathbf{b} + \lambda\mathbf{a}$

$\overrightarrow{OE} = \mathbf{a} + \mu\mathbf{b}$

area of triangle OCE

$$= \frac{1}{2}\left|\overrightarrow{OC} \times \overrightarrow{EC}\right|$$



$= \frac{1}{2}\left|(\mathbf{a}+\mathbf{b})\times(1-\mu)\mathbf{b}\right|$

$= \frac{1-\mu}{2}\left|\mathbf{a}\times\mathbf{b}\right|$ since $0<\mu<1$

area of triangle *ODE*

$= \frac{1}{2}\left|\overrightarrow{OD}\times\overrightarrow{OE}\right|$

$= \frac{1}{2}\left|(\mathbf{b}+\lambda\mathbf{a})\times(\mathbf{a}+\mu\mathbf{b})\right|$

$= \frac{1}{2}\left|(\mathbf{b}\times\mathbf{a})+\lambda\mu(\mathbf{a}\times\mathbf{b})\right|$

$= \frac{1}{2}\left|(\mathbf{b}\times\mathbf{a})-\lambda\mu(\mathbf{b}\times\mathbf{a})\right|$

$= \frac{1-\lambda\mu}{2}\left|\mathbf{b}\times\mathbf{a}\right|$

$= \frac{1-\lambda\mu}{2}\left|\mathbf{a}\times\mathbf{b}\right|$ since $0<\mu<1$ and $0<\lambda<1$

area of triangle *ODE* = *k* (area of triangle *OCE*)

$\frac{1-\lambda\mu}{2}\left|\mathbf{a}\times\mathbf{b}\right| = k\left(\frac{1-\mu}{2}\right)\left|\mathbf{a}\times\mathbf{b}\right|$

$1-\lambda\mu = k(1-\mu)$

$k = \frac{1-\lambda\mu}{1-\mu}$

Note :

- explanation $0<\mu<1$ and $0<\lambda<1$ should be seen.

- For area, must have modulus if written as $\lambda\mu-1$ and $\mu-1$. i.e $|\lambda\mu-1|$, $|\mu-1|$.

- For *k*, if written as $k = \left|\frac{1-\lambda\mu}{1-\mu}\right|$, must proceed to

$k = \frac{1-\lambda\mu}{1-\mu}$

or

$k = -\left(\frac{1-\lambda\mu}{1-\mu}\right)$ (rejected as $k>0$)

(ii)

Given $OF:FC = 6:1$ and $DF:FE = 3:4$.

By Ratio Theorem,

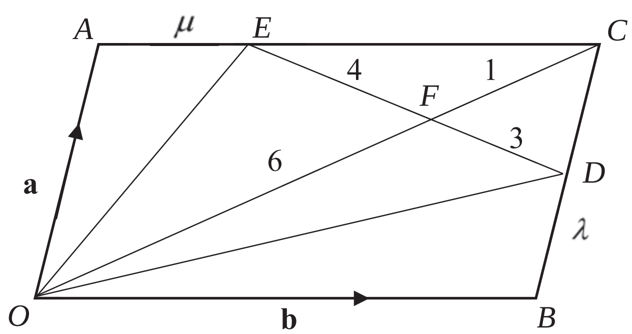
$\overrightarrow{OF} = \frac{3\overrightarrow{OE}+4\overrightarrow{OD}}{7}$

$\frac{6}{7}(\mathbf{a}+\mathbf{b}) = \frac{1}{7}\left[3(\mathbf{a}+\mu\mathbf{b})\right]+\frac{1}{7}\left[4(\mathbf{b}+\lambda\mathbf{a})\right]$

$\mathbf{a}$: $6 = 3+4\lambda \Rightarrow \lambda = \frac{3}{4}$

$\mathbf{b}$: $6 = 3\mu+4 \Rightarrow \mu = \frac{2}{3}$

$k = \frac{1-\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}$



Q4

(i)

Consider $y=\dfrac{x^2+5}{x-2}$ and the line $y=k$.

$k=\dfrac{x^2+5}{x-2}$

$kx-2k=x^2+5$

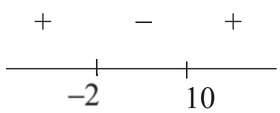
$x^2-kx+2k+5=0$

For $y=\dfrac{x^2+5}{x-2}$ and the line $y=k$ not to intersect,

$k^2-4(2k+5)<0$

$k^2-8k-20<0$

$(k-10)(k+2)<0$



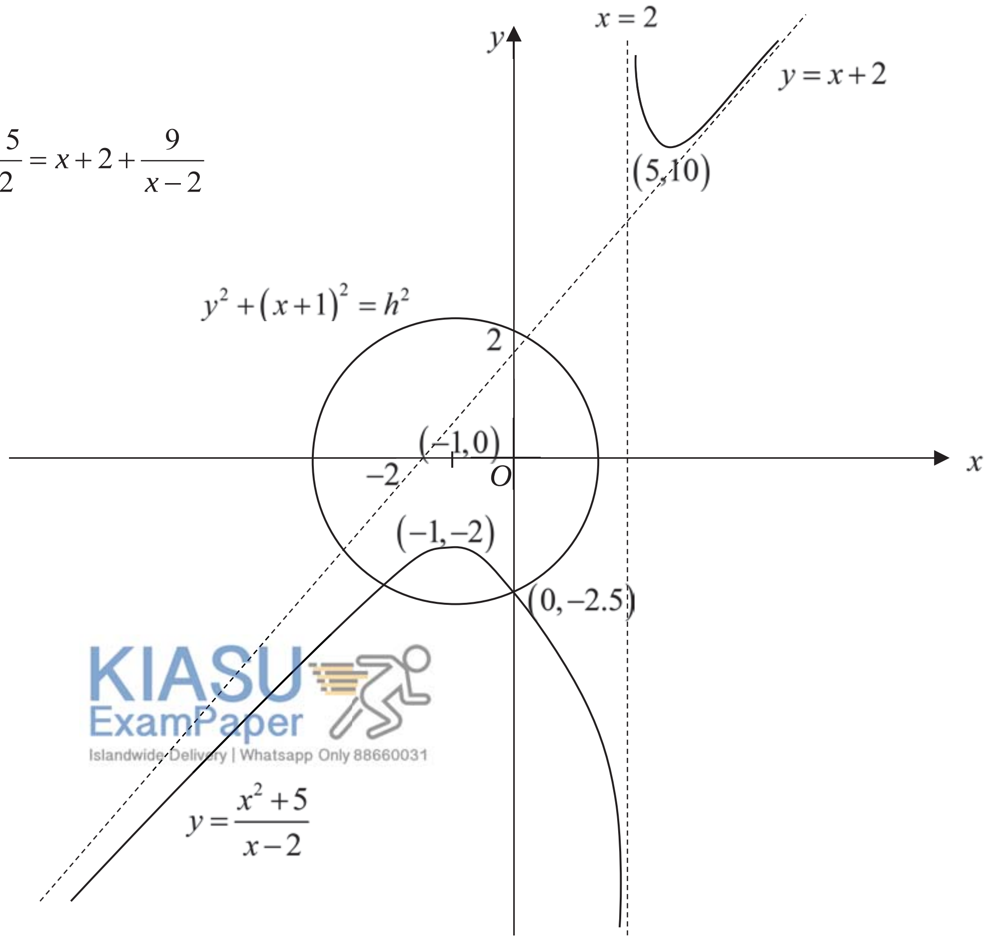
$-2<k<10$

$-2<y<10$

C cannot lie between -2 and 10.

(ii)

$y=\dfrac{x^2+5}{x-2}=x+2+\dfrac{9}{x-2}$



(iii)

$$\left(x^2+5\right)^2+\left(x+1\right)^2\left(x-2\right)^2=h^2\left(x-2\right)^2$$

$$\frac{\left(x^2+5\right)^2}{\left(x-2\right)^2}+\left(x+1\right)^2=h^2$$

$$y^2+\left(x+1\right)^2=h^2$$

Consider distance from centre of circle to y-intercept of C

$$|h|=\sqrt{\left(-1-0\right)^2+\left(0-\left(-2.5\right)\right)^2}=\frac{1}{2}\sqrt{29}\ \text{ or } 2.69$$

$$h^2>\frac{29}{4}$$

$$h<-\frac{1}{2}\sqrt{29}\ \text{ or }\ h>\frac{1}{2}\sqrt{29}$$

Q5

(a)(i)

$$w_1^{\ 3}=\left(-3+\sqrt{5}\,\mathrm{i}\right)^3$$

$$=\left(-3\right)^3+3\left(-3\right)^2\left(\left(\sqrt{5}\right)\mathrm{i}\right)+3\left(-3\right)\left(\left(\sqrt{5}\right)\mathrm{i}\right)^2+\left(\left(\sqrt{5}\right)\mathrm{i}\right)^3$$

$$=-27+27\sqrt{5}\,\mathrm{i}+45-5\sqrt{5}\,\mathrm{i}$$

$$=18+22\sqrt{5}\,\mathrm{i}$$

(ii)

Since $w_1=-3+\sqrt{5}\,\mathrm{i}$ is a root,

$$4w_1^{\ 3}+pw_1^{\ 2}+qw_1-14=0$$

$$4(18+22\sqrt{5}\,\mathrm{i})+p\left(-3+\sqrt{5}\,\mathrm{i}\right)^2+q\left(-3+\sqrt{5}\,\mathrm{i}\right)-14=0$$

$$4(18+22\sqrt{5}\,\mathrm{i})+p\left(9-6\sqrt{5}\,\mathrm{i}-5\right)+q\left(-3+\sqrt{5}\,\mathrm{i}\right)-14=0$$

$$72+4p-3q-14+\left(88\sqrt{5}-6\sqrt{5}p+\sqrt{5}q\right)\mathrm{i}=0$$

Comparing real and imaginary parts,

$4p-3q+58=0$ ---------(1)

$88\sqrt{5}-6\sqrt{5}p+\sqrt{5}q=0$

$\therefore 88-6p+q=0$------(2)

$(2)\times 3: 264-18p+3q=0$ ------(3)

Solving (1) and (3) , $14p=322$

$p=23$, $q=50$.

(iii)

Since $w_1 = -3+\sqrt{5}\,\mathrm{i}$ is a root, and polynomial equation has real coeffcients, $w_1^* = -3-\sqrt{5}\,\mathrm{i}$ is also a root.

$$\begin{aligned} 4w^3+23w^2+50w-14 &= \left(w-\left(-3+\sqrt{5}\,\mathrm{i}\right)\right)\left(w-\left(-3-\sqrt{5}\,\mathrm{i}\right)\right)g(w) \\ &= \left((w+3)+\sqrt{5}\,\mathrm{i}\right)\left((w+3)-\sqrt{5}\,\mathrm{i}\right)g(w) \\ &= \left((w+3)^2-\left(\sqrt{5}\,\mathrm{i}\right)^2\right)g(w) \\ &= \left(w^2+6w+14\right)(4w-1) \end{aligned}$$

When $4w^3+23w^2+50w-14=0$,

Therefore, other two roots are $-3-\sqrt{5}\,\mathrm{i}$ and $\frac{1}{4}$.

(b)

$$z=-1-\sqrt{3}\,\mathrm{i}=2\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}$$

$$\frac{z^*}{z^n}=\frac{2\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}}{2^n\mathrm{e}^{-\frac{2}{3}n\pi\mathrm{i}}}=2^{1-n}\mathrm{e}^{\frac{2}{3}\pi(n+1)\mathrm{i}}=2^{1-n}\left(\cos\left(\frac{2}{3}\pi(n+1)\right)+\mathrm{i}\sin\left(\frac{2}{3}\pi(n+1)\right)\right)$$

$\frac{z^*}{z^n}$ is imaginary: $\cos\left(\frac{2}{3}\pi(n+1)\right)=0$

$$\frac{2}{3}\pi(n+1)=\frac{(2k+1)}{2}\pi, \text{ where } k\in\mathbb{Z}$$

$$n=\frac{3k}{2}-\frac{1}{4}$$

Note :

Cosine is zero when

$\theta=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2}......$

$\pm1,\pm2,\pm3,......$ odd number

Q6

(i)

$$\mathrm{f}(x)=\mathrm{e}^{2x}-9\mathrm{e}^{-2x}$$

$$\mathrm{f}'(x)=2\mathrm{e}^{2x}+18\mathrm{e}^{-2x}$$

Since $\mathrm{e}^{2x}>0$ and $\mathrm{e}^{-2x}>0$ for all x, $\mathrm{f}'(x)=2\mathrm{e}^{2x}+18\mathrm{e}^{-2x}>0$ for all x.

(ii)

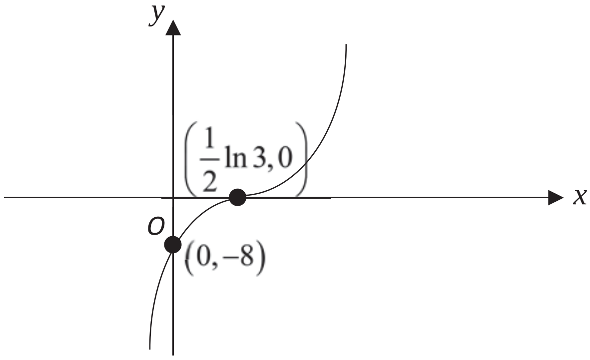
$$\mathrm{f}''(x)=4\mathrm{e}^{2x}-36\mathrm{e}^{-2x}$$

For $y=\mathrm{f}(x)$ to be concave upward, $\mathrm{f}''(x)=4\mathrm{e}^{2x}-36\mathrm{e}^{-2x}>0 \Rightarrow \mathrm{e}^{2x}-9\mathrm{e}^{-2x}>0$ -(1)

For $\mathrm{f}(x)>0$, $e^{2x}-9e^{-2x}>0$ which is the same as (1) (Shown)

$$\begin{aligned}
e^{2x}-9e^{-2x}&>0\\
e^{4x}-9&>0\\
e^{4x}&>9\\
x&>\frac{1}{4}\ln 9=\frac{1}{2}\ln 3
\end{aligned}$$

(iii)



(iv)

$$\int_0^2 \left|e^{2x}-9e^{-2x}\right| dx$$

$$=-\int_0^{\frac{1}{2}\ln 3} e^{2x}-9e^{-2x}\,dx+\int_{\frac{1}{2}\ln 3}^{2} e^{2x}-9e^{-2x}\,dx$$

$$=-\left[\frac{1}{2}e^{2x}+\frac{9}{2}e^{-2x}\right]_0^{\frac{1}{2}\ln 3}+\left[\frac{1}{2}e^{2x}+\frac{9}{2}e^{-2x}\right]_{\frac{1}{2}\ln 3}^{2}$$

$$=-\left[\left(\frac{1}{2}e^{2\left(\frac{1}{2}\ln 3\right)}+\frac{9}{2}e^{-2\left(\frac{1}{2}\ln 3\right)}\right)-\left(\frac{1}{2}e^{2(0)}+\frac{9}{2}e^{-2(0)}\right)\right]+\left[\left(\frac{1}{2}e^{2(2)}+\frac{9}{2}e^{-2(2)}\right)-\left(\frac{1}{2}e^{2\left(\frac{1}{2}\ln 3\right)}+\frac{9}{2}e^{-2\left(\frac{1}{2}\ln 3\right)}\right)\right]$$

$$=-\left[\frac{3}{2}+\frac{9}{2}\left(\frac{1}{3}\right)-\frac{1}{2}-\frac{9}{2}\right]+\left[\frac{1}{2}e^{4}+\frac{9}{2}e^{-4}-\frac{3}{2}-\frac{9}{2}\left(\frac{1}{3}\right)\right]$$

$$=-1+\frac{1}{2}e^{4}+\frac{9}{2}e^{-4}$$

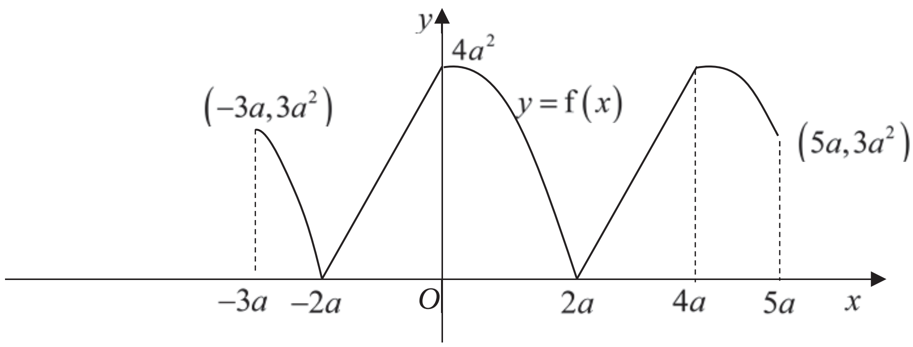
Q7

(i)

Since $\mathrm{f}(x)=\mathrm{f}(x+4a)$,

$\mathrm{f}(2019a)=\mathrm{f}(3a)=2a(3a-2a)=2a^2$

(ii)



(iii)

Since $R_f = \left[0, 4a^2\right] \not\subset D_g = (2a, 4a)$, gf does not exist.

(iv) Please note that g(2*a*) = 0, g(4*a*) = 2*a*, so f takes 0 to 2*a*

$$fg(x) = 4a^2 - \left(\sqrt{4a^2 - (x-2a)^2}\right)^2 = (x-2a)^2$$

$$fg : x \mapsto (x-2a)^2, \qquad 2a < x < 4a,$$

(v)

$$(fg)^{-1}(27) = \frac{7}{2}a$$

$$fg\left(\frac{7}{2}a\right) = 27$$

$$\left(\frac{7}{2}a - 2a\right)^2 = 27$$

$$\frac{9}{4}a^2 = 27$$

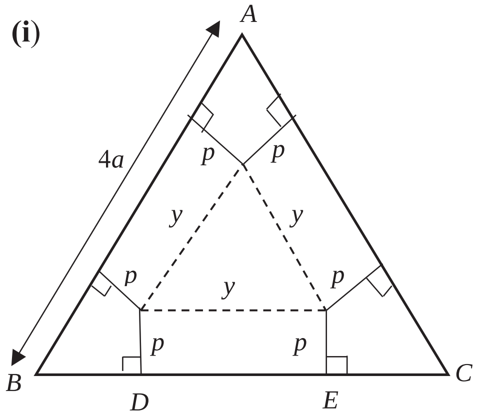
$$a^2 = 12$$

$$a = \pm 2\sqrt{3}$$

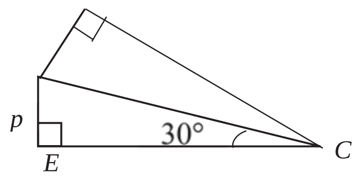
Since $a > 0$, $a = 2\sqrt{3}$

Q8

(i)



Since ABC is an equilateral triangle, $\angle ABC = \angle BCA = \angle CAB = 60^0$



length of BD = length of $CE = \dfrac{p}{\tan 30^0} = \sqrt{3}p$

Thus $y = 4a - 2\sqrt{3}p$

Volume of the prism, $V = \left(\dfrac{1}{2}y^2 \sin 60^0\right)p$

$$= \frac{1}{2}\left(4a - 2\sqrt{3}p\right)^2 \times \frac{\sqrt{3}}{2} \times p$$

$$= \frac{\sqrt{3}}{4}p\left(4a - 2\sqrt{3}p\right)^2$$

$$= \frac{\sqrt{3}}{4}p(2)^2\left(2a - \sqrt{3}p\right)^2$$

$$= \sqrt{3}p\left(2a - \sqrt{3}p\right)^2 \text{ cm}^3 \text{ (Shown)}$$

(ii)

$$\frac{\mathrm{d}V}{\mathrm{d}p} = \sqrt{3}\left(2a - \sqrt{3}p\right)^2 + 2\sqrt{3}p\left(2a - \sqrt{3}p\right)\left(-\sqrt{3}\right)$$

$$= 4\sqrt{3}a^2 - 12ap + 3\sqrt{3}p^2 - 12ap + 6\sqrt{3}p^2$$

$$= 9\sqrt{3}p^2 - 24ap + 4\sqrt{3}a^2$$

$$\frac{\mathrm{d}V}{\mathrm{d}p} = 0$$

$$9\sqrt{3}p^2 - 24ap + 4\sqrt{3}a^2 = 0$$

$$p = \frac{-(-24a) \pm \sqrt{(-24a)^2 - 4(9\sqrt{3})\left(4\sqrt{3}a^2\right)}}{2(9\sqrt{3})}$$

$$p = \frac{24a \pm \sqrt{144a^2}}{18\sqrt{3}} = \frac{2a}{3\sqrt{3}} \text{ or } \frac{2a}{\sqrt{3}}$$

When $p=\dfrac{2a}{3\sqrt{3}}$, $y=4a-2\sqrt{3}\left(\dfrac{2a}{3\sqrt{3}}\right)=4a-\dfrac{4}{3}a=\dfrac{8}{3}a>0$

When $p=\dfrac{2a}{\sqrt{3}}$, $y=4a-2\sqrt{3}\left(\dfrac{2a}{\sqrt{3}}\right)=4a-4a=0$ (NA as $y\neq 0$)

Or

When $p=\dfrac{2a}{3\sqrt{3}}$, $V=\sqrt{3}\left(\dfrac{2a}{3\sqrt{3}}\right)\left(2a-\sqrt{3}\left(\dfrac{2a}{3\sqrt{3}}\right)\right)^2=\dfrac{2a}{3}\left(2a-\dfrac{2a}{3}\right)^2=\dfrac{32}{27}a^3>0$

When $p=\dfrac{2a}{\sqrt{3}}$, $V=\sqrt{3}\left(\dfrac{2a}{\sqrt{3}}\right)\left(2a-\sqrt{3}\left(\dfrac{2a}{\sqrt{3}}\right)\right)^2=2a\left(2a-2a\right)^2=0$ (NA as $V\neq 0$)

Therefore, $p=\dfrac{2a}{3\sqrt{3}}=\dfrac{2\sqrt{3}a}{9}$ cm

(iii)

$\dfrac{\mathrm{d}V}{\mathrm{d}p}=9\sqrt{3}p^2-24ap+4\sqrt{3}a^2$

$\dfrac{\mathrm{d}^2V}{\mathrm{d}p^2}=18\sqrt{3}p-24a$

When $p=\dfrac{2a}{3\sqrt{3}}=\dfrac{2\sqrt{3}a}{9}$, $\dfrac{\mathrm{d}^2V}{\mathrm{d}p^2}=18\sqrt{3}\left(\dfrac{2a}{3\sqrt{3}}\right)-24a=12a-24a=-12a<0$ (max)

Hence, V is maximum when $p=\dfrac{2a}{3\sqrt{3}}=\dfrac{2\sqrt{3}a}{9}$

Maximum volume of the prism = $\dfrac{\sqrt{3}}{4}p\left(4a-2\sqrt{3}p\right)^2=\dfrac{32}{27}a^3$ cm^3

$\dfrac{3}{4}$ of the maximum volume = $\dfrac{3}{4}\times\dfrac{32}{27}\times a^2=\dfrac{8}{9}a^3$ cm^3

Maximum cost that the housewife has to pay = $\dfrac{8}{9}a^3\times 0.4=\dfrac{16}{45}a^3$ cents.

9

(a)

$$\int \frac{e^{\frac{1}{x}}}{x^2}\,\mathrm{d}x=-\int -\frac{1}{x^2}e^{\frac{1}{x}}\,\mathrm{d}x$$

$$=-e^{\frac{1}{x}}+C$$

(b)

$$\int \cos kx \cos(k+2)x \,\mathrm{d}x = \frac{1}{2}\int 2\cos(k+2)x\cos kx\,\mathrm{d}x$$

$$= \frac{1}{2}\int \left[\cos(2k+2)x + \cos 2x\right]\mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{\sin(2k+2)x}{2k+2} + \frac{\sin 2x}{2}\right] + C$$

(c)

$$u = \tan^{-1}(3x) \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{3}{1+9x^2} \qquad v = \frac{1}{2}x^2$$

$$\int x\tan^{-1}(3x)\,\mathrm{d}x$$

$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{3}{2}\int \frac{x^2}{1+9x^2}\,\mathrm{d}x$$

$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}\int 1 - \frac{1}{1+9x^2}\,\mathrm{d}x$$

$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}\int 1\,\mathrm{d}x + \frac{1}{6}\int \frac{1}{9\left(\frac{1}{9}+x^2\right)}\,\mathrm{d}x$$

$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}x + \frac{1}{18}\tan^{-1}(3x) + C$$

Q10
(a)(i)
Using AP, $v_n = 44\ 000 + (n-1)(335) = 43665 + 335n$

(ii)
Using $v_n > 53\ 500$

$43665 + 335n > 53\ 500$

$n > 29.358$

least $n = 30$
Average speed of 53 500 km/hr was reached in March 2001.

(iii)
$T =$ 1 month = 30(24) hours = 720 hours , 3 years = 36 months

Distance = $720\left[\frac{36}{2}\left(2(44000) + 35(335)\right)\right]$ = 1 292 436 000 km

(b)(i)
GP : 0.44, 0.44(2.03) , $0.44(2.03)^2$, ………

longest orbital period = $0.44(2.03)^{66} = 8.67 \times 10^{19}$ days

(ii)

$$\left|S_n - 0.44(2.03)^{19}\right| < 5\times10^6$$

$$\left|\frac{0.44(2.03^n - 1)}{2.03-1} - 0.44(2.03)^{19}\right| < 5\times10^6$$

Using GC (table) , largest $n = 23$

OR:

$$-5\times10^6 < \frac{0.44(2.03^n - 1)}{2.03-1} - 0.44(2.03)^{19} < 5\times10^6$$

$$-5\times10^6 + 306110.3422 < \frac{0.44\left(2.03^n\right)}{1.03} < 5\times10^6 + 306110.3422$$

Since $\dfrac{0.44\left(2.03^n\right)}{1.03}$ is always positive, $\dfrac{0.44\left(2.03^n\right)}{1.03} < 5\times10^6 + 306110.3422$

Solving, $n < 23.0707$

largest $n = 23$

Q1

$$a_{n+1} = a_n + 3^n - n$$

$$\sum_{r=0}^{n-1}\left(a_{r+1} - a_r\right) = \sum_{r=0}^{n-1}\left(3^r - r\right)$$

$$= \sum_{r=0}^{n-1} 3^r - \sum_{r=0}^{n-1} r$$

$$\begin{aligned}
&\cancel{a_1} - a_0 \\
&+\cancel{a_2} - \cancel{a_1} \\
&+\cancel{a_3} - \cancel{a_2} \\
&+ \\
&\quad\vdots \\
&+\cancel{a_{n-2}} - \cancel{a_{n-3}} = \frac{1(3^n - 1)}{3-1} - \frac{(n-1)n}{2} \\
&+\cancel{a_{n-1}} - \cancel{a_{n-2}} \\
&+a_n - \cancel{a_{n-1}}
\end{aligned}$$

$$a_n - a_0 = \frac{(3^n - 1)}{2} - \frac{(n-1)n}{2}$$

$$a_n = a_0 + \frac{(3^n - 1)}{2} - \frac{(n-1)n}{2}$$

$$= \frac{3}{5} - \frac{1}{2} + \frac{3^n}{2} - \frac{(n-1)n}{2}$$

$$= \frac{1}{10} + \frac{3^n}{2} - \frac{(n-1)n}{2}$$

Q2

(a)(i)

$$\ln(\cos 3x) \approx \ln\left(1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!}\right) = \ln\left(1 + \left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)\right)$$

$$\approx \left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right) - \frac{\left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)^2}{2} + \frac{\left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)^3}{3}$$

$$\approx \left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right) - \frac{1}{2}\left(\frac{81}{4}x^4 - \frac{243}{8}x^6\right) + \frac{1}{3}\left(-\frac{729}{8}x^6\right)$$

$$= -\frac{9}{2}x^2 - \frac{27}{4}x^4 - \frac{81}{5}x^6$$

(a)(ii)

Differentiating wrt x,

$$\frac{-3\sin 3x}{\cos 3x} \approx -9x - 27x^3 - \frac{486}{5}x^5$$

$$-3\tan 3x \approx -9x - 27x^3 - \frac{486}{5}x^5$$

$$\tan 3x \approx 3x + 9x^3 + \frac{162}{5}x^5$$

(b)

$$\frac{e^{\tan x}}{(2+x)^2}$$

$$= e^{\tan x}(2+x)^{-2}$$

$$= \frac{1}{4}e^{\tan x}\left(1+\frac{x}{2}\right)^{-2}$$

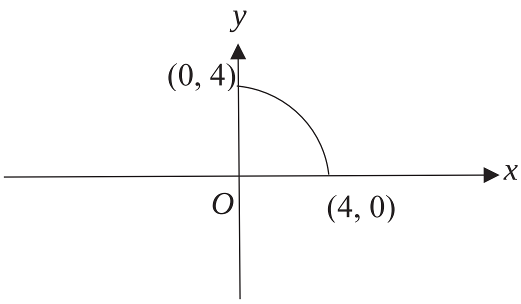
$$\approx \frac{1}{4}e^{x}\left(1+\frac{x}{2}\right)^{-2}$$

$$\approx \frac{1}{4}\left(1+x+\frac{1}{2}x^2\right)\left(1-x+\frac{3}{4}x^2\right)$$

$$= \frac{1}{4}\left(1+\frac{1}{4}x^2\right)$$

Q3

(i)



(ii)

$x = 4\sin 2t$ $\qquad$ $y = 4\cos 2t$

$\frac{dx}{dt} = 8\cos 2t$ $\qquad$ $\frac{dy}{dt} = -8\sin 2t$,

$\frac{dy}{dx} = -\tan 2t$

(a) For tangents parallel to y-axis ($\frac{dy}{dx}$ is undefined): $2t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{4}$

(b) For tangent parallel to x-axis ($\frac{dy}{dx} = 0$): $2t = 0 \Rightarrow t = 0$

(iii)

Given that gradient of tangent = $-\dfrac{1}{\sqrt{3}}$

$-\tan 2t = -\dfrac{1}{\sqrt{3}}$,

$2t = \dfrac{\pi}{6}$

$\therefore \quad t = \dfrac{\pi}{12}$

When $t = \dfrac{\pi}{12}$, $x = 4\sin 2\left(\dfrac{\pi}{12}\right) = 2$, $y = 4\cos 2\left(\dfrac{\pi}{12}\right) = 2\sqrt{3}$

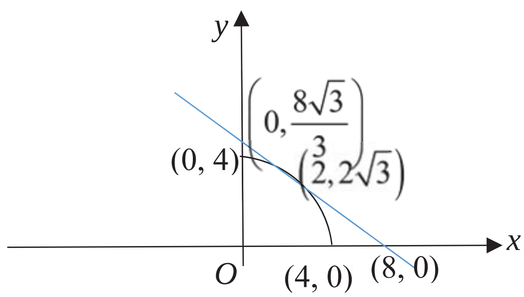
Equation of tangent at $P(2,\ 2\sqrt{3})$:

$y - 2\sqrt{3} = -\dfrac{1}{\sqrt{3}}(x-2)$

$y = -\dfrac{\sqrt{3}}{3}x + \dfrac{8}{3}\sqrt{3} = \dfrac{\sqrt{3}}{3}(8-x)$, where $a = \sqrt{3}$, $b = 8$.

(iv)

When $x = 0$, $y = \dfrac{8\sqrt{3}}{3}$. When $y = 0$, $x = 8$



Required area

= Area of the triangle – Area of the quadrant of the circle

$= \int_0^8 \dfrac{\sqrt{3}}{3}(8-x)\ \mathrm{d}x - \dfrac{1}{4}\pi(4)^2$

$= \left(\dfrac{1}{2}\times 8\times\dfrac{8\sqrt{3}}{3}\right) - \dfrac{1}{4}\pi(4)^2 = \left(\dfrac{32\sqrt{3}}{3} - 4\pi\right)$ units ^{2}

Q4

$\dfrac{\mathrm{d}}{\mathrm{d}u}\left[u^2\dfrac{\mathrm{d}x}{\mathrm{d}u}\right] = u^2\dfrac{\mathrm{d}^2x}{\mathrm{d}u^2} + 2u\dfrac{\mathrm{d}x}{\mathrm{d}u}$

$2u^2\dfrac{\mathrm{d}^2x}{\mathrm{d}u^2} + 4u\dfrac{\mathrm{d}x}{\mathrm{d}u} = 15u + 12$ becomes $2\dfrac{\mathrm{d}}{\mathrm{d}u}\left[u^2\dfrac{\mathrm{d}x}{\mathrm{d}u}\right] = 15u + 12$

Thus $u^2\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}\displaystyle\int 15u + 12\ \mathrm{d}u$

$u^2\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}\left[\dfrac{15u^2}{2} + 12u\right] + C$

$x = 0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}u} = 1$ when $u = 1$

$C = -\dfrac{35}{4}$

$u^2\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}\left[\dfrac{15u^2}{2} + 12u\right] - \dfrac{35}{4}$,

$$\frac{dx}{du}=\frac{1}{2}\left[\frac{15}{2}+\frac{12}{u}\right]-\frac{35}{4u^2}$$

$$x=\int\frac{1}{2}\left[\frac{15}{2}+\frac{12}{u}\right]-\frac{35}{4u^2}\ du$$

$$=\ \frac{15}{4}u+6\ln|u|+\frac{35}{4u}+D$$

$x=0$ when $u=1$

$\therefore\ D=-\frac{25}{2}$

$$x=\frac{15}{4}u+6\ln|u|+\frac{35}{4u}-\frac{25}{2}$$

Q5

(i)

Method 1

$$\begin{pmatrix}8\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}1\\2\\0\end{pmatrix}=8-8+0=0$$

$$\begin{pmatrix}8\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}2\\3\\-4\end{pmatrix}=16-12-4=0$$

Since direction vector of l is perpendicular to two vectors parallel to p, l is perpendicular to p.

Method 2

$$\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}-8\\4\\-1\end{pmatrix}=-\begin{pmatrix}8\\-4\\1\end{pmatrix}$$ which is parallel to l

l is perpendicular to p.

$$\begin{pmatrix}1\\0\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}-10\\0\\4\end{pmatrix}+t\begin{pmatrix}8\\-4\\1\end{pmatrix}$$

$x:\quad 1+\lambda+2\mu=-10+8t\Rightarrow\lambda+2\mu-8t=-11\quad(1)$

$y:\quad 2\lambda+3\mu=-4t\Rightarrow 2\lambda+3\mu+4t=0\quad(2)$

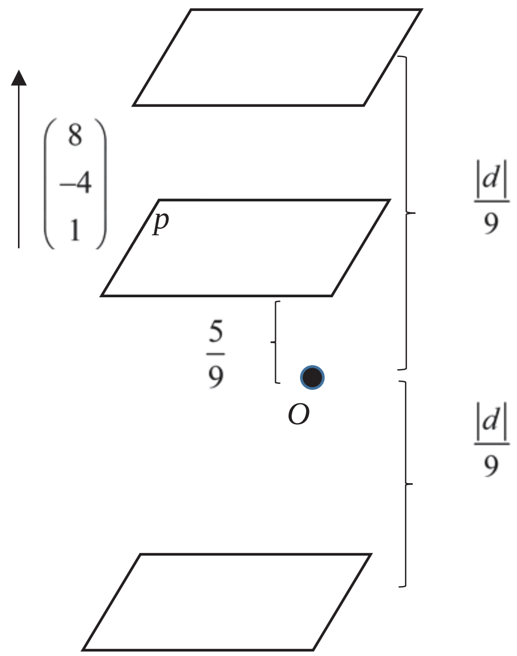
$z:\quad -3-4\mu=4+t\Rightarrow -4\mu-t=7\quad(3)$

$\lambda=1,\quad\mu=-2,\quad t=1$

(ii)

Method 1

Equation of p is $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=\begin{pmatrix}1\\0\\-3\end{pmatrix}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=5$



Let the equations of the required planes be $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=d$

Given distance between p and planes = 2

$\frac{|d-5|}{9}=2$

$|d-5|=18$

$d-5=18$ or $d-5=-18$

$d=23$ or $d=-13$

Cartesian equations of the required planes are

$8x-4y+z=23$ and $8x-4y+z=-13$

Method 2

Equation of p is $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=\begin{pmatrix}1\\0\\-3\end{pmatrix}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=5$

$\mathbf{r}\bullet\frac{1}{9}\begin{pmatrix}8\\-4\\1\end{pmatrix}=\frac{5}{9}$

Equations of the required planes are

$\mathbf{r}\bullet\frac{1}{9}\begin{pmatrix}8\\-4\\1\end{pmatrix}=\frac{5}{9}+2$ and $\mathbf{r}\bullet\frac{1}{9}\begin{pmatrix}8\\-4\\1\end{pmatrix}=\frac{5}{9}-2$

$\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=23$ and $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=-13$

Cartesian equations of the required planes are

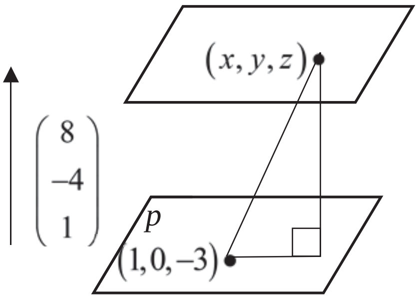
$8x-4y+z=23$ and $8x-4y+z=-13$

Method 3

Let a point on the required planes be (x, y, z) .

Consider point $(1, 0, -3)$ on *p*.

Given distance between *p* and planes = 2



$$\frac{1}{9}\left|\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}\right] \bullet \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}\right| = 2$$

$$\frac{1}{9}\left|\begin{pmatrix} x-1 \\ y \\ z+3 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}\right| = 2$$

$$|8x - 8 - 4y + z + 3| = 18$$

$$|8x - 4y + z - 5| = 18$$

$8x - 4y + z - 5 = 18$ or $8x - 4y + z - 5 = -18$

Cartesian equations of the required planes are

$8x - 4y + z = 23$ and $8x - 4y + z = -13$

Q6

(i)

Whether a box contains voucher is independent of any other boxes.

The probability of a box containing a voucher is constant.

(ii)

Let *X* be the number of vouchers obtained out of 9 boxes of Brand *A* cereal

$$X \sim \mathrm{B}(9, 0.35)$$

$$\mathrm{P}(X \le 3) = 0.60889 \approx 0.609$$

(iii)

Required probability = $\mathrm{P}(X = 7) \times 0.35 = 0.0034251 \approx 0.00343$

Alternative

Required probability = ${}^{9}C_{7}\,(0.35)^{8}\,(0.65)^{2} = 0.0034251 \approx 0.00343$

Let *W* be the number of vouchers obtained out of 10 boxes of Brand *B* cereal

$$W \sim \mathrm{B}(10, p)$$

$$\mathrm{P}(W \le 1) = 0.4845$$

$$\binom{10}{0} p^{0}(1-p)^{10} + \binom{10}{1} p^{1}(1-p)^{9} = 0.4845$$

$$(1-p)^{10} + 10p(1-p)^{9} = 0.4845$$

$$p = 0.16667 \approx 0.167$$

Q7

(i)
Central Limit Theorem states that sample means will follow a normal distribution approximately when sample size is big enough.

(ii)

$\sum(x-390)=120$ and $\sum(x-390)^2=3100$

$\bar{x}=\dfrac{120}{50}+390=392.4$

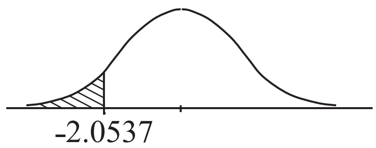
$s^2=\dfrac{1}{50-1}\left[3100-\dfrac{120^2}{50}\right]=57.3877551\approx 57.388$

$\mathrm{H}_0:\mu=m$ vs $\mathrm{H}_1:\mu<m$

Since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}(m, \dfrac{57.388}{50})$ approximately

Level of significance: 2%

Critical region: Reject H_0 when $z<-2.0537$



Consider $\dfrac{392.4-m}{\sqrt{\dfrac{57.388}{50}}}<-2.0537$

$m>394.6$ (5 s.f.)

The least possible value of m is 395grams.

Q8

(a)

Case 1: Last digit is 5

{2,4,6,8}					{5}

Number of different 6-digit numbers = $4\times 4! = 96$

Case 2: Last digit is 0

{2,4,5,6,8}					{0}

Number of different 6-digit numbers = $5! = 120$

$\therefore$ Required number of different 6-digit numbers = 96 + 120 = 216

(b)

Case 1: All letters are different.

Number of 4-letter code words = ${}^7C_4\times 4!=840$

Case 2: One pair of repeated letters.

Number of 4-letter code words = ${}^2C_1\times {}^6C_2\times\dfrac{4!}{2!}=360$

Case 3: Two pairs of repeated letters.

Number of 4-letter code words = $\dfrac{4!}{2!2!}=6$

$\therefore$ Total number of 4-letter code words = 840 + 360 + 6 = 1206

Q9

(i)

Let A be the mass of a Grade A strawberry

$$A \sim \mathrm{N}\left(18, 3^2\right)$$

$$\mathrm{P}(17 < A < 20) = 0.37806 \approx 0.378$$

(ii)

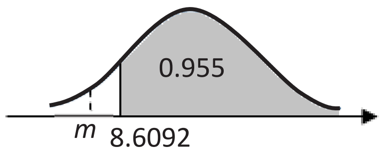
Let B be the mass of a Grade B strawberry

$$B \sim \mathrm{N}\left(12, 2^2\right)$$

$$\mathrm{P}(B > m) \geq 0.955$$

$$m \leq 8.6092$$

Greatest value of $m = 8.60$



(iii)

Let $X = \left(A_1 + \ldots + A_{12}\right) - \left(B_1 + \ldots + B_{15}\right)$

$$\mathrm{E}(X) = 12(18) - 15(12) = 36$$

$$\mathrm{Var}(X) = 12\left(3^2\right) + 15\left(2^2\right) = 168$$

$$X \sim \mathrm{N}(36, 168)$$

$$\mathrm{P}(X > 0) = 0.99726 \approx 0.997$$

(iv)

The masses of strawberries are independent of each other.

Q10

(i)

	$4n$-4 Cards	3 Cards	1 Card
Score (+)	**1**	**2**	**4**
1	2	3	5
2	3	4	6
4	5	6	Nil

$$\mathrm{P}(W = 10)$$

$$= \mathrm{P}(\text{Score} > 4)$$

$$= \mathrm{P}(1,4) + \mathrm{P}(4,1) + \mathrm{P}(2,4) + \mathrm{P}(4,2)$$

$$= \left(\frac{4n-4}{4n}\right)\left(\frac{1}{4n-1}\right) \times 2 + \left(\frac{3}{4n}\right)\left(\frac{1}{4n-1}\right) \times 2$$

$$= \frac{8n-8+6}{4n(4n-1)}$$

$$= \frac{1}{2n}$$

(ii)

$$\frac{1}{2n}=\frac{1}{8}$$
$$n=4$$

$$\begin{aligned}&\mathrm{P}(W=-2)\\&=\mathrm{P}(\text{Score}<4)\\&=\mathrm{P}(1,1)+\mathrm{P}(1,2)+\mathrm{P}(2,1)\\&=\left(\frac{12}{16}\right)\left(\frac{11}{15}\right)+\left(\frac{12}{16}\right)\left(\frac{3}{15}\right)\times 2\\&=\frac{17}{20}\end{aligned}$$

$$\begin{aligned}&\mathrm{P}(W=0)\\&=\mathrm{P}(\text{Score}=4)\\&=\mathrm{P}(2,2)\\&=\left(\frac{3}{16}\right)\left(\frac{2}{15}\right)\\&=\frac{1}{40}\end{aligned}$$

Hence, the probability distribution of W is

W	-2	0	10
$\mathrm{P}(W=w)$	$\frac{17}{20}$	$\frac{1}{40}$	$\frac{1}{8}$

(iii)

$$\begin{aligned}\mathrm{E}(W)&=(-2)\left(\frac{17}{20}\right)+(0)\left(\frac{1}{40}\right)+(10)\left(\frac{1}{8}\right)\\&=-\frac{9}{20}\end{aligned}$$

$$\begin{aligned}\mathrm{E}(W^2)&=(-2)^2\left(\frac{17}{20}\right)+(0)^2\left(\frac{1}{40}\right)+(10)^2\left(\frac{1}{8}\right)\\&=\frac{159}{10}\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(W)&=\mathrm{E}(W^2)-\left[\mathrm{E}(W)\right]^2\\&=\frac{159}{10}-\left(-\frac{9}{20}\right)^2\\&=\frac{6279}{400}\quad\text{or } 15.6975 \text{ (exact)}\end{aligned}$$

Since $\mathrm{E}(W)=-\frac{9}{20}<0$, it is expected that she will lose money. Hence, Kathryn should not play the game.

(iv)

Since $n = 50$ is large, by Central Limit Theorem, $\overline{W} \sim \mathrm{N}\left(-\frac{9}{20}, \frac{15.6975}{50}\right)$ approximately

$\mathrm{P}\left(\overline{W} \leq 1\right) = 0.99517 \approx 0.995$

Q11

(i)

Given $\mathrm{P}(A|B) = \frac{7}{10}$

$$\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{7}{10}$$

$\therefore\ \mathrm{P}(B) = \frac{10}{7}\mathrm{P}(A \cap B)$ -----(1)

Given $\mathrm{P}(B|A) = \frac{4}{15}$

$$\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{4}{15}$$

$\therefore\ \mathrm{P}(A) = \frac{15}{4}\mathrm{P}(A \cap B)$ -----(2)

Given that $\mathrm{P}(A \cup B) = \frac{3}{5}$

$\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{3}{5}$ -----(3)

Substitute (1) and (2) into (3):

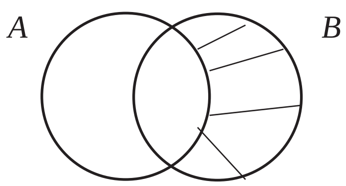
$$\frac{15}{4}\mathrm{P}(A \cap B) + \frac{10}{7}\mathrm{P}(A \cap B) - \mathrm{P}(A \cap B) = \frac{3}{5}$$

$\therefore\ \mathrm{P}(A \cap B) = \frac{28}{195}$

(ii)

$\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$$= \frac{10}{7}\left(\frac{28}{195}\right) - \frac{28}{195} = \frac{4}{65}$$



<u>Alternative Method 1:</u>

$\mathrm{P}(A' \cap B) = \mathrm{P}(A \cup B) - \mathrm{P}(A) = \frac{3}{5} - \left(\frac{15}{4} \times \frac{28}{195}\right) = \frac{4}{65}$

<u>Alternative Method 2:</u>

$\mathrm{P}(A'|B) = 1 - \mathrm{P}(A|B) = 1 - \frac{7}{10} = \frac{3}{10}$

$\therefore\ \mathrm{P}(A' \cap B) = \mathrm{P}(A'|B) \times \mathrm{P}(B) = \frac{3}{10} \times \left(\frac{10}{7} \times \frac{28}{195}\right) = \frac{4}{65}$

(iii)

$$P(A)=\frac{15}{4}\left(\frac{28}{195}\right)=\frac{7}{13}$$

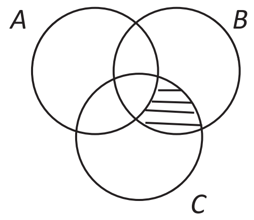
Since events A and C are independent,

$$P(A'\cap C)=P(A')\times P(C)=\left(1-\frac{7}{13}\right)\times\frac{3}{10}=\frac{9}{65}$$

(iv)

$P(A'\cap B\cap C)\le P(A'\cap C)$

$\therefore\ 0\le P(A'\cap B\cap C)\le\frac{9}{65}$

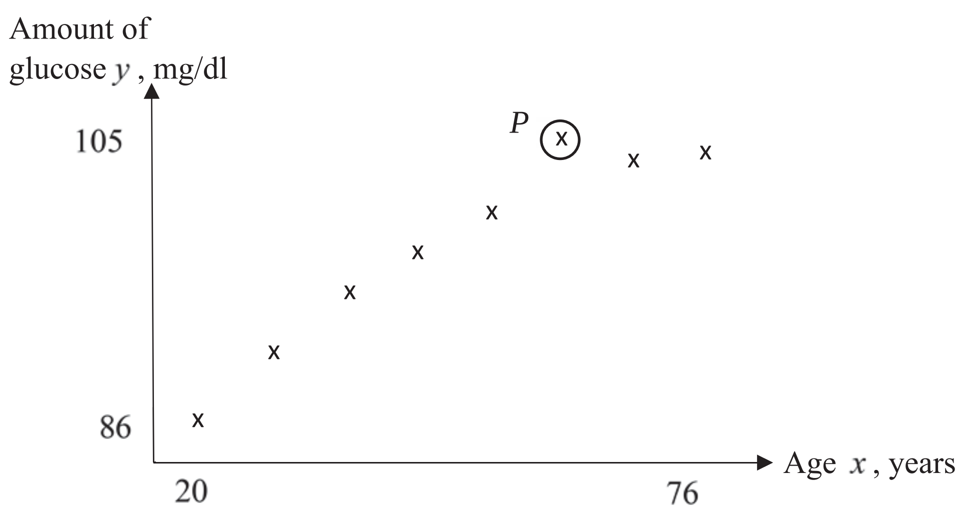


Q12

(i)

$r=0.954$. Since the r value is close to 1, there is a strong positive linear correlation between x and y. A linear model may be appropriate.

(ii) and (iii)



(iv)

From the scatter diagram, excluding P, it can be observed that as x increases, y also increases but at a decreasing rate, hence a linear model $y=ax+b$ may not be appropriate.

(v)

From GC,

$r=0.998$

$y=44.281+13.972\ln x$

$c=44.3$ $\quad$ $d=14.0$

(vi)

When $x=60$, $y=44.281+13.972\ln 60=101.5$ (nearest 0.5)

$r=0.998$ is close to 1. $x=60$ lies within the given data range, hence interpolation is being done. The linear model $y=a+b\ln x$ still holds, hence, the estimate is reliable.

(vii)

From GC, mean of $\ln x$ = 3.7864

$\bar{y} = 43.942 + 14.079(3.7864) = 97.251$

$\bar{y} = 97.251 = \frac{1}{8}(86.0 + 90.5 + 94.5 + 97.5 + 100 + 103.5 + 104 + y)$

$y = 102.0$ (nearest 0.5)

H

CANDIDATE NAME

CLASS

ADMISSION NUMBER

2019 Preliminary Exams
Pre-University 3

MATHEMATICS **9758/01**

Paper 1 **3 September 2019**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Qn No.	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	*	Total
Score													
Max Score	6	6	6	8	8	9	10	10	12	12	13		100

This document consists of **24** printed pages.

[Turn over

1 The nth term of a sequence $u_1,\ u_2,\ u_3,\ \ldots$ is given by $u_n = n^2 + \dfrac{1}{n!}$.

(i) Show that $u_n - u_{n-1} = 2n - 1 + \dfrac{1-n}{n!}$. [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{2n}\left(2r - 1 + \frac{1-r}{r!}\right)$. [3]

(iii) State, with a reason, if the series in part **(ii)** converges. [1]

2 It is given that

$$\mathrm{f}(x) = \begin{cases} 3 - x & \text{for } 0 < x \le 2, \\ \dfrac{1}{6}\left(x^2 + 2\right) & \text{for } 2 < x \le 4, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+4)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \le x \le 10$. [3]

(ii) Find the volume of revolution when the region bounded by the graph of $y = \mathrm{f}(x)$, the lines $x = -1$, $x = 2$ and the x-axis is rotated completely about the x-axis. [3]

3 **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos x^2\right)$. [2]

(ii) Hence find $\int x^3 \sin x^2 \ \mathrm{d}x$. [4]

4 **(a)** The curve C has equation $y = \mathrm{f}(x)$.

(i) Given that $\mathrm{f}(x) = \dfrac{x^2 + 3x + 4}{x+1}$, sketch the curve C, showing the equations of the asymptotes and the coordinates of any turning points and any points of intersection with the axes. [3]

(ii) Hence, state the range of values of x where $\mathrm{f}'(x) > 0$. [2]

(b) The curve with equation $y = \mathrm{g}(x)$ is transformed by a stretch with scale factor 2 parallel to the x-axis, followed by a translation of 1 unit in the negative x-direction and followed by a translation of 1 unit in the positive y-direction. The resulting curve has equation $y = \dfrac{x^2 + 3x + 4}{x+1}$. Find $\mathrm{g}(x)$. [3]

5 **(i)** On the same axes, sketch the graphs of $y=\dfrac{1}{|x-a|}$ and $y=-b(x-a)$, where a and b are positive constants and $ab>\dfrac{1}{a}$, stating clearly any axial intercepts and equations of any asymptotes. [3]

(ii) Given that the solution to the inequality $\dfrac{1}{|x-a|}>-b(x-a)$ is $\dfrac{1}{2}<x<1$ or $x>1$, find the values of a and b. [4]

(iii) Using the values of a and b found in part **(ii)**, write down the solution to the inequality $\dfrac{1}{x-a}>-b(x-a)$. [1]

6 **(i)** Given that $\mathrm{f}(x)=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ where a is a constant, find $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$ in terms of a. Hence write down the first two non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. Give the coefficients in terms of e. [5]

(ii) The first two non-zero terms in the Maclaurin series for $\mathrm{f}(x)$ are equal to the first two non-zero terms in the series expansion of $\dfrac{1}{\sqrt{b+x^2}}$, where b is a constant. By using appropriate expansions from the List of Formulae (MF26), find the possible values of a and b in terms of e. [4]

7 **(a)** Find the exact value of $\displaystyle\int_0^1 \frac{x}{2-x^2}\,\mathrm{d}x$. [3]

(b) The expression $\dfrac{x^2}{(2-x)^2}$ can be written in the form $A+\dfrac{B}{2-x}+\dfrac{C}{(2-x)^2}$.

(i) Find the values of A, B and C. [3]

(ii) Show that $\displaystyle\int_0^1 \frac{x^2}{(2-x)^2}\,\mathrm{d}x=p+q\ln 2$, where p and q are constants to be found. [4]

8 **(a)** Referred to the origin O, the point Q has position vector $\mathbf{q}$ such that

$$\mathbf{q} = 2\mathbf{i} - \frac{3}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}.$$

(i) Find the acute angle between $\mathbf{q}$ and the y-axis. [2]

It is given that a vector $\mathbf{m}$ is perpendicular to the xy-plane and its magnitude is 1.

(ii) With reference to the xy-plane, explain the geometrical meaning of $|\mathbf{q} \cdot \mathbf{m}|$ and state its value. [2]

(b) Referred to the origin O, the point R has position vector $\mathbf{r}$ given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where λ is a positive constant and $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. It is known that $\mathbf{c}$ is a non-zero vector that is not parallel to $\mathbf{a}$ or $\mathbf{b}$. Given that $\mathbf{c} \times \mathbf{a} = \lambda\mathbf{b} \times \mathbf{c}$, show that $\mathbf{r}$ is parallel to $\mathbf{c}$. [2]

It is also given that $\mathbf{a}$ is a unit vector that is perpendicular to $\mathbf{b}$ and $|\mathbf{b}| = 2$.

By considering $\mathbf{r} \cdot \mathbf{r}$, show that $|\mathbf{c}| = k\sqrt{4\lambda^2 + 1}$, where k is a non-zero constant. [4]

9 The function f is defined by $\mathrm{f} : x \mapsto 1 + 2\mathrm{e}^{-x^2}$, $x \in \mathbb{R}$.

(i) Show that f does not have an inverse. [2]

(ii) The domain of f is further restricted to $x \le k$, state the largest value of k for which the function f^{-1} exists. [1]

In the rest of the question, the domain of f is $x \in \mathbb{R}$, $x \le k$, with the value of k found in part **(ii)**.

(iii) Find $\mathrm{f}^{-1}(x)$. [3]

The function g has an inverse such that the range of g^{-1} is given by $(1, 3]$.

(iv) Explain clearly why the composite function gf exists. [2]

It is given that the composite function gf is defined by $\mathrm{gf}(x) = x$.

(v) State the domain and range of gf. [2]

(vi) By considering $\mathrm{gf}(-2)$, find the exact value of $\mathrm{g}^{-1}(-2)$. [2]

10 As a raindrop falls due to gravity, its mass decreases with time due to evaporation. The rate of change of the mass of a raindrop, m grams, with respect to time, t seconds, is a constant c.

(i) **(a)** Write down a differential equation relating m, t and c. State, with a reason, whether c is a positive or negative constant. [2]

(b) Initially the mass of a raindrop is 0.05 grams and, after a further 60 s, the mass of the raindrop is 0.004 grams. Find m in terms of t. [3]

In recent years, scientists are looking for alternative sources of sustainable energy to meet our energy needs. One approach aims to extract kinetic energy from rain to harvest energy. In order to test for the viability of this approach, scientists need to find the maximum kinetic energy that can be harvested from a falling raindrop.

It is known that the kinetic energy K, in millijoules, of an object falling from rest is given by

$$K = \frac{mg^2t^2}{2},$$

where m grams is the mass of the object at time t seconds and g is a positive constant known as the gravitational acceleration.

In the rest of the question, use your answer in part **(i)(b)** as the mass of a raindrop at time t.

(ii) **(a)** Show by differentiation that the maximum value of the kinetic energy of a raindrop falling from rest is pg^2, where p is a constant to be found . Give your answer correct to 2 decimal places. [5]

(b) It is given that $g = 10$. The kinetic energy of a raindrop falling from rest is 1 millijoules when it hits the surface of the ground. Given that it takes more than one minute for a raindrop to fall from rest to the surface of the ground, find the time taken for a raindrop to hit the surface of the ground. [2]

11 In this question, the distance is measured in metres and time is in seconds.

A radio-controlled airplane takes off from ground level and is assumed to be travelling at a steady speed in a straight line. The position vector of the airplane t seconds after it takes off is given by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ refers to the position vector of the point where it departs and $\mathbf{b}$ is known as its velocity vector.

(i) Given that the airplane reaches the point P with coordinates $(-9,\ 4,\ 6)$ after 6 seconds and its velocity vector is $-2\mathbf{i}+\mathbf{j}+\mathbf{k}$, find the coordinates of the point where it departs. [1]

Paul stands at the point C with coordinates $(-7,\ 5,\ 2)$ to observe the airplane.

(ii) Find the shortest distance from Paul's position to the flight path of the airplane. [3]

While the airplane is in the air, a drone is seen flying at a steady speed in a straight line with equation

$$\frac{2x-1}{-6}=\frac{y+7}{4}=\frac{z-10}{k}.$$

(iii) Show that the equation of the flight path of the drone can be written as $\mathbf{r}=\mathbf{p}+\lambda\mathbf{q}$, where λ is a non-negative constant and $\mathbf{p}$ and $\mathbf{q}$ are vectors to be determined, leaving your answer in terms of k. [1]

(iv) Given that the flight paths of the radio-controlled airplane and the drone intersect, find k. [3]

At P, the airplane suddenly changes its speed and direction. The position vector of the airplane s seconds after it leaves P is given by

$$\mathbf{r}=\begin{pmatrix}-9\\4\\6\end{pmatrix}+s\begin{pmatrix}3\\2\\-1\end{pmatrix},\ \text{where } s\in\mathbb{R},\ s\geq 0.$$

It travels at a steady speed in a straight line towards an inclined slope, which is assumed to be a plane with equation

$$x-y+7z=2.$$

(v) Determine if the new flight path is perpendicular to the inclined slope. [2]

(vi) The airplane eventually collides with the slope. Find the coordinates of the point of collision. [3]

End of Paper

H

CANDIDATE NAME

CLASS

ADMISSION NUMBER

2019 Preliminary Exams
Pre-University 3

MATHEMATICS **9758/02**

Paper 2 **18 September 2019**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
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You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Qn No.	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	*	Total
Score													
Max Score	**3**	**7**	**9**	**9**	**12**	**5**	**8**	**11**	**11**	**12**	**13**		**100**

This document consists of **25** printed pages.

[Turn over

Section A: Pure Mathematics [40 marks]

1 It is known that the nth term of a sequence is given by

$$p_n = 3^{n-1} + a,$$

where a is a constant.

Find $\sum_{r=3}^{n} p_r$. [3]

2 **(i)** An architect places 25 rectangular wooden planks in a row as a design for part of the facade of a building. The lengths of the first 18 planks form an arithmetic progression and the first plank has length 4 m. Given that the sum of the first three planks is 11.46 m, find the length of the 18th plank. [2]

(ii) For the last 7 wooden planks, each plank has a length that is $\frac{5}{4}$ of the length of the previous plank. Find the length of the 25th plank. [2]

(iii) The even-numbered planks (2nd plank, 4th plank, 6th plank and so on) are painted blue. Find the total length of the planks that are painted blue. [3]

3 **(a)** Find the general solution for the following differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{e}^{-5x+3} + \sin x .$$ [3]

(b) By using the substitution $z = x + \frac{\mathrm{d}y}{\mathrm{d}x}$, show that the following differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - x + 1 = 0$$

can be reduced to

$$\frac{\mathrm{d}z}{\mathrm{d}x} = z .$$ [2]

Hence, given that when $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, find y in terms of x. [4]

4 The curve C has parametric equations

$$x = t^3 - t^2, \quad y = t^2 + 2t - 3,$$

where $t \geq -1$.

(i) Sketch the curve C, stating clearly any axial intercepts and the coordinates of any end-points. [2]

The point P on the curve C has parameter $t = 2$.

(ii) Without using a calculator, find the equation of tangent to C at P. [3]

(iii) Without using a calculator, find the area of the region bounded by C, the tangent to C at P and the y-axis. [4]

5 **(a)** One of the roots of the equation $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ is bi, where a and b are positive real constants.

(i) Find the values of a and b and hence find the other roots of the equation. [4]

(ii) Deduce the roots of the equation $w^4 + 2w^3 + aw^2 + 8w + 40 = 0$. [2]

(b) **(i)** The complex number $-6 + \left(2\sqrt{3}\right)\mathrm{i}$ is denoted by w. Without using a calculator, find an exact expression of w^n in modulus-argument form. [3]

(ii) Hence find the two smallest positive integers n such that $w^n w^*$ is purely imaginary. [3]

Section B: Probability and Statistics [60 marks]

6 The government of Country X uses a particular method to create a unique Identification Number (IN) for each of its citizens. The IN consists of 4 digits from 0 to 9 followed by one of the ten letters A – J. The digits in the IN can be repeated.

(i) Find the number of different Identification Numbers that can be created. [1]

Suppose the letter at the end of the IN is determined by the following steps:

Step 1: Find the sum of all the digits in the IN.

Step 2: Divide the sum by 10 and note the remainder.

Step 3: Use the table below to determine the letter that corresponds to the remainder.

Remainder	0	1	2	3	4	5	6	7	8	9
Letter	A	B	C	D	E	F	G	H	I	J

(ii) It is known that all citizens who are born in the year 1990 must have the digit '9' appearing twice in their IN and the sum of all the digits in their IN is at least 20. Eric, who is born in the year 1990, has 'I' as the last letter of his IN. Find the number of INs that could possibly belong to Eric. [4]

7 In a carnival, a player begins a game by rolling a fair 12-sided die which consists of 3 red faces, 7 blue faces and 2 white faces. When the die thrown comes to rest, the colour of the uppermost face of the die is noted. If the colour of the uppermost face is red, a ball is picked from box A that contains 3 red and 4 blue balls. If the colour of the uppermost face is blue, a ball is picked from box B that contains 2 red, 3 blue and 3 green balls. Otherwise, the game ends. A mystery gift is only given when the colour of the uppermost face of the die is the same as the colour of the ball picked.

It is assumed that Timothy plays the game only once.

(i) Find the probability that the colour of the uppermost face of the die is blue and a red ball is picked. [1]

(ii) Find the probability the mystery gift is given to Timothy. [2]

(iii) Given that Timothy did not win the mystery gift, find the probability that a red ball is picked. [3]

One of the rules of the game is changed such that if the colour of the uppermost face is white when the die thrown comes to rest, the player gets to roll the die again. The other rules of the game still hold.

Alicia plays the game once. Find the probability that she obtains the mystery gift in the end. [2]

8 **(a)** Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A), (B) and (C) below. In each case your diagram should include 6 points, approximately equally spaced with respect to x, with all x- and y-values positive. The letters a, b, c, d, e and f represent constants.

(A) $y = a + bx^2$, where a and b are positive.

(B) $y = c + d \ln x$, where c is positive and d is negative.

(C) $y = e + \dfrac{f}{x}$, where e is positive and f is negative. [3]

(b) The following table shows the population of a certain country, P, in millions, at various times, t years after the year 1990.

t	4	10	12	14	15	18	20	24
P	3.65	3.89	3.95	4.02	4.15	4.30	4.45	4.95

(i) Draw the scatter diagram for these values, labelling the axes. Give a reason why a linear model may not be appropriate. [2]

(ii) Explain which of the three cases in part **(a)** is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. [2]

(iii) A statistician wants to predict the population of the country in the year 2020. Use the case that you identified in part **(b)(ii)** to find the equation of a suitable regression line, and use your equation to find the required prediction. [3]

(iv) Comment on the reliability of the statistician's prediction. [1]

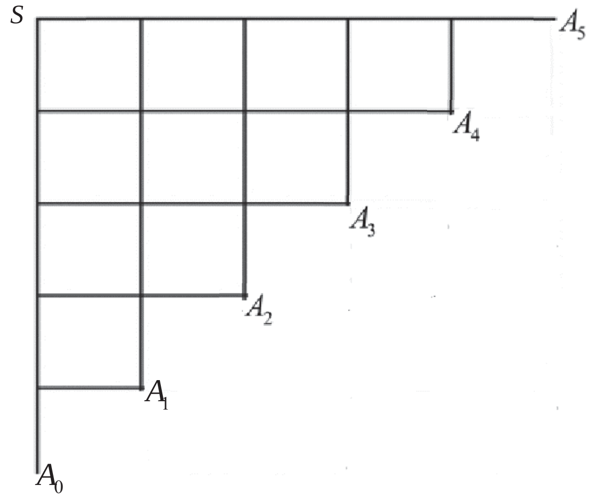
9 A chemist is conducting experiments to analyse the amount of active ingredient A in a particular type of health supplement tablets. Each tablet is said to contain an average amount of 50 mg of active ingredient A. A random sample of 40 tablets is taken and the amount of active ingredient A per tablet is recorded. The sample mean is 50.6 mg and the sample variance is 2.15 mg^2 respectively.

(i) Explain the meaning of 'a random sample' in the context of the question. [1]

(ii) Test, at the 1% level of significance, whether the mean amount of active ingredient A in a tablet has changed. You should state your hypotheses and define any symbols you use. [6]

In a revised formula of the same type of health supplement tablets, the amount of active ingredient A now follows a normal distribution with population variance 1.5 mg^2. The chemist wishes to test his claim that the mean amount of active ingredient A per tablet is more than 50 mg. A second random sample of n such tablets is analysed and its mean is found to be 50.4 mg. Find the set of values that n can take such that the chemist's claim is valid at 2.5% level of significance. [4]

10



A particle moves one step each time either to the right or downwards through a network of connected paths as shown above. The particle starts at S, and, at each junction, randomly moves one step to the right with probability p, or one step downwards with probability q, where $q = 1 - p$. The steps taken at each junction are independent. The particle finishes its journey at one of the 6 points labelled A_i, where $i = 0,\ 1,\ 2,\ 3,\ 4,\ 5$ (see diagram). Let $\{\ X = i\ \}$ be the event that the particle arrives at point A_i.

(i) Show that $\mathrm{P}(X = 2) = 10p^2q^3$. [2]

(ii) After experimenting, it is found that the particle will end up at point A_2 most of the time. By considering the mode of X or otherwise, show that $\dfrac{1}{3} < p < \dfrac{1}{2}$. [4]

The above setup is a part of a two-stage computer game.

- If the particle lands on A_0, the game ends immediately and the player will not win any points.
- If the particle lands on A_i, where i = 2 or 4, then the player gains 2 points.
- If the particle lands on A_i, where i = 1, 3 or 5, then the player proceeds on to the next stage, where there is a probability of 0.4 of winning the stage. If he wins the stage, he gains 5 points; otherwise he gains 3 points.

Let Y be the number of points gained by the player when one game is played.

(iii) If $p = 0.4$, determine the probability distribution of Y. [4]

(iv) Hence find the expectation and variance of Y. [2]

11 The two most popular chocolates sold by the *Dolce* chocolatier are the dark truffles and salted caramel ganaches and their masses have independent normal distributions. The masses, in grams, of dark truffles have the distribution $\mathrm{N}(17, 1.3^2)$.

(i) Find the probability that the total mass of 4 randomly chosen dark truffles is more than 70 g. [2]

(ii) The dark truffles are randomly packed into boxes of 4. In a batch of 20 boxes, find the probability that there are more than 3 boxes of dark truffles that have a mass more than 70 g. State an assumption you made in your calculations. [4]

The masses of salted caramel ganaches are normally distributed such that the proportion of them having a mass less than 12 grams is the same as the proportion of them having a mass greater than 15 grams. It is also given that 97% of the salted caramel ganaches weigh at most 15 grams.

(iii) Find the mean and variance of the masses of salted caramel ganaches. [3]

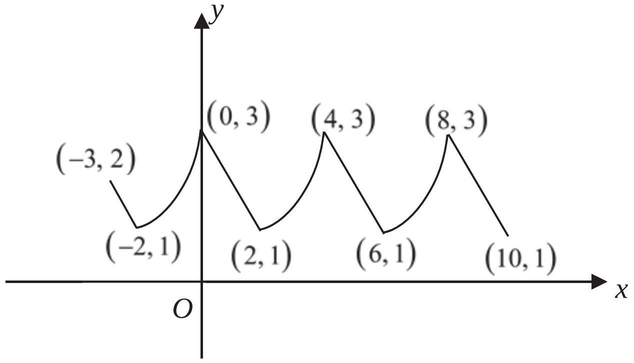
Dark truffles are sold at \$0.34 per gram and the salted caramel ganaches are sold at \$0.28 per gram.

(iv) Find the probability that the total cost of 6 randomly chosen salted caramel ganaches is less than the total cost of 4 randomly chosen dark truffles. State the distribution you use and its parameters. [4]

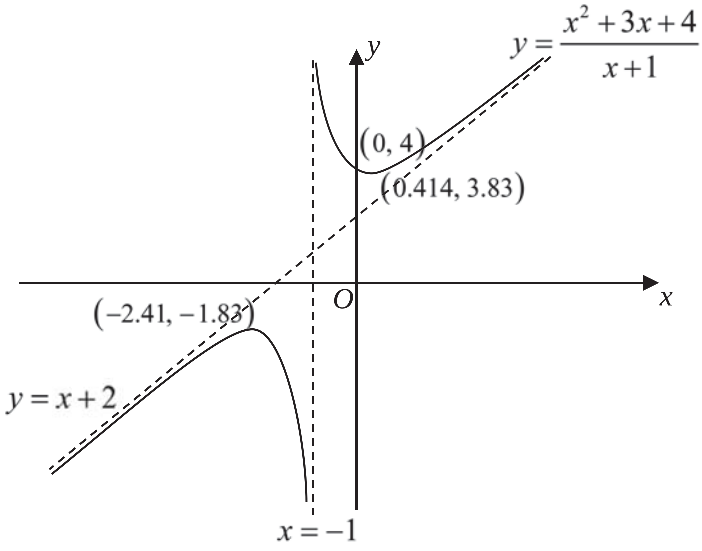
End of Paper

Solution to Paper 9758/01

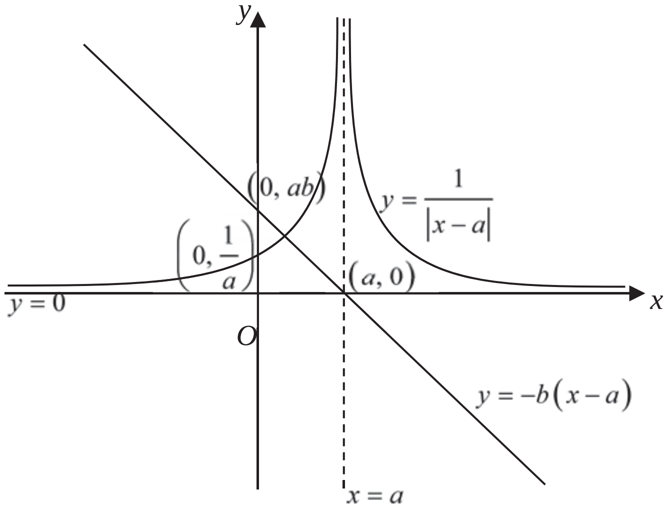
Qn	Solution
1(i)	$u_n = n^2 + \frac{1}{n!}$ $\text{LHS} = u_n - u_{n-1}$ $= n^2 + \frac{1}{n!} - \left((n-1)^2 + \frac{1}{(n-1)!}\right)$ $= n^2 + \frac{1}{n!} - n^2 + 2n - 1 - \frac{1}{(n-1)!}$ $= 2n - 1 + \frac{1}{n!} - \frac{n}{(n-1)! \times n}$ $= 2n - 1 + \frac{1-n}{n!}$ $= \text{RHS}$
1(ii)	$\sum_{r=2}^{2n} 2r - 1 + \frac{1-r}{r!}$ $= \sum_{r=2}^{2n} u_r - u_{r-1}$ $= \quad u_2 - u_1$ $+ \ u_3 - u_2$ $+ \ u_4 - u_3$ $+ \ \ldots$ $+ \ u_{2n-2} - u_{2n-3}$ $+ \ u_{2n-1} - u_{2n-2}$ $+ \ u_{2n} - u_{2n-1}$ $= u_{2n} - u_1$ $= (2n)^2 + \frac{1}{(2n)!} - (1+1)$ $= 4n^2 - 2 + \frac{1}{(2n)!}$
1(iii)	As $n \to \infty$, $4n^2 - 2 \to \infty$, $\frac{1}{(2n)!} \to 0$. Hence $\sum_{r=2}^{2n} 2r - 1 + \frac{1-r}{r!}$ does not converge. It diverges.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>2(i)</td><td>
</td></tr>
<tr><td>2(ii)</td><td>$$\text{Volume} = \pi\int_0^2 (3-x)^2\ \mathrm{d}x + \pi\int_3^4 \left(\frac{x^2+2}{6}\right)^2\ \mathrm{d}x$$
$$= 45.5$$</td></tr>
<tr><td>3(i)</td><td>$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x^2 = -2x\sin x^2$$</td></tr>
<tr><td>3(ii)</td><td>$$\int x^3 \sin x^2\ \mathrm{d}x$$
$$= -\frac{1}{2}\int\left(-2x\sin x^2\right)\left(x^2\right)\ \mathrm{d}x$$
$$= -\frac{1}{2}\left[x^2\cos x^2 - \int (2x)\left(\cos x^2\right)\ \mathrm{d}x\right]$$
$$= -\frac{x^2}{2}\cos x^2 + \int x\left(\cos x^2\right)\ \mathrm{d}x$$
$$= -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\int 2x\left(\cos x^2\right)\ \mathrm{d}x$$
$$= -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2 + c$$
$u = x^2 \quad v' = -2x\sin x^2$
$u' = 2x \quad v = \cos x^2$

Note:
$\frac{\mathrm{d}}{\mathrm{d}x}\sin x^2 = 2x\cos x^2$</td></tr>
</table>

Qn	Solution
4a(i)	$y=\dfrac{x^2+3x+4}{x+1}$ $=x+2+\dfrac{2}{x+1}$ Vertical asymptote: $x=-1$, Oblique asymptote: $y=x+2$ 
4a(ii)	$x<-2.41$ or $x>0.414$
4(b)	**Method 1a:** Applying transformations backward We are given the transformations in the order of A, B and C, and the resulting curve $y=\dfrac{x^2+3x+4}{x+1}$. In order to find the original curve $y=\mathrm{g}(x)$, we use our resulting curve, apply the C', B' then A', where A', B' and C' are the "opposites" of A, B and C respectively. Original Curve $\underset{A'}{\overset{A}{\rightleftarrows}}$ ♠ $\underset{B'}{\overset{B}{\rightleftarrows}}$ ♥ $\underset{C'}{\overset{C}{\rightleftarrows}}$ Resulting Curve, where ♠ and ♥ represent intermediate curves in the entire process. Hence, we have C': Translate 1 unit in negative *y*-direction (Replace *y* by $y+1$); B': Translate 1 unit in positive *x*-direction (Replace *x* by $x-1$); A': Stretch with scale factor $\dfrac{1}{2}$ parallel to *x*-axis (Replace *x* by $2x$).

Qn	Solution
	$y=\frac{x^2+3x+4}{x+1}\xrightarrow[C']{\text{Replace } y \text{ by } y+1} y+1=\frac{x^2+3x+4}{x+1}$ (i.e. $y=\frac{x^2+2x+3}{x+1}$) $y=\frac{x^2+2x+3}{x+1}\xrightarrow[B']{\text{Replace } x \text{ by } x-1} y=\frac{(x-1)^2+2(x-1)+3}{(x-1)+1}$ $=\frac{x^2-2x+1+2x-2+3}{x}$ $=\frac{x^2+2}{x}$ $=x+\frac{2}{x}$ $y=x+\frac{2}{x}\xrightarrow[A']{\text{Replace } x \text{ by } 2x} y=2x+\frac{2}{2x}$ $=2x+\frac{1}{x}=\mathrm{g}(x)$ **Method 1b:** Applying transformations backward **Note that:** Hence, we have C': Translate 1 unit in negative *y*-direction (Replace *y* by $y+1$); B': Translate 1 unit in positive *x*-direction (Replace *x* by $x-1$); A': Stretch with scale factor $\frac{1}{2}$ parallel to *x*-axis (Replace *x* by $2x$). Note that $y=\frac{x^2+3x+4}{x+1}=x+2+\frac{2}{x+1}$ $y=x+2+\frac{2}{x+1}\xrightarrow[C']{\text{Replace } y \text{ by } y+1} y+1=x+2+\frac{2}{x+1}$ (i.e. $y=x+1+\frac{2}{x+1}$) $y=x+1+\frac{2}{x+1}\xrightarrow[B']{\text{Replace } x \text{ by } x-1} y=(x-1)+1+\frac{2}{(x-1)+1}$ $=x+\frac{2}{x}$ $y=x+\frac{2}{x}\xrightarrow[A']{\text{Replace } x \text{ by } 2x} y=2x+\frac{2}{2x}$ $=2x+\frac{1}{x}=\mathrm{g}(x)$ • **Method 2:** Start from the original curve $y=\mathrm{g}(x)$ Original Curve $y=\mathrm{g}(x)\xrightarrow{A}\spadesuit\xrightarrow{B}\heartsuit\xrightarrow{C}$ Resulting Curve $y=\frac{x^2+3x+4}{x+1}$

Qn	Solution
	A: Stretch with scale factor 2 parallel to x-axis (Replace x by $\frac{1}{2}x$); B: Translate 1 unit in negative x-direction (Replace x by $x+1$); C: Translate 1 unit in positive y-direction (Replace y by $y-1$). $y=\mathrm{g}(x) \xrightarrow[\text{by } \frac{1}{2}x]{\text{Replace } x} y=\mathrm{g}\left(\frac{x}{2}\right) \xrightarrow[\text{by } x+1]{\text{Replace } x} y=\mathrm{g}\left(\frac{x+1}{2}\right) \xrightarrow[\text{by } y\text{-}1]{\text{Replace } y} y-1=\mathrm{g}\left(\frac{x+1}{2}\right)$ $y=\mathrm{g}\left(\frac{x+1}{2}\right)+1$ Let $\mathrm{g}\left(\frac{x+1}{2}\right)+1=\frac{x^2+3x+4}{x+1}$ $\mathrm{g}\left(\frac{x+1}{2}\right)+1=x+2+\frac{2}{x+1}$ $\mathrm{g}\left(\frac{x+1}{2}\right)=x+1+\frac{2}{x+1}$ Note that we need to progress till we reach $\mathrm{g}(x)$. To do that, we need to replace x by $2x-1$ in $\mathrm{g}\left(\frac{x+1}{2}\right)=x+1+\frac{2}{x+1}$. $\mathrm{g}\left(\frac{(2x-1)+1}{2}\right)=(2x-1)+1+\frac{2}{(2x-1)+1}$ $\therefore \mathrm{g}(x)=2x+\frac{1}{x}$
5(i)	
5(ii)	**Method 1**: From graph in **(i)**, one of the solution to $\frac{1}{\lvert x-a\rvert}>-b(x-a)$ is $x>a$. By observation, since $x>1$, $a=1$

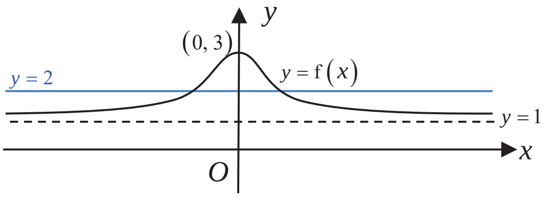
Qn	Solution
	When $x=\frac{1}{2}, y=\frac{1}{\left\|\frac{1}{2}-1\right\|}=2$ Subst $x=\frac{1}{2}$ and $y=2$ into $y=-b(x-1)$ $2=-b\left(\frac{1}{2}-1\right)$ $b=4$ ----- **<u>Method 2</u>**: From graph in **(i)**, one of the solution to $\frac{1}{\|x-a\|}>-b(x-a)$ is $x>a$. By observation, since $x>1$, $a=1$ To find the *x*-coordinate of the point of intersection: $\frac{1}{-(x-1)}=-b(x-1)$ $(x-1)^2=\frac{1}{b}$ $x-1=\pm\frac{1}{\sqrt{b}}$ When $x=\frac{1}{2}$, $\frac{1}{2}-1=\pm\frac{1}{\sqrt{b}}$ $-\frac{1}{2}=-\frac{1}{\sqrt{b}}$ or $-\frac{1}{2}=\frac{1}{\sqrt{b}}$ (rej. since $\sqrt{b}>0$) $\sqrt{b}=2$ $b=4$
5(iii)	$x>1$
6(i)	**<u>Method 1</u>**: $\mathrm{f}(x)=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ $\mathrm{f}'(x)=a\cos\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ $\mathrm{f}''(x)=a^2\cos^2\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}+a\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}\left[-a\sin\left(ax+\frac{\pi}{2}\right)\right]$ $\mathrm{f}''(x)=a^2\cos^2\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}-a^2\sin\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ ----

Qn	Solution
	<u>Method 2</u>: Let $y=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ $\Rightarrow \ln y=\sin\left(ax+\frac{\pi}{2}\right)$ Differentiate w.r.t. x: $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=a\cos\left(ax+\frac{\pi}{2}\right)$ Differentiate w.r.t. x: $\frac{1}{y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(-\frac{1}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-a^2\sin\left(ax+\frac{\pi}{2}\right)$ --- **<u>Method 3:</u>** $\mathrm{f}(x)=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}=\mathrm{e}^{\sin ax\cos\frac{\pi}{2}+\cos ax\sin\frac{\pi}{2}}=\mathrm{e}^{\cos ax}$ $\mathrm{f}'(x)=-(\sin ax)\mathrm{e}^{\cos ax}$ $\mathrm{f}''(x)=\mathrm{e}^{\cos ax}(-a\cos ax)+(\sin^2 ax)\mathrm{e}^{\cos ax}$ --- Hence $\mathrm{f}(0)=\mathrm{e}^{\sin\left(\frac{\pi}{2}\right)}=\mathrm{e}$ $\mathrm{f}'(0)=0$ $\mathrm{f}''(0)=-a^2\mathrm{e}$ By Maclaurin Series, $\mathrm{f}(x)=\mathrm{f}(0)+x\,\mathrm{f}'(0)+\frac{x^2}{2!}\mathrm{f}''(0)+\ldots$ $\mathrm{f}(x)=\mathrm{e}-\frac{a^2\mathrm{e}}{2}x^2+\ldots$ Or it can be presented as $\mathrm{f}(x)\approx\mathrm{e}-\frac{a^2\mathrm{e}}{2}x^2$
6(ii)	$\frac{1}{\sqrt{b+x^2}}=\left(b+x^2\right)^{-\frac{1}{2}}$ $=b^{-\frac{1}{2}}\left(1+\frac{x^2}{b}\right)^{-\frac{1}{2}}$ $=\frac{1}{\sqrt{b}}\left(1+\left(-\frac{1}{2}\right)\left(\frac{x^2}{b}\right)+\ldots\right)$ $=\frac{1}{\sqrt{b}}-\frac{x^2}{2b\sqrt{b}}+\ldots$

Qn	Solution
	Comparing the constant term, $\dfrac{1}{\sqrt{b}}=\mathrm{e} \quad \Rightarrow \quad b=\dfrac{1}{\mathrm{e}^2}$ Comparing coefficient of x^2, $-\dfrac{1}{2b\sqrt{b}}=\dfrac{-a^2\mathrm{e}}{2}$ $a^2\mathrm{e}=\dfrac{1}{\left(\dfrac{1}{\mathrm{e}^2}\right)\left(\dfrac{1}{\mathrm{e}}\right)}$ $a^2=\mathrm{e}^2$ $a=\pm\mathrm{e}$
7(a)	$\displaystyle\int_0^1 \frac{x}{2-x^2}\,\mathrm{d}x=-\frac{1}{2}\int_0^1 \frac{-2x}{2-x^2}\,\mathrm{d}x$ $\displaystyle=-\frac{1}{2}\left[\ln\left\|2-x^2\right\|\right]_0^1$ $\displaystyle=-\frac{1}{2}(\ln 1-\ln 2)$ $\displaystyle=-\frac{1}{2}\ln\frac{1}{2} \quad \text{or} \quad \frac{1}{2}\ln 2$
7(b)i)	$x^2=A(2-x)^2+B(2-x)+C$ $=Ax^2-4Ax+4A+x^2+2B-Bx+C$ $=Ax^2+(-4A-B)x+4A+2B+C$ By comparing coefficients, $A=1$, $-4(1)-B=0\Rightarrow B=-4$, $4(1)+2(-4)+C=0\Rightarrow C=4$
7(b)ii)	$\displaystyle\int_0^1 \frac{x^2}{(2-x)^2}\,\mathrm{d}x=\int_0^1 1-\frac{4}{2-x}+\frac{4}{(2-x)^2}\,\mathrm{d}x$ $\displaystyle=\left[x+4\ln\left\|2-x\right\|+\frac{4}{2-x}\right]_0^1$ $\displaystyle=1+4\ln(1)+\frac{4}{2-1}-4\ln 2-2$ $=3-4\ln 2$ $p=3,\ q=-4$

Qn	Solution
8(a)i)	$\overrightarrow{OQ} = \begin{pmatrix} 2 \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$ Acute angle $= \cos^{-1} \dfrac{\left\| \begin{pmatrix} 2 \\ -1.5 \\ -0.5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\|}{\sqrt{6.5}\sqrt{1}}$ $= \cos^{-1}\left(\dfrac{1.5}{\sqrt{6.5}}\right)$ $= 54.0^\circ$ (or 0.942 rad)
8(a)ii)	xy-plane: $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$ Since $\mathbf{m}$ is perpendicular to xy-plane, $\mathbf{m} \mathbin{//} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Since $\|\mathbf{m}\| = 1$, $\mathbf{m} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $\|\mathbf{q} \cdot \mathbf{m}\|$ refers to the perpendicular distance from Q to xy-plane. $\|\mathbf{q} \cdot \mathbf{m}\| = \left\| \begin{pmatrix} 2 \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\| = \frac{1}{2}$

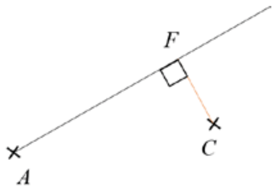
8 (b)	$\mathbf{c} \times \mathbf{a} = \lambda \mathbf{b} \times \mathbf{c}$
	$\mathbf{c} \times \mathbf{a} - \lambda \mathbf{b} \times \mathbf{c} = \mathbf{0}$
	$\mathbf{c} \times \mathbf{a} + \mathbf{c} \times \lambda \mathbf{b} = \mathbf{0}$
	$\mathbf{c} \times (\mathbf{a} + \lambda \mathbf{b}) = \mathbf{0}$
	$\mathbf{c} \times \mathbf{r} = \mathbf{0}$
	$\therefore$ $\mathbf{r} \mathbin{//} \mathbf{c}$ (shown)
	
	Since $\mathbf{c} \mathbin{//} \mathbf{r}$, $\mu \mathbf{c} = \mathbf{r}$.

Qn	Solution
	$\mathbf{r} \bullet \mathbf{r} = \mu\mathbf{c} \bullet \mu\mathbf{c}$ $\mathbf{r} \bullet \mathbf{r} = \mu^2 \lvert\mathbf{c}\rvert^2$ Since a is a unit vector, $\lvert\mathbf{a}\rvert = 1$. $\mathbf{r} \bullet \mathbf{r} = (\mathbf{a}+\lambda\mathbf{b}) \bullet (\mathbf{a}+\lambda\mathbf{b})$ $= \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \lambda\mathbf{b} + \lambda\mathbf{b} \bullet \mathbf{a} + \lambda\mathbf{b} \bullet \lambda\mathbf{b}$ $= \lvert\mathbf{a}\rvert^2 + 2\lambda(\mathbf{a} \bullet \mathbf{b}) + \lambda^2 \lvert\mathbf{b}\rvert^2$ $= 1 + 4\lambda^2$ $\mu^2 \lvert\mathbf{c}\rvert^2 = 1 + 4\lambda^2$ $\lvert\mathbf{c}\rvert^2 = \frac{1}{\mu^2}(1+4\lambda^2)$ $\lvert\mathbf{c}\rvert = \sqrt{\frac{1}{\mu^2}(1+4\lambda^2)}$ (rej. $-\sqrt{\frac{1}{\mu^2}(1+4\lambda^2)}$ since $\lvert\mathbf{c}\rvert > 0$) $\lvert\mathbf{c}\rvert = \sqrt{\frac{1}{\mu^2}}\left(\sqrt{(1+4\lambda^2)}\right)$ $\lvert\mathbf{c}\rvert = k\sqrt{(1+4\lambda^2)}$ where $k = \sqrt{\frac{1}{\mu^2}}$ (or $k = \frac{1}{\lvert\mu\rvert}$)
9(i)	**<u>Method 1: Horizontal Line Test</u>** $\mathrm{f} : x \mapsto 1 + 2\mathrm{e}^{-x^2}$, $x \in \mathbb{R}$  Since there exists **a/the horizontal line** $y = 2$ that intersects the graph of $y = \mathrm{f}(x)$ **more than once**, f is **not one-one**, f does not have an inverse. **<u>Method 2: Counterexample</u>** Since $\mathrm{f}(-1) = \mathrm{f}(1) = 1 + \frac{2}{\mathrm{e}}$, f is **not one-one**, f does not have an inverse.
9(ii)	Largest value of k is 0.
9(iii)	Let $y = 1 + 2\mathrm{e}^{-x^2}$ $y - 1 = 2\mathrm{e}^{-x^2}$ $\mathrm{e}^{-x^2} = \frac{y-1}{2}$ $-x^2 = \ln\left(\frac{y-1}{2}\right)$ $x^2 = -\ln\left(\frac{y-1}{2}\right)$ $x = \pm\sqrt{-\ln\left(\frac{y-1}{2}\right)}$

Qn	Solution
	Since $x \le 0$ (restricted domain of f), $x = -\sqrt{-\ln\left(\frac{y-1}{2}\right)}$ $\mathrm{f}^{-1}(x) = -\sqrt{-\ln\left(\frac{x-1}{2}\right)}$ OR $-\sqrt{\ln\left(\frac{2}{x-1}\right)}$
9(iv)	Range of $\mathrm{f} = (1,\ 3]$ Domain of g = Range of $\mathrm{g}^{-1} = (1,\ 3]$ Since Range of f $\subseteq$ Domain of g, gf exists.
9(v)	Domain of gf = Domain of f = $(-\infty,\ 0]$ Range of gf = $(-\infty,\ 0]$
9(vi)	Given that $\mathrm{gf}(x) = x$, $\mathrm{gf}(-2) = -2$ $\mathrm{g}^{-1}\mathrm{gf}(-2) = \mathrm{g}^{-1}(-2)$ $\mathrm{g}^{-1}(-2) = \mathrm{f}(-2)$ $= 1 + 2\mathrm{e}^{-4}$
10(i) a)	**<u>Method 1:</u>** $\frac{\mathrm{d}m}{\mathrm{d}t} = c$ where c is a <u>negative</u> constant because the mass of the raindrop is decreasing with time due to evaporation. **<u>Method 2:</u>** $\frac{\mathrm{d}m}{\mathrm{d}t} = -c$ where c is a <u>positive</u> constant because the mass of the raindrop is decreasing with time due to evaporation.
10(i) b)	$\frac{\mathrm{d}m}{\mathrm{d}t} = c$ $m = ct + D$ When $t = 0$, $m = 0.05$, $0.05 = c(0) + D \Rightarrow D = 0.05$ When $t = 60$, $m = 0.004$, $0.004 = c(60) + 0.05$ $c = -0.00076667$ (5s.f.) $c = -0.000767$ (3s.f.) $\therefore m = -0.000767t + 0.05$
10(ii) a)	$K = \frac{mg^2t^2}{2}$ From (i)(b), $m = -0.00076667t + 0.05$,

Qn	Solution
	$K=\dfrac{(-0.00076667t+0.05)g^2t^2}{2}$ $=-0.00038333g^2t^3+0.025g^2t^2$ At stationary points, $\dfrac{\mathrm{d}K}{\mathrm{d}t}=-0.00115g^2t^2+0.05g^2t=0$ $t\left(-0.00115g^2t+0.05g^2\right)=0$ $t=0$ (rejected since $K=0$ when $t=0$) or $t=43.478$ $\dfrac{\mathrm{d}^2K}{\mathrm{d}t^2}=-0.0023g^2t+0.05g^2$ When $t=43.478$, $\dfrac{\mathrm{d}^2K}{\mathrm{d}t^2}=-0.0500g^2$ <0 (since $g^2>0$) $\therefore t=43.478$ gives maximum K. When $t=43.478$, $K=15.75g^2$, where $p=15.75$ (2d.p.)
10(ii) b)	$K=-0.00038333(10)^2t^3+0.025(10)^2t^2$ $=-0.038333t^3+2.5t^2$ At surface of ground, $1=-0.038333t^3+2.5t^2$ Using GC, $t=-0.629,\ 0.636$ or 65.2 Since $t>60$, $\therefore t=65.2$ (3 s.f.) Required time taken = 65.2 s. (3 s.f.)
11(i)	Let A be the point where it departs from the ground. $\mathbf{r}=\overrightarrow{OA}+t\begin{pmatrix}-2\\1\\1\end{pmatrix}$ $\begin{pmatrix}-9\\4\\6\end{pmatrix}=\overrightarrow{OA}+6\begin{pmatrix}-2\\1\\1\end{pmatrix}$ $\overrightarrow{OA}=\begin{pmatrix}-9\\4\\6\end{pmatrix}-6\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}3\\-2\\0\end{pmatrix}$ Coordinates: $A(3,\ -2,\ 0)$

Qn	Solution
11(ii)	



Shortest distance from C to the flight path (line) refers to the perpendicular distance from C to the flight path.

Method 1 (via foot of perpendicular)

Let F be the foot of perpendicular from C to the flight path

$$\mathbf{r}=\begin{pmatrix}3\\-2\\0\end{pmatrix}+t\begin{pmatrix}-2\\1\\1\end{pmatrix} \quad \text{for } t\in\mathbb{R},\ t\geq 0$$

$$\overrightarrow{OF}=\begin{pmatrix}3\\-2\\0\end{pmatrix}+t\begin{pmatrix}-2\\1\\1\end{pmatrix} \quad \text{for some } t\in\mathbb{R},\ t\geq 0$$

$$\overrightarrow{CF}=\overrightarrow{OF}-\overrightarrow{OC}=\begin{pmatrix}3-2t\\-2+t\\t\end{pmatrix}-\begin{pmatrix}-7\\5\\2\end{pmatrix}$$

$$=\begin{pmatrix}10-2t\\-7+t\\t-2\end{pmatrix}$$

Since $\overrightarrow{CF}\perp$ flight path, $\begin{pmatrix}10-2t\\-7+t\\t-2\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\1\end{pmatrix}=0$

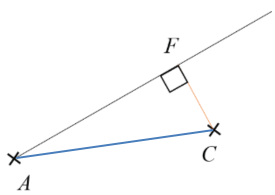
$-20+4t-7+t+t-2=0$

$t=\dfrac{29}{6}$ (or 4.8333)

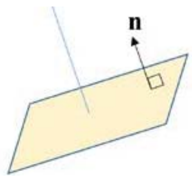
$$\overrightarrow{CF}=\begin{pmatrix}10-2\left(\frac{29}{6}\right)\\-7+\left(\frac{29}{6}\right)\\\left(\frac{29}{6}\right)-2\end{pmatrix}=\begin{pmatrix}\frac{1}{3}\\-\frac{13}{6}\\\frac{17}{6}\end{pmatrix}$$

$$\left|\overrightarrow{CF}\right|=\sqrt{\left(\frac{1}{3}\right)^2+\left(-\frac{13}{6}\right)^2+\left(\frac{17}{6}\right)^2}=\sqrt{\frac{77}{6}}$$

$\left|\overrightarrow{CF}\right|=3.58236=3.58$ m (3 sf)

Qn	Solution
	Method 2 (Vector product)
	$\overrightarrow{OC} = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$
	$\overrightarrow{AC} = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \\ 2 \end{pmatrix}$
	$\text{Perpendicular distance} = \dfrac{\left\lvert \overrightarrow{AC} \times \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rvert}$
	$= \dfrac{\left\lvert \begin{pmatrix} -10 \\ 7 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rvert}{\sqrt{4+1+1}}$
	$= \dfrac{\left\lvert \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix} \right\rvert}{\sqrt{6}}$
	$= \dfrac{\sqrt{77}}{\sqrt{6}}$
	$= 3.58236 = 3.58 \text{ m} \ \ (3 \text{ sf})$
	Method 3 (Pythagoras Theorem)
	$\overrightarrow{OC} = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$
	$\overrightarrow{AC} = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \\ 2 \end{pmatrix}$ 

Qn	Solution
	$\left\lvert \overrightarrow{AF} \right\rvert = \dfrac{\left\lvert \overrightarrow{AC} \bullet \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rvert}$ $= \dfrac{\left\lvert \begin{pmatrix} -10 \\ 7 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rvert}{\sqrt{4+1+1}}$ $= \dfrac{29}{\sqrt{6}}$ Perpendicular distance $= \sqrt{\left\lvert \overrightarrow{AC} \right\rvert^2 - \left\lvert \overrightarrow{AF} \right\rvert^2}$ $= \sqrt{\left((-10)^2 + 7^2 + 2^2\right) - \left(\dfrac{29}{\sqrt{6}}\right)^2}$ $= 3.58236 = 3.58$ m (3 sf)
11(iii)	Let $\lambda = \dfrac{2x-1}{-6} = \dfrac{y+7}{4} = \dfrac{z-10}{k}$ $\dfrac{2x-1}{-6} = \lambda \quad \Rightarrow \quad x = \dfrac{1}{2} - 3\lambda$ $\dfrac{y+7}{4} = \lambda \quad \Rightarrow \quad y = -7 + 4\lambda$ $\dfrac{z-10}{k} = \lambda \quad \Rightarrow \quad z = 10 + k\lambda$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ k \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} \frac{1}{2} \\ -7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ k \end{pmatrix}$, where $\lambda \in \mathbb{R}$, $\lambda \geq 0$
11(iv)	Given that both flight paths (lines) intersect,

Qn	Solution
	$\begin{pmatrix} \frac{1}{2} \\ -7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ k \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{2} - 3\lambda \\ -7 + 4\lambda \\ 10 + k\lambda \end{pmatrix} = \begin{pmatrix} 3 - 2t \\ -2 + t \\ t \end{pmatrix}$ $\frac{1}{2} - 3\lambda = 3 - 2t \quad \Rightarrow \quad -3\lambda + 2t = \frac{5}{2}$ ---(1) $-7 + 4\lambda = -2 + t \quad \Rightarrow \quad 4\lambda - t = 5$ ---(2) $10 + k\lambda = t$ Using GC to solve simultaneously, $\lambda = 2.5, \quad t = 5$ $10 + k(2.5) = 5$ $k = -2$
11(v)	Equation of line: $\mathbf{r} = \begin{pmatrix} -9 \\ 4 \\ 6 \end{pmatrix} + s \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ Equation of plane: $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 2$  If the new flight path is perpendicular to the slope, it will be parallel to the normal vector of the slope. **Method 1:** If **a** and **b** are parallel, then **a** $= k$ **b** for all $k \in \mathbb{R}$. Since $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$ for all $k \in \mathbb{R}$, $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$. Thus the new flight path and the inclined slope are not perpendicular to each other.

Qn	Solution
	Method 2: If **a** and **b** are parallel, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. **Note:** **0** means a zero vector, not a constant 0. If the new flight path (line) is perpendicular to the slope (plane), then this means the direction vector of the line and the normal vector of the plane is parallel. This is to then show that the vector (cross) product of the direction vector of the line and the normal vector of the plane is **0**. $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 13 \\ -22 \\ -5 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. This shows that $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$. Thus the new flight path and the inclined slope are not perpendicular to each other. **Method 3:** If **a** and **b** are parallel, then the angle between **a** and **b** is 0°. If the new flight path (line) is perpendicular to the slope (plane), then this means the direction vector of the line and the normal vector of the plane is parallel. This is to then show that the angle between the line and the plane is 0°. $\sin\theta = \dfrac{\left\| \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} \right\|}{\sqrt{14}\sqrt{51}} \Rightarrow \theta = 13.0^{\circ} \neq 0^{\circ}$ This shows that $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$. Thus the new flight path and the inclined slope are not perpendicular to each other.
11(vi)	Equation of line: $\mathbf{r} = \begin{pmatrix} -9 \\ 4 \\ 6 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ Equation of plane: $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 2$

Qn	Solution
	$\begin{pmatrix} -9+3s \\ 4+2s \\ 6-s \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 2$ $-9+3s-4-2s+42-7s=2$ $6s=27$ $s=4.5$ $\mathbf{r} = \begin{pmatrix} -9 \\ 4 \\ 6 \end{pmatrix} + 4.5\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ 13 \\ \frac{3}{2} \end{pmatrix}$ Coordinates: $\left(\frac{9}{2},\ 13,\ \frac{3}{2}\right)$

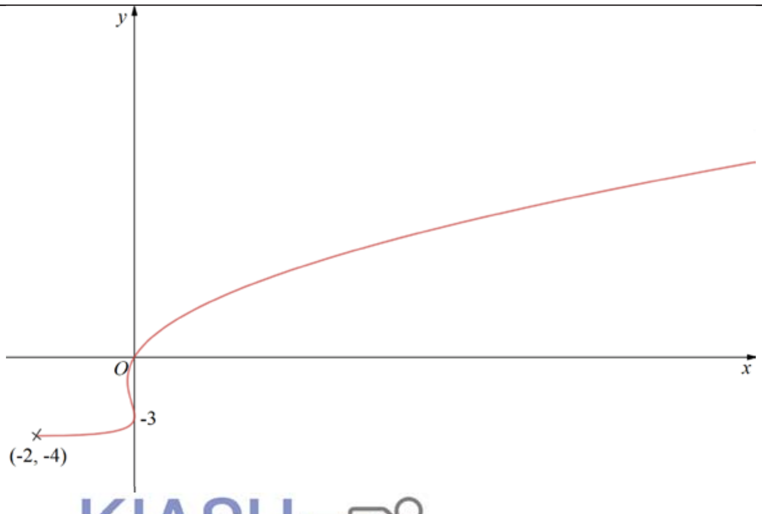
Solution to Paper 9758/02

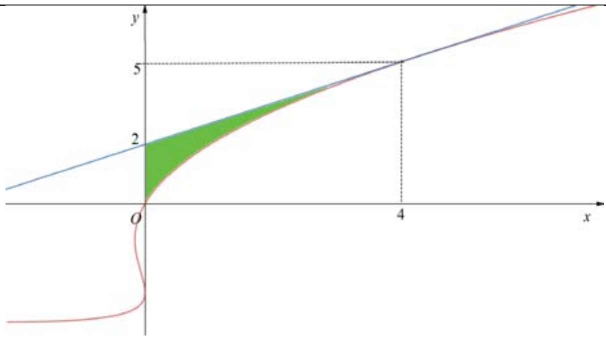
Section A: Pure Mathematics

Qn	Solution
1	$p_n = 3^{n-1} + a$ **Method 1** $\sum_{r=3}^{n} p_r = \sum_{r=3}^{n}\left(3^{r-1}+a\right) = \sum_{r=3}^{n}\left(3^{r-1}\right) + \sum_{r=3}^{n} a$ $= \left[3^2 + 3^3 + 3^4 + \ldots + 3^{n-1}\right] + (n-3+1)a$ $= \frac{3^2\left(3^{n-2}-1\right)}{3-1} + (n-3+1)a$ $= \frac{9}{2}\left(3^{n-2}-1\right) + (n-2)a$ **Method 2** $\sum_{r=3}^{n} p_r = \sum_{r=3}^{n}\left(3^{r-1}+a\right)$ $= \frac{1}{3}\sum_{r=3}^{n}\left(3^{r}\right) + \sum_{r=3}^{n} a$ $= \frac{1}{3}\left(3^3 + 3^4 + 3^5 + \ldots + 3^n\right) + (n-3+1)a$ $= \frac{1}{3}\left(\frac{3^3\left(3^{n-2}-1\right)}{3-1}\right) + (n-3+1)a$ $= \frac{9}{2}\left(3^{n-2}-1\right) + (n-2)a$

Qn	Solution
2(i)	Let T_n be the length of the nth plank Given that it is an AP: Let $a = 4,\ S_3 = 11.46$. Let d be the common difference. $S_3 = 11.46 = \frac{3}{2}\left(2(4)+2d\right)$ $d = -0.18$ $T_{18} = 4+17\left(-0.18\right)$ $= 0.94$ Hence the length of the 18th plank is 0.94 m.
(ii)	The length of the remaining **7** planks follows a GP: first term of GP: $T_{19} = 0.94\left(\frac{5}{4}\right)$ and $r = \frac{5}{4}$. Length of last plank $= T_{25}$ $= ar^{n-1}$ $= T_{19}\left(\frac{5}{4}\right)^{7-1}$ $= 0.94\left(\frac{5}{4}\right)\left(\frac{5}{4}\right)^{6}$ $= 0.94\left(\frac{5}{4}\right)^{7}$ $= 4.4822$ $= 4.48$ m (3 sf)
(iii)	**<u>Method 1</u>** Total length of blue planks $= \underbrace{T_2+T_4+T_6+\ldots+T_{18}}_{\text{Sum of AP with } a=T_2,\ d=-0.36,\ l=T_{18}} + \underbrace{T_{20}+T_{22}+T_{24}}_{\text{Sum of GP with } a=T_{20},\ r=\left(\frac{5}{4}\right)^2}$ $= \frac{9}{2}\left[\left(4-0.18\right)+0.94\right]+0.94\left(\frac{5}{4}\right)^2\left(\frac{\left(\left(\frac{5}{4}\right)^2\right)^3-1}{\left(\frac{5}{4}\right)^2-1}\right)$ $= 21.42+7.34948$ $= 28.8$ m (3 sf) **<u>Method 2</u>** Total length of blue planks

Qn	Solution
	$=\frac{9}{2}\left[2(4-0.18)+(9-1)(2(-0.18))\right]$ $+0.94\left(\frac{5}{4}\right)^2\left(\frac{\left(\left(\frac{5}{4}\right)^2\right)^3-1}{\left(\frac{5}{4}\right)^2-1}\right)$ $=21.42+7.34948$ $=28.8$ m (3 sf)
3(a)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^{-5x+3}+\sin x$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\int\left(\mathrm{e}^{-5x+3}+\sin x\right)\,\mathrm{d}x=-\frac{1}{5}\mathrm{e}^{-5x+3}-\cos x+c$ $y=\int\left(-\frac{1}{5}\mathrm{e}^{-5x+3}-\cos x+c\right)\mathrm{d}x$ $y=\frac{1}{25}\mathrm{e}^{-5x+3}-\sin x+cx+d$ where $c,\ d$ are arbitrary constants.
3(b)	$z=x+\frac{\mathrm{d}y}{\mathrm{d}x}\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=1+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}-x+1=0\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+1=\left(\frac{\mathrm{d}y}{\mathrm{d}x}+x\right)$ Thus, replacing accordingly, $\frac{\mathrm{d}z}{\mathrm{d}x}=z$ (Shown) $\frac{\mathrm{d}z}{\mathrm{d}x}=z$ $\Rightarrow\int\frac{1}{z}\,\mathrm{d}z=\int 1\,\mathrm{d}x$ $\ln\lvert z\rvert=x+c$ **Method 1:** $\lvert z\rvert=\mathrm{e}^{x+c}$ $z=\pm\mathrm{e}^{x+c}$ $z=A\mathrm{e}^x$ where $A=\pm\mathrm{e}^c$ Since $z=x+\frac{\mathrm{d}y}{\mathrm{d}x}$, when $x=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=1\Rightarrow z=1$ $z=A\mathrm{e}^x\Rightarrow 1=A\mathrm{e}^0\Rightarrow A=1$ $z=\mathrm{e}^x$

Qn	Solution
	Thus, $z = \mathrm{e}^x \Rightarrow x + \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - x$ $\Rightarrow y = \mathrm{e}^x - \frac{1}{2}x^2 + d$ When $x = 0$, $y = 1 \Rightarrow 1 = 1 + d \Rightarrow d = 0$ Hence, $y = \mathrm{e}^x - \frac{1}{2}x^2$ **<u>Method 2:</u>** $\ln\lvert z\rvert = x + c$ $\Rightarrow \ln 1 = 0 + c \Rightarrow c = 0$ Thus, $z = \pm\mathrm{e}^x \Rightarrow x + \frac{\mathrm{d}y}{\mathrm{d}x} = \pm\mathrm{e}^x$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - x$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^x - x$ (rej since $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\rvert_{x=0} \neq 1$) $\Rightarrow y = \mathrm{e}^x - \frac{1}{2}x^2 + d$ When $x = 0$, $y = 1$ $\Rightarrow 1 = 1 + d \Rightarrow d = 0$ Hence, $y = \mathrm{e}^x - \frac{1}{2}x^2$
4(i)	
(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 - 2t, \frac{\mathrm{d}y}{\mathrm{d}t} = 2t + 2$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t+2}{3t^2 - 2t}$ At $t = 2$, $x = 4,\ y = 5,\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{4}$

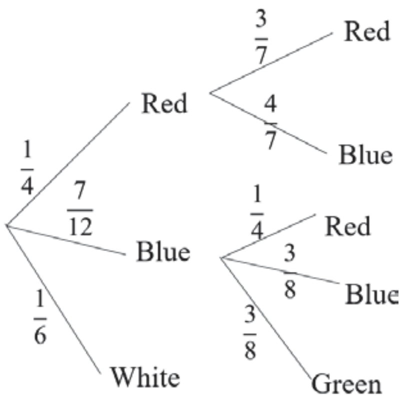
Qn	Solution
	Equation of tangent: $y-5=\frac{3}{4}(x-4)$ $y=\frac{3}{4}x+2$
(iii)	 **<u>Method 1: Integrate wrt *y*</u>** When $y=0 \Rightarrow t^2+2t-3=0$ $\Rightarrow t=-3$ (rej) or $t=1$ When $y=5 \Rightarrow t=2$ (from part (ii)) Area $=\int_0^5 x\,\mathrm{d}y-\frac{1}{2}(4)(5-2)$ $=\int_1^2 (t^3-t^2)(2t+2)\,\mathrm{d}t-6$ $=\int_1^2 2t^4-2t^2\ \mathrm{d}t-6$ $=\left[\frac{2t^5}{5}-\frac{2t^3}{3}\right]_1^2-6$ $=\frac{26}{15}$ **<u>Method 2: Integrate wrt *x*</u>** When $x=0 \Rightarrow t^3-t^2=0$ $\Rightarrow t=0$ or $t=1$ $\Rightarrow y=-3$ (not this point) or $y=0$ When $x=4 \Rightarrow t=2$ (from part (ii)) Area $=\frac{1}{2}(2+5)(4)-\int_0^4 y\,\mathrm{d}x$ $=14-\int_1^2 (t^2+2t-3)(3t^2-2t)\,\mathrm{d}t$ $=14-\int_1^2 3t^4+4t^3-13t^2+6t\ \mathrm{d}t$ $=14-\left[\frac{3t^5}{5}+\frac{4t^4}{4}-\frac{13t^3}{3}+\frac{6t^2}{2}\right]_1^2$ $=\frac{26}{15}$

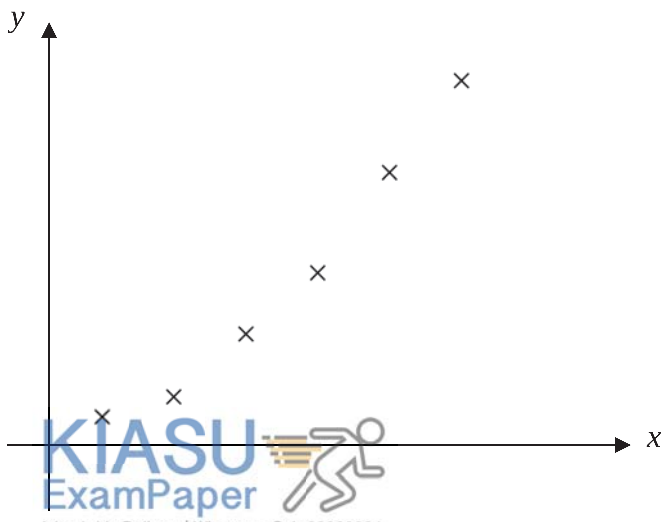
Qn	Solution
5(a)(i)	**Method 1** $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ Since bi is a root, $(b\mathrm{i})^4 - 2(b\mathrm{i})^3 + a(b\mathrm{i})^2 - 8(b\mathrm{i}) + 40 = 0$ $b^4 + 2b^3\mathrm{i} - ab^2 - 8b\mathrm{i} + 40 = 0$ By comparing real and imaginary parts, $2b^3 - 8b = 0$ $2b(b^2 - 4) = 0$ Since $b \neq 0$, $b = -2$ (rej) or $b = 2$ When $b = 2$, $b^4 - ab^2 + 40 = 0$ $16 - 4a + 40 = 0$ $a = 14$ $z^4 - 2z^3 + 14z^2 - 8z + 40 = 0$ Using GC, Other roots are $z = 1 - 3\mathrm{i},\ 1 + 3\mathrm{i},\ -2\mathrm{i}$. **Method 2** $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ Since all coefficients are real, $-b$i is also a root. Quadratic factor: $(z - b\mathrm{i})(z + b\mathrm{i}) = z^2 + b^2$ $z^4 - 2z^3 + az^2 - 8z + 40 = (z^2 + b^2)(z^2 + cz + d)$ $z^4 - 2z^3 + az^2 - 8z + 40 = z^4 + cz^3 + (d + b^2)z^2 + cb^2 z + b^2 d$ By comparison of: Coefficient of z^3 : $c = -2$ Coefficient of z : $-8 = (-2)b^2 \Rightarrow b = 2$ (since $b > 0$) Constant: $40 = 4d \Rightarrow d = 10$ Coefficient of z^2 : $a = 10 + 4 = 14$ $z^4 - 2z^3 + 14z^2 - 8z + 40 = 0$ Using GC, Other roots are $z = 1 - 3\mathrm{i},\ 1 + 3\mathrm{i},\ -2\mathrm{i}$ **Method 3** $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ Since bi is a root, $(b\mathrm{i})^4 - 2(b\mathrm{i})^3 + a(b\mathrm{i})^2 - 8(b\mathrm{i}) + 40 = 0$ $b^4 + 2b^3\mathrm{i} - ab^2 - 8b\mathrm{i} + 40 = 0$

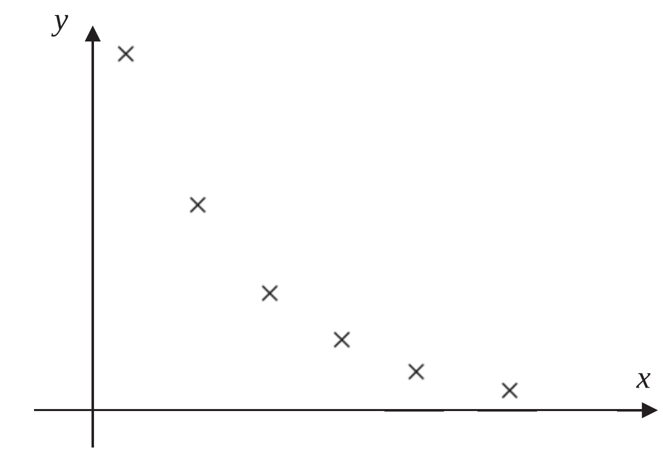
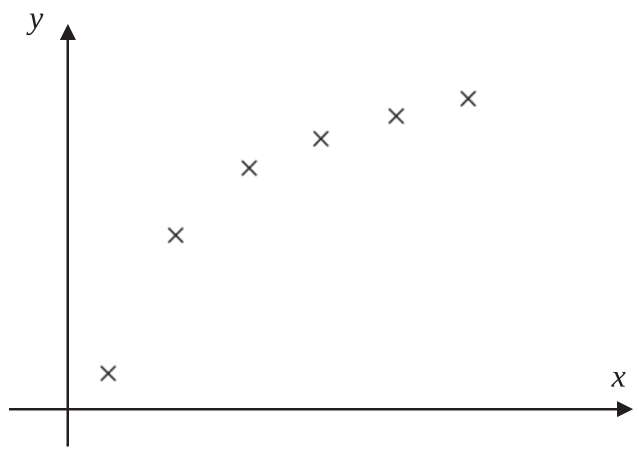
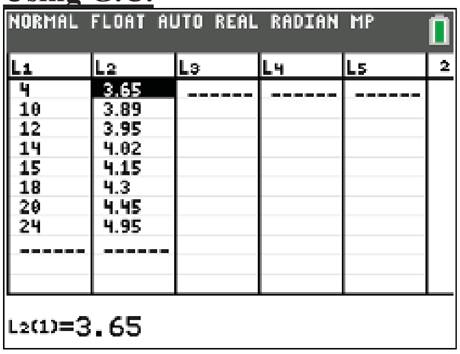
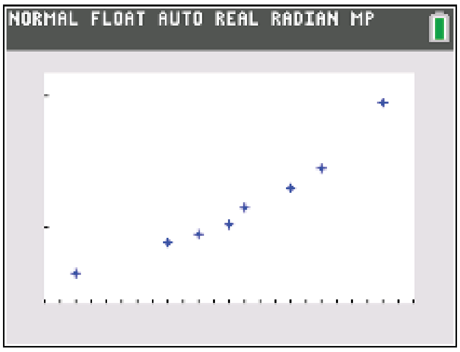
Qn	Solution
	By comparing real and imaginary parts, $2b^3-8b=0$ $2b(b^2-4)=0$ Since $b>0$, $b=-2$ (rejected) or $b=2$ When $b=2$, $b^4-ab^2+40=0$ $16-4a+40=0$ $a=14$ $z^4-2z^3+14z^2-8z+40=0$ Since all coefficients are real, –2i is also a root. Quadratic factor: $(z-2\mathrm{i})(z+2\mathrm{i})=z^2+4$ $z^4-2z^3+az^2-8z+40=(z^2+4)(z^2-2z+10)=0$ $z^2-2z+10=0$ $z=\dfrac{2\pm\sqrt{4-4(10)}}{2}=\dfrac{2\pm\sqrt{36\mathrm{i}^2}}{2}=1\pm3\mathrm{i}$ Hence, other roots are $z=1-3\mathrm{i},\ 1+3\mathrm{i},\ -2\mathrm{i}$.
5(a) (ii)	$w^4+2w^3+aw^2+8w+40=0$ Let $z=-w$ For $z^4-2z^3+14z^2-8z+40=0$ $\Rightarrow\ (-w)^4-2(-w)^3+14(-w)^2-8(-w)+40=0$ $w^4+2w^3+14w^2+8w+40=0$ $-w=1-3\mathrm{i},\ 1+3\mathrm{i},\ -2\mathrm{i}$ or $2\mathrm{i}$ $w=-1+3\mathrm{i},\ -1-3\mathrm{i},\ 2\mathrm{i}$ or $-2\mathrm{i}$
5(b) (i)	$w=-6+(2\sqrt{3})\mathrm{i}$ $\|w\|=\sqrt{(-6)^2+(2\sqrt{3})^2}=\sqrt{48}$ (or $4\sqrt{3}$) $\arg(w)=\pi-\tan^{-1}\left(\dfrac{2\sqrt{3}}{6}\right)=\dfrac{5\pi}{6}$ $\|w^n\|=\|w\|^n=(\sqrt{48})^n=48^{\frac{n}{2}}$ (or $(4\sqrt{3})^n$) $\arg(w^n)=n\arg(w)=\dfrac{5n\pi}{6}$ $w^n=48^{\frac{n}{2}}\mathrm{e}^{\mathrm{i}\left(\frac{5n\pi}{6}\right)}=(4\sqrt{3})^n\mathrm{e}^{\mathrm{i}\left(\frac{5n\pi}{6}\right)}$ Or $w^n=48^{\frac{n}{2}}\left[\cos\left(\dfrac{5n\pi}{6}\right)+\mathrm{i}\sin\left(\dfrac{5n\pi}{6}\right)\right]$

Qn	Solution
5(b) (ii)	$\arg\left(w^n w^*\right)=\frac{5n\pi}{6}-\frac{5\pi}{6}=\frac{5(n-1)\pi}{6}$ For $w^n w^*$ to be purely imaginary $\arg\left(w^n w^*\right)=\frac{\pi}{2},\ \frac{\pi}{2}\pm\pi,\ \frac{\pi}{2}\pm 2\pi,\ldots$ $\arg\left(w^n w^*\right)=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$ $\arg\left(w^n w^*\right)=\frac{(2k+1)\pi}{2},\ k\in\mathbb{Z}$ $\frac{5(n-1)\pi}{6}=\frac{(2k+1)\pi}{2}$ $n-1=\frac{2k+1}{2}\left(\frac{6}{5}\right)$ $n=1+\frac{3(2k+1)}{5},\ k\in\mathbb{Z}$ $n=4,\ 10$ (when $k=2$ and $k=7$)

Section B: Probability and Statistics

Qn	Solution
6(i)	Total no. of possible IN $=10^4 \times 10 = 100\ 000$ ways
6(ii)	Since the last letter of his IN is I, i.e. the remainder is 8, then the possible sum is 28 (since the sum must be at least 20) The possible sets of digits are $\{9,9,a,b\}$, where $a+b=10$. The possible values of $\{a,b\}$ are $\{2,8\}$, $\{3,7\}$, $\{4,6\}$ and $\{5,5\}$. Hence, no. of ways $=3\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!2!}\right)=$ 42 ways Hence, there are a total of 42 ways.
7(i)	 P(blue face shown and ball is red) $=\frac{7}{12}\left(\frac{1}{4}\right)$ $=\frac{7}{48}=0.146$ (3 sf)
7(ii)	P(mystery gift given) =P(all red) + P(all blue) $=\frac{1}{4}\left(\frac{3}{7}\right)+\left(\frac{7}{12}\right)\left(\frac{3}{8}\right)$ $=\frac{73}{224}$ $=0.32589 = 0.326$ (3 sf)

Qn	Solution
7(iii)	$\text{P}\left(\text{one red ball picked} \mid \text{did not win mystery gift}\right)$ $= \dfrac{\text{P}\left(\text{one red ball picked and did not win mystery gift}\right)}{1-\dfrac{73}{224}}$ $= \dfrac{7}{48} \div \left(1-\dfrac{73}{224}\right)$ $= \dfrac{98}{453}$ $= 0.216 \text{ (3sf)}$
7 Last part	P(mystery gift given) = P(gift given in the first roll) + P(white face in 1^{st} roll and gift given in the second roll) + P(white face in 1^{st} 2 rolls and gift given in the third roll) $+\ldots$ $= \underbrace{\dfrac{73}{224} + \left(\dfrac{1}{6}\right)\left(\dfrac{73}{224}\right) + \left(\dfrac{1}{6}\right)^2\left(\dfrac{73}{224}\right) + \left(\dfrac{1}{6}\right)^3\left(\dfrac{73}{224}\right) + \ldots}_{\text{Sum of infinity with } a=\frac{73}{224} \text{ and } r=\frac{1}{6}}$ $= \dfrac{73}{224}\left[\dfrac{1}{1-\dfrac{1}{6}}\right]$ $= \dfrac{219}{560}$ $= 0.391 \text{ (3 sf)}$
8 (a)	

Qn	Solution
	 
8 (b) (i)	Using G.C.  NORMAL FLOAT AUTO REAL RADIAN MP L1: 4, 10, 12, 14, 15, 18, 20, 24 L2: 3.65, 3.89, 3.95, 4.02, 4.15, 4.3, 4.45, 4.95 L2(1)=3.65 

<table>
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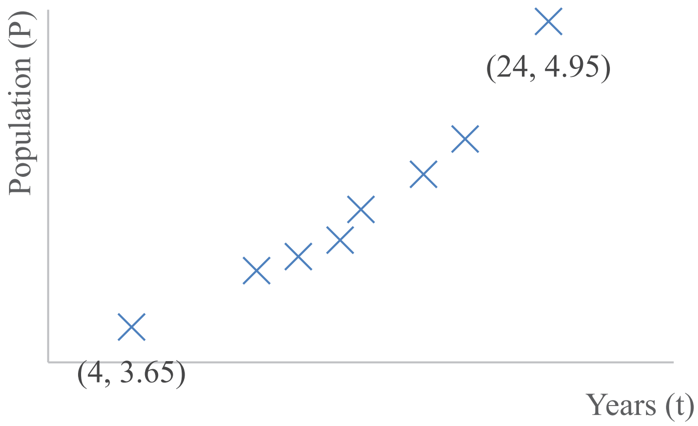


A linear model may not be appropriate as the points do not lie close to a straight line. (Or the points do not follow a linear trend.)
</td></tr>
<tr><td>8 (b)(ii)</td><td>
<u>Using G.C.</u>

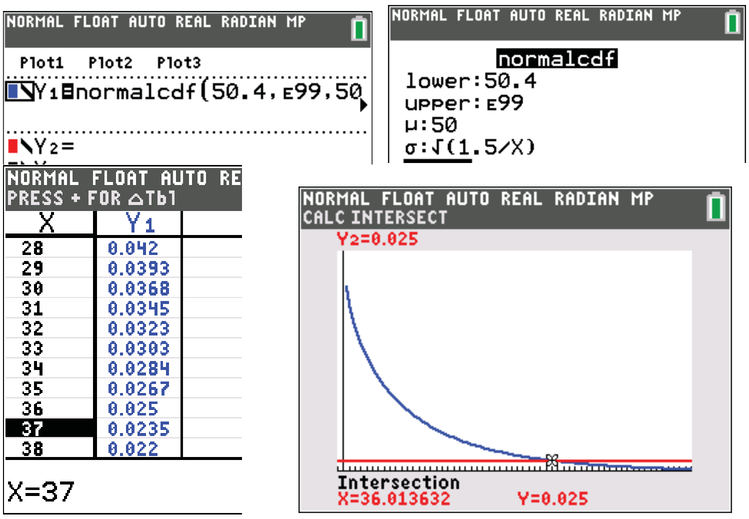
Note that from the scatter diagram suggests that as t increases, P increases at an increasing rate.
Hence, Case (A) is most appropriate.

Using G.C., $r = 0.995$ (3 s.f.)
</td></tr>
<tr><td>8 (b) (iii)</td><td>
Using G.C.,
$P = 3.6211 + 0.0022167t^2$
$P = 3.62 + 0.00222t^2$ (3 s.f.)

In the year 2020, $t = 30$
Hence, when $t = 30$, $P = 3.6211 + 0.0022167(30)^2$
$= 5.6161$
Hence, the predicted population is approximately 5.62 million (3 s.f.)
</td></tr>
<tr><td>8 (b) (iv)</td><td>
The statistician's prediction is not be reliable as $t = 30$ is not within data range.

</td></tr>
</table>

Qn	Solution
9(i)	A random sample means that each of the tablets were **selected independently** from each other *and* **each** tablet has an **equal chance** of being **selected**.
9(ii)	Let X be the amount in grams of active ingredient A in a particular type of health supplement tablets. Define μ : Let μ be the **population mean amount/mass of active ingredient** ***A*** in a particular type of health supplement tablets. Given that sample mean $\bar{x} = 50.6$, sample variance $= 2.15$ **Unbiased estimate** of population variance $= \frac{40}{39}(2.15) = 2.2051$ $\text{H}_0 : \mu = 50$ $\text{H}_1 : \mu \neq 50$ Under H$_0$, since ***n* = 40 is large, by Central Limit Theorem,** $\bar{X} \sim \text{N}\left(50,\ \frac{2.2051}{40}\right)$ approximately. Use *z*-test at $\alpha = 0.01$ Using GC, *p*-value $= 0.010605 = 0.0106$ (3 sf) Since *p*-value $> \alpha$, do not reject H$_0$. There is **insufficient evidence** at 1% level of significance to conclude that the mean amount of active ingredient *A* has changed (or is not 50 mg).
9 last part	Let Y be the amount in grams of active ingredient A in the revised formula $\text{H}_0 : \mu = 50$ $\text{H}_1 : \mu > 50$ Under H$_0$, $\bar{Y} \sim \text{N}\left(50,\ \frac{1.5}{n}\right)$ Use *z*-test at $\alpha = 0.025$. **<u>Method 1</u>** Test-statistic distribution: $Z = \frac{\bar{Y} - 50}{\sqrt{\frac{1.5}{n}}} \sim \text{N}(0,\ 1)$

Qn	Solution
	Corresponding test-statistic: $z=\dfrac{50.4-50}{\sqrt{\dfrac{1.5}{n}}}=\dfrac{0.4\sqrt{n}}{\sqrt{1.5}}$ Critical region: $z\ge 1.95996$ Since the chemist claim is valid, H_0 is rejected. Test-statistic falls within the critical region. $\dfrac{0.4\sqrt{n}}{\sqrt{1.5}}\ge 1.95996$ $n\ge 36.0135$ $n\ge 37$ $\{n\in\mathbb{Z}:n\ge 37\}$ **<u>Method 2 (Using GC, Table/Graph method)</u>** For H_0 to be rejected, p - value ≤ 0.025. p-value $=\mathrm{P}\left(\bar{Y}>50.4\right)\le 0.025$  $n\ge 37$ $\{n\in\mathbb{Z}:n\ge 37\}$
10 (i)	$\mathrm{P}\left(X=2\right)=\mathrm{P}\left(\text{point } A_2\right)$ $=\mathrm{P}\left(\text{2 steps to the right and 3 steps downwards}\right)$ $={}^5C_2\,p^2q^3$ $=10p^2q^3$ (Shown)
10 (ii)	Note that $\mathrm{P}\left(X=3\right)=\mathrm{P}\left(\text{point } A_3\right)$ $={}^5C_3\,p^3q^2=10p^3q^2$ Since the particle will end up at point A_2 most of the time, then mode occurs at $X=2$.

Qn	Solution
	$\therefore \mathrm{P}(X=3)<\mathrm{P}(X=2)$ $10p^3q^2<10p^2q^3$ $\frac{q}{p}>1 \Rightarrow 1-p>p$ $2p<1 \Rightarrow p<0.5$ In the same way, $\mathrm{P}(X=2)>\mathrm{P}(X=1)$ $10p^2q^3>5pq^4$ $2p>q$ $2p>1-p$ $3p>1 \Rightarrow p>\frac{1}{3}$ Combining both inequalities, $\frac{1}{3}<p<\frac{1}{2}$ (Shown)
10 (iii)	see table and working below
10 (iv)	Using G.C. $\mathrm{E}(Y)=2.744192=2.74$ (3 sf) $\mathrm{Var}(Y)=1.3626^2=1.8567=1.86$ (3 sf)

Working for 10 (iii):

Y	0	2	3	5
$\mathrm{P}(Y=y)$	0.07776	0.4224	0.299904	0.199936

$\mathrm{P}(Y=0)=\mathrm{P}(X=0)={}^5C_0(0.4)^0(0.6)^5=0.07776$

$\mathrm{P}(Y=2)=\mathrm{P}(X=i,\text{ where } i \text{ is even})$

$=\mathrm{P}(X=2)+\mathrm{P}(X=4)$

$=10(0.4)^2(0.6)^3+{}^5C_4(0.4)^4(0.6)^1=0.4224$

$\mathrm{P}(Y=3)=\mathrm{P}(X=i,\text{ where } i \text{ is odd and lose the game})$

$=0.6(\mathrm{P}(X=1)+\mathrm{P}(X=3)+\mathrm{P}(X=5))$

$=0.6\left[{}^5C_1(0.4)^1(0.6)^4+{}^5C_3(0.4)^3(0.6)^2+{}^5C_5(0.4)^5(0.6)^0\right]$

$=0.299904$

$\mathrm{P}(Y=5)=\mathrm{P}(X=i,\text{ where } i \text{ is odd and win the game})$

$=0.4(\mathrm{P}(X=1)+\mathrm{P}(X=3)+\mathrm{P}(X=5))$

$=0.4\left[{}^5C_1(0.4)^1(0.6)^4+{}^5C_3(0.4)^3(0.6)^2+{}^5C_5(0.4)^5(0.6)^0\right]$

$=0.199936$

Qn	Solution
11(i)	Let X be the mass, in g, of a randomly chosen dark truffle. $X \sim \mathrm{N}\left(17,\ 1.3^2\right)$ Let $T = X_1 + X_2 + X_3 + X_4$ $\mathrm{E}(T) = 4(17) = 68$ $\mathrm{Var}(T) = 4\left(1.3^2\right) = 6.76$ $T \sim \mathrm{N}(68,\ 6.76)$ $\mathrm{P}(T > 70) = 0.220878 = 0.221$ (3 sf)
11(ii)	Let Y be the **number of boxes** that weigh more than 70 g, out of 20 boxes. $Y \sim \mathrm{B}(20,\ 0.220878)$ $\mathrm{P}(Y > 3) = 1 - \mathrm{P}(Y \le 3)$ $= 0.67448$ $= 0.674$ (3 sf) **Assumption:** The mass of the empty box is negligible. (**Other possible answer:** The **event that a mass of a box of 4 dark truffles** has mass more than 70g is **independent** of other boxes.)
11(iii)	Let W be the mass, in g, of a randomly chosen salted caramel ganache. $W \sim \mathrm{N}\left(\mu,\ \sigma^2\right)$ Given $\mathrm{P}(W < 12) = \mathrm{P}(W > 15)$ and $\mathrm{P}(W \le 15) = 0.97$ **Method 1** By symmetry, $\mu = \dfrac{12+15}{2} = 13.5$ $\mathrm{P}(W \le 15) = 0.97$ $\mathrm{P}\left(Z \le \dfrac{15-13.5}{\sigma}\right) = 0.97$ $\dfrac{1.5}{\sigma} = 1.88079$ $\sigma = 0.797537$ $\sigma^2 = 0.636065$ $\sigma^2 = 0.636$ (3 sf)

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td><u>Method 2</u>
$\mathrm{P}(W < 12) = 0.03$ | $\mathrm{P}(W \le 15) = 0.97$
$\mathrm{P}\left(Z < \frac{12-\mu}{\sigma}\right) = 0.03$ | $\mathrm{P}\left(Z < \frac{15-\mu}{\sigma}\right) = 0.03$
$\frac{12-\mu}{\sigma} = -1.88079$ | $\frac{15-\mu}{\sigma} = 1.88079$
$\mu - 1.88079\sigma = 12$ ---(1) | $\mu + 1.88079\sigma = 15$ ---(2)

Solving equation (1) and (2),
$\mu = 13.5, \quad \sigma = 0.797537$
$\sigma^2 = 0.636065$
$\sigma^2 = 0.636$ (3 sf)</td></tr>
<tr><td>11(iv)</td><td>$W \sim \mathrm{N}(13.5,\ 0.636065)$
Find $\mathrm{P}\left(0.28(W_1 + W_2 + \ldots + W_6) < 0.34T\right)$.

Let $S = 0.28(W_1 + W_2 + \ldots + W_6) - 0.34T$
$\mathrm{E}(S) = 0.28(6)(13.5) - 0.34(68) = -0.44$
$\mathrm{Var}(S) = 0.28^2(6)(0.636065) + 0.34^2(6.76) = 1.08065$

$S \sim \mathrm{N}(-0.44,\ 1.08065)$
$\mathrm{P}(S < 0) = 0.66394 = 0.664$ (3 sf)</td></tr>
</table>

NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

Candidate Name	

CT Class	1	8			Centre Number/ Index Number					/				

MATHEMATICS 9758/01

Paper 1

2nd September 2019

3 Hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	Mark
1	
2	
3	
4	
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6	
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9	
10	
11	
12	
Total	

This document consists of **6** printed pages.

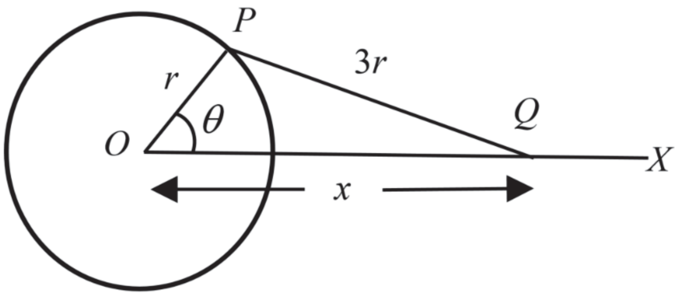
NANYANG JUNIOR COLLEGE
Internal Examinations

[Turn Over

1 The curve with equation $y^2 = x^2 + 9$ is transformed by a stretch with scale factor 2 parallel to the x-axis, followed by a translation of 4 units in the negative x-direction, followed by a translation of $\frac{1}{2}$ units in the positive y-direction.

Find the equation of the new curve and state the equations of any asymptote(s). Sketch the new curve, indicating the coordinates of any turning points. [6]

2 The diagram shows a mechanism for converting rotational motion into linear motion. The point P, on the circumference of a disc of radius r, rotates about a fixed point O. The point Q moves along the line OX, and P and Q are connected by a rod of fixed length $3r$. As the disc rotates, the point Q is made to slide backwards and forwards along OX. At time t, angle POQ is θ, measured anticlockwise from OX, and the distance OQ is x.



(i) Show that $x = r\left(\cos\theta + \sqrt{9 - \sin^2\theta}\right)$. [2]

(ii) State the maximum value of x. [1]

(iii) Express x as a polynomial in θ if θ is sufficiently small for θ^3 and higher powers of θ are to be neglected. [3]

3 Without using a calculator, solve the inequality $\dfrac{x^2 - 3x + 4}{x + 2} \geq 2x + 1$. Hence solve the inequality $\dfrac{a^{2x} + 3a^x + 4}{a^x - 2} \leq 2a^x - 1$ where $a > 2$. [6]

4 **(i)** Using double angle formula, prove that $\sin^4\theta = \frac{1}{8}\left(3 - 4\cos 2\theta + \cos 4\theta\right)$. [2]

(ii) By using the substitution $x = 2\cos\theta$, find the exact value of $\int_0^2 \left(4 - x^2\right)^{\frac{3}{2}}\,\mathrm{d}x$. [4]

5 Relative to the origin O, the points A, B, and C, have non-zero position vectors $\mathbf{a}$, $\mathbf{b}$, and $3\mathbf{a}$ respectively. D lies on AB such that $AD = \lambda AB$, where $0 < \lambda < 1$.

(i) Write down a vector equation of the line OD. [1]

(ii) The point E is the midpoint of BC. Find the value of λ if E lies on the line OD. Show that the area of ΔBED is given by $k\left|\mathbf{a}\times\mathbf{b}\right|$, where k is a constant to be determined. [5]

6 The function f is given by $\mathrm{f}: x \mapsto 2x^2 + 4x + k$ for $-5 \le x < a$, where a and k are constants and $k > 2$.

(i) State the largest value of a for the inverse of f to exist. [1]

For the value of a found in **(i)**,

(ii) find $\mathrm{f}^{-1}(x)$ and the domain of f^{-1}, leaving your answer in terms of k, [3]

(iii) on the same diagram, sketch the graphs of $y = \mathrm{ff}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$, labelling your graphs clearly. Determine the number of solutions to $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. [4]

7 A spherical tank with negligible thickness and internal radius a cm contains water. At time t s, the water surface is at a height x cm above the lowest point of the tank and the volume of water in the tank, V cm^3, is given by $V = \frac{1}{3}\pi x^2(3a - x)$. Water flows from the tank, through an outlet at its lowest point, at a rate $\pi k\sqrt{x}$ cm^3 s^{-1}, where k is a positive constant.

(i) Show that $\left(2ax - x^2\right)\frac{\mathrm{d}x}{\mathrm{d}t} = -k\sqrt{x}$. [2]

(ii) Find the general solution for t in terms of x, a and k. [3]

(iii) Find the ratio $T_1 : T_2$, where T_1 is the time taken to empty the tank when initially it is completely full, and T_2 is the time taken to empty the tank when initially it is half full. [4]

8 A curve C has equation $y^2 + xy = 4$, where $y > 0$.

(i) Without using a calculator, find the coordinates of the point on C at which the gradient is $-\frac{1}{5}$. [4]

(ii) Variables z and y are related by the equation $y^2 + z^2 = 10y$, where $z > 0$. Given that x increases at a constant rate of 0.5 unit/s, find the rate of change of z when $x = 3$. [5]

9 **(a)** The complex numbers z and w satisfy the simultaneous equations

$$|z| - w^* = -3 - \sqrt{2}\text{i} \text{ and } w^* + w + 5z = 1 + 20\text{i},$$

where w^* is the complex conjugate of w. Find the value of z and the corresponding value of w. [4]

(b) It is given that 8i is a root of the equation $\text{i}z^3 + (8 - 2\text{i})z^2 + az + 40 = 0$ where a is a complex number.

(i) Find a. [2]

(ii) Hence, find the other roots of the equation, leaving your answer in the form $a + b\text{i}$ where a and b are real constants. [3]

(iii) Deduce the number of real roots the equation $z^3 - (8 - 2\text{i})z^2 + a\text{i}z + 40 = 0$ has. [1]

10 For this question, you may use the results $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.

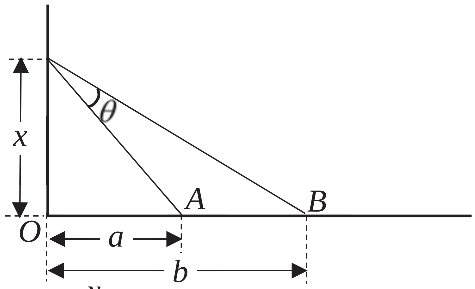
(i) Find $\sum_{r=1}^{n} r^2(2r-1)$ in terms of n. [2]

(ii) Find $\sum_{r=1}^{n} r^2(r-1)$ in terms of n. Hence find $\sum_{r=2}^{n-1} r(r+1)^2$ in terms of n. [5]

(iii) Without using a graphing calculator, find the sum of the series

$$4(25) - 5(36) + 6(49) - 7(64) + \cdots - 59(3600).$$ [3]

11 A boy is playing a ball game on a field. He arranges two cones A and B along the end of the field such that the cones are a and b metres respectively from one corner, O, of the field as shown in the diagram below. The boy stands along the edge of the field at x metres from O and kicks the ball between the two cones. The angle that the two cones subtends at the position of the boy is denoted by θ.



(i) Show that $\tan\theta = (b-a)\dfrac{x}{x^2+ab}$. [2]

(ii) It is given that $a=15$ and $b=20$. Find by differentiation, the value of x such that θ is at a maximum. [3]

(iii) It is given instead that the boy gets two friends to vary the position of both cones A and B along the end of the field such that $5 \le a \le 12$ and $b = 2a$, and the boy moves along the edge of the field such that his distance from cone A remains unchanged at 18 metres. Sketch a graph that shows how θ varies with a and find the largest possible value of θ. [4]

(iv) The boy runs until he is at a distance k metres from the goal line that is formed by the two cones and kicks the ball toward the goal line. The path of the ball is modelled by the equation $h = -\left(\dfrac{1}{10}k+2\right)^2+6$, where k is the distance of the ball from the goal line and h its corresponding height above the ground respectively. Find the angle that the path of the ball makes with the horizontal at the instant the ball crosses the goal line. [3]

12 In the study of force field, we are often interested in whether the work done in moving an object from point A to point B is independent of the path taken. If a force field is such that the work done is independent of the path taken, it is said to be a *conservative* field.

A force field **F** can be regarded as a vector $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ where M and N are functions of x and y. The path that the object is moving along is denoted by C. The work done in moving the object along the curve C from the point where $x = a$ to the point where $x = b$ is given by

$$W = \int_a^b \left[M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} \right] \mathrm{d}x,$$

where $y = \mathrm{f}(x)$ is the equation of the curve C.

(i) Sketch the curve C with equation $y^2 = 4(1-x)$, for $x \le 1$. [2]

(ii) Find an expression of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y. [1]

(iii) The points P and Q are on C with $x = 1$ and $x = -3$ respectively and Q is below the x-axis. Find the equation of the line PQ. [2]

For the rest of the question, the force field is given by $\mathbf{F} = x^2\mathbf{i} + xy^2\mathbf{j}$.

(iv) Show that the work done in moving an object along the curve C from Q to P is given by the integral $\displaystyle\int_{-3}^{1} \left(x^2 + 4x(1-x)^{\frac{1}{2}} \right) \mathrm{d}x$. Hence evaluate the exact work done in moving the object along the curve C from Q to P. [4]

(v) Find the work done in moving an object along the line PQ from Q to P to 2 decimal places. [2]

(vi) Determine, with reason, whether **F** is a conservative force field. [1]

–––––END OF PAPER–––––

NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

Candidate Name													
CT Class	1	8			Centre Number/ Index Number					/			

MATHEMATICS 9758/02

Paper 2

16th September 2019

3 Hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	Mark
1	
2	
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Total	

This document consists of **5** printed pages.

NANYANG JUNIOR COLLEGE
Internal Examinations

[Turn Over

Section A: Pure Mathematics [40 marks]

1 **(i)** Show that $\dfrac{1}{(n-1)!}-\dfrac{3}{n!}+\dfrac{2}{(n+1)!}=\dfrac{An^2+Bn+C}{(n+1)!}$ where A, B and C are constants to be determined. [2]

(ii) Hence find $\displaystyle\sum_{n=1}^{N}\frac{n^2-2n-1}{5(n+1)!}$ in terms of N. [3]

(iii) Give a reason why the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-2n-1}{5(n+1)!}$ converges and write down its value. [2]

2 The curve C has parametric equations $x=6t^2$, $y=\dfrac{2t}{\sqrt{1-t^2}}$, $0<t<1$.

(i) A line is tangent to the curve C at point A and passes through the origin O. Show that the line has equation $y=\dfrac{2}{3}x$. [4]

The region R is bounded by the curve and the tangent line in **(i)**.

(ii) Find the area of R. [3]

(iii) Write down the Cartesian equation of the curve C. [1]

(iv) Find the exact volume of the solid of revolution generated when R is rotated completely about the x-axis, giving your answer in the form $(a\ln b-c)\pi$, where constants a, b, c are to be determined. [4]

3 When a ball is dropped from a height of H m above the ground, it will rebound to a height of eH m where $0<e<1$. The height of each successive bounce will be e times of that of its previous height. It is also known that the time taken between successive bounce is given by $t=0.90305\sqrt{h}$ where h is the maximum height of the ball from the ground between these bounces. We can assume that there is negligible air resistance.
A ball is now dropped from a height of 10 m from the ground. Let t_n be the time between the n^{th} and $(n+1)^{\text{th}}$ bounce.

(i) Show that the total distance travelled by the ball just before the n^{th} bounce is $\dfrac{10(1+e-2e^n)}{1-e}$. [3]

(ii) Show that t_n is a geometric sequence. State the common ratio for this sequence. [3]

(iii) Find in terms of e the total distance the ball will travel and the time taken when it comes to rest. You may assume that between any two bounces, the time taken for the ball to reach its maximum height is the same as the time it takes to return to the ground. [3]

4 Referred to the origin, the points A and B have position vectors $-\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and $3\mathbf{i}+\mathbf{k}$ respectively. The plane π has equation $\mathbf{r}=\begin{pmatrix}5\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\2\\a\end{pmatrix}$, and the line l has equation $\mathbf{r}=\begin{pmatrix}3\\0\\a\end{pmatrix}+t\begin{pmatrix}4a\\4\\1\end{pmatrix}$, where a is a constant and λ, μ, and t are parameters.

(i) Show that for all real values of a, l is parallel to π. [2]

(ii) Find the value of a such that l and π have common points. [2]

For the rest of the question, let $a=1$.

(iii) Find the projection of $\overrightarrow{AB}$ onto π. [3]

(iv) Let F be the foot of perpendicular from A to π. The point C lies on AF extended such that $\angle ABF=\angle CBF$. Find a cartesian equation of the plane that contains C and l. [3]

(v) Let D be a point on l. Find the largest possible value of the non-reflex angle $\angle ADC$. [2]

Section B: Probability and Statistics [60 marks]

5 This question is about arrangements of all nine letters in the word ADDRESSEE.

(i) Find the number of different arrangements of the nine letters. [1]

(ii) Find the number of different arrangements that can be made with both the D's together and both the S's together. [2]

(iii) Find the number of different arrangements that can be made where the E's are separated by at least one letter and the D's are together. [2]

(iv) Find the number of different arrangements that can be made where the E's are not together, S's are not together and the D's are not together. [4]

6 Emergency flares are simple signalling devices similar to fireworks and they are designed to communicate a much more direct message in an emergency, for example, distress at sea.

A company categorised their stocks of emergency flares as 1-year old, 5-year old and 10-year old. The probabilities of successful firing of 1-year old, 5-year old and 10-year old emergency flares are 0.995, 0.970 and 0.750 respectively.

(i) Find the probability that, out of 100 randomly chosen 1-year old flares, at most 2 fail to fire successfully. [1]

(ii) One-year old flares are packed into boxes of 100 flares. Find the probability that, out of 50 randomly chosen boxes of 1-year old flares, not more than 48 of these boxes will have at most 2 flares that will fail to fire successfully in each box. [3]

(iii) Seven flares are chosen at random, of which one is 5 years old and six are 10 years old. Find the probability that

(a) the 5-year old flare fails to fire successfully and at least 4 of the 10-year old flares fire successfully, [2]

(b) at least 4 of the 7 flares fire successfully. [3]

7 With the move towards automated services at a bank, only two cashiers will be deployed to serve customers wanting to withdraw or deposit cash. For each cashier, the bank observed that the time taken to serve a customer is a random variable having a normal distribution with mean 150 seconds and standard deviation 45 seconds.

(i) Find the probability that the time taken for a randomly chosen customer to be served by a cashier is more than 180 seconds. [1]

(ii) One of the two cashiers serves two customers, one straight after the other. By stating a necessary assumption, find the probability that the total time taken by the cashier is less than 200 seconds. [3]

(iii) During peak-hour on a particular day, one cashier has a queue of 4 customers and the other cashier has a queue of 3 customers, and the cashiers begin to deal with customers at the front of their queues. Assuming that the time taken by each cashier to serve a customer is independent of the other cashier, find the probability that the 4 customers in the first queue will all be served before the 3 customers in the second queue are all served. [3]

8 To study if the urea serum content, u mmol per litre, depends on the age of a person, 10 patients of different ages, x years, admitted into the Accident and Emergency Department of a hospital are taken for study by a medical student. The results are shown in the table below.

Age, x (years)	37	44	56	60	64	71	74	77	81	89
Urea, u (mmol/l)	4.2	5.1	4.9	5.7	7.4	7.0	6.8	6.2	7.8	9.6

(i) Draw a scatter diagram of these data. [1]

(ii) By calculating the relevant product moment correlation coefficients, determine whether the relationship between u and x is modelled better by $u = ax + b$ or by $u = ae^{bx}$. Explain how you decide which model is better, and state the equation in this case. [5]

(iii) Explain why we can use the equation in **(ii)** to estimate the age of the patient when the urea serum is 7 mmol per litre. Find the estimated age of the patient when the urea serum is 7 mmol per litre [2]

(iv) The units for the urea serum is now given in mmol per decilitre.

(a) Give a reason if the product moment correlation coefficient calculated in **(ii)** will be changed. [1]

(b) Given that 1 decilitre is equal to 0.1 litre, re-write your equation in **(ii)** so that it can be used when the urea serum is given in mmol per decilitre. [1]

9 A game is played with 18 cards, each printed with a number from 1 to 6 and each number appears on exactly 3 cards. A player draws 3 cards without replacement. The random variable X is the number of cards with the same number.

(i) Show that $\mathrm{P}(X=2)=\dfrac{45}{136}$ and determine the probability distribution of X. [3]

(ii) Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X)=0.922$ correct to 3 significant figures. [3]

(iii) 40 games are played. Find the probability that the average number of cards with the same number is more than 1. [2]

(iv) In each game, Sam wins $\$(a+10)$ if there are cards with the same number, otherwise he loses $\$a$. Find the possible values of a, where a is an integer, such that Sam's expected winnings per game is positive. [4]

10 In the manufacturing of a computer device, there is a process which coats a computer part with a material that is supposed to be 100 microns thick. If the coating is too thin, the proper insulation of the computer device will not occur and it will not function reliably. Similarly, if the coating is too thick, the device will not fit properly with other computer components.

The manufacturer has calibrated the machine that applies the coating so that it has an average coating depth of 100 microns with a standard deviation of 10 microns. When calibrated this way, the process is said to be "in control".

Due to wear out of mechanical parts, there is a tendency for the process to drift. Hence the process has to be monitored to make sure that it is in control.

(i) After running the process for a reasonable time, a random sample of 50 computer devices is drawn. The sample mean is found to be 103.4 microns. Test at the 5% level of significance whether the sample suggests that the process is not in control. State any assumptions for this test to be valid. [4]

(ii) To ease the procedure of checking, the supervisor of this process would like to find the range of values of the sample mean of a random sample of size 50 that will suggests that the process is not in control at 5% level of significance. Find the required range of values of the sample mean, leaving your answer to 1 decimal places. [3]

On another occasion, a random sample of 40 computer devices is taken. The data can be summarised by

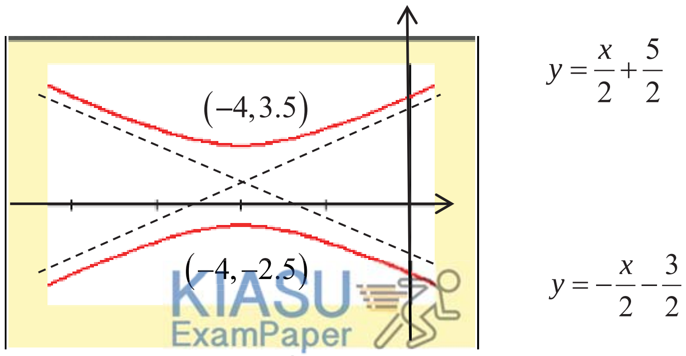
$$\Sigma(y-100)=164, \quad \Sigma(y-100)^2=9447.$$

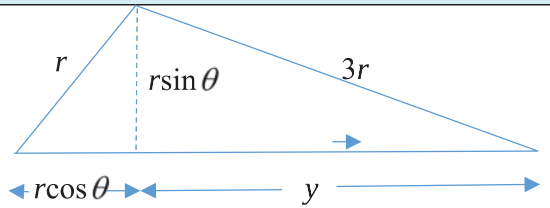
(iii) Calculate the unbiased estimate for the population mean and population variance of the thickness of a coating on the computer device. [2]

(iv) Give, in context, a reason why we may not be able to use 10 microns for the standard deviation of the thickness of a coating on the computer device. [1]

(v) Assume that the standard deviation has changed, test at the 4% level of significance whether the sample suggests that the process is not in control. [3]

———END OF PAPER———

Qn	
1	$y^2 = x^2 + 9$ Scale 2 parallel to x $y^2 = \left(\frac{x}{2}\right)^2 + 9$ Translate by -4 units parallel to x $y^2 = \left(\frac{(x+4)}{2}\right)^2 + 9$ Translate by 1/2 units parallel to y $\left(y - \frac{1}{2}\right)^2 = \left(\frac{(x+4)}{2}\right)^2 + 9$ Equations of asymptotes $y = \frac{1}{2} \pm \frac{x+4}{2} = \frac{x}{2} + \frac{5}{2}, -\frac{x}{2} - \frac{3}{2}$ 

Qn	
2(i)	 Using Pythagoras' Theorem, $y=\sqrt{9r^2-r^2\sin^2\theta}$ $\therefore x=r\cos\theta+r\sqrt{9-\sin^2\theta}\ =x=r\left[\cos\theta+\sqrt{\left(9-\sin^2\theta\right)}\right]$
2(ii)	Max $x=4r$
2(iii)	$x\approx r\left[\left(1-\frac{\theta^2}{2}\right)+\left(9-\theta^2\right)^{\frac{1}{2}}\right]$ $\approx r\left[1-\frac{\theta^2}{2}+3\left(1-\frac{\theta^2}{9}\right)^{\frac{1}{2}}\right]$ $\approx r\left[1-\frac{\theta^2}{2}+3\left(1-\frac{1}{2}\left(\frac{\theta^2}{9}\right)\right)\right]$ $=r\left(4-\frac{2}{3}\theta^2\right)$

Qn

3

$$\frac{x^2-3x+4}{x+2}\geq 2x+1$$

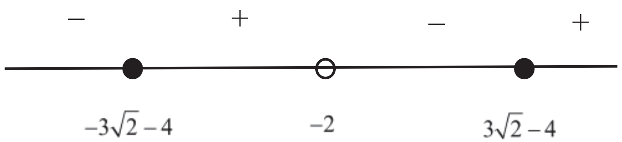
$$\frac{x^2-3x+4-(2x+1)(x+2)}{x+2}\geq 0$$

$$\frac{-x^2-8x+2}{x+2}\geq 0$$

$$\frac{x^2+8x-2}{x+2}\leq 0$$

$$\frac{(x+4)^2-18}{x+2}\leq 0$$

$$\frac{\left(x+4-3\sqrt{2}\right)\left(x+4+3\sqrt{2}\right)}{x+2}\leq 0$$



$x\leq -3\sqrt{2}-4$ or $-2<x\leq 3\sqrt{2}-4$

Replacing x by $-a^x$

$$\frac{a^{2x}+3a^x+4}{-a^x+2}\geq -2a^x+1$$

$$\frac{a^{2x}+3a^x+4}{a^x-2}\leq 2a^x-1$$

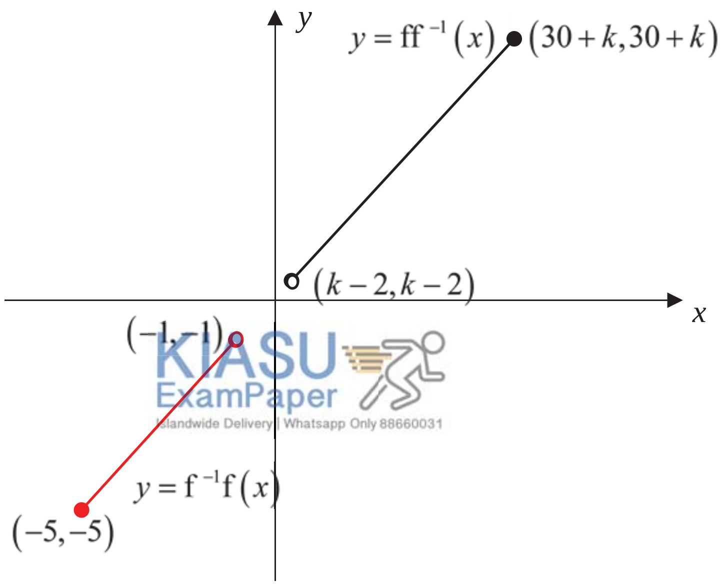
Since $-a^x<0$

$-a^x\leq -3\sqrt{2}-4$ or $-2<-a^x\leq 3\sqrt{2}-4$

$$x\geq \frac{\ln\left(3\sqrt{2}+4\right)}{\ln a} \quad \text{or} \quad x<\frac{\ln 2}{\ln a}$$

Qn	
4(i)	$\sin^4\theta = \frac{1}{4}\left(2\sin^2\theta\right)^2$ $= \frac{1}{4}(1-\cos 2\theta)^2$ $= \frac{1}{4}\left(1-2\cos 2\theta+\cos^2 2\theta\right)$ $= \frac{1}{4}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)$ $= \frac{1}{8}(3-4\cos 2\theta+\cos 4\theta)$
4(ii)	Let $x = 2\cos\theta$. Thus $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -2\sin\theta$. When $x=0$, $\theta=\frac{\pi}{2}$; when $x=2$, $\theta=0$. $\int_0^2 \left(4-x^2\right)^{\frac{3}{2}}\,\mathrm{d}x = \int_{\frac{\pi}{2}}^{0}\left(4-4\cos^2\theta\right)^{\frac{3}{2}}(-2\sin\theta)\,\mathrm{d}\theta$ $= 2\int_0^{\frac{\pi}{2}} \sin\theta\left(4\sin^2\theta\right)^{\frac{3}{2}}\,\mathrm{d}\theta$ $= 16\int_0^{\frac{\pi}{2}} \sin^4\theta\,\mathrm{d}\theta$ $= 2\int_0^{\frac{\pi}{2}} (3-4\cos 2\theta+\cos 4\theta)\,\mathrm{d}\theta$ $= 2\left[3\theta-2\sin 2\theta+\frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$ $= 3\pi$

Qn	
5(i)	$\overrightarrow{OD}=\lambda b+(1-\lambda)a, \quad \lambda\in\mathbb{R}$ $l_{OD}: r=s(\lambda b+(1-\lambda)a), \qquad s\in\mathbb{R}$
5(ii)	$\overrightarrow{OE}=s(\lambda\mathbf{b}+(1-\lambda)\mathbf{a}), \qquad$ for some $s,\lambda\in\mathbb{R}$. $\overrightarrow{OE}=\frac{1}{2}(\mathbf{b}+3\mathbf{a})$ $s(\lambda\mathbf{b}+(1-\lambda)\mathbf{a})=\frac{1}{2}(\mathbf{b}+3\mathbf{a})$ Since **a** and **b** are non-zero and non-parallel ($\lambda>0$), $s\lambda=\frac{1}{2}$ $s(1-\lambda)=\frac{3}{2}$ Solving, $\lambda=\frac{1}{4}$. **Area of** $\Delta BED=\frac{1}{2}\left\|\overrightarrow{BE}\times\overrightarrow{BD}\right\|$ $=\frac{1}{2}\|(-\frac{1}{2}\mathbf{b}+\frac{3}{2}\mathbf{a})\times(-\frac{3}{4}\mathbf{b}+\frac{3}{4}\mathbf{a})\|$ $=\frac{1}{2}\|\frac{3}{8}\mathbf{b}\times\mathbf{b}-\frac{3}{8}\mathbf{b}\times\mathbf{a}-\frac{9}{8}\mathbf{a}\times\mathbf{b}+\frac{9}{8}\mathbf{a}\times\mathbf{a}\|$ $=\frac{1}{2}\left\|\frac{3}{8}\mathbf{a}\times\mathbf{b}-\frac{9}{8}\mathbf{a}\times\mathbf{b}\right\|$ $=\frac{3}{8}\|\mathbf{a}\times\mathbf{b}\|$ $k=\frac{3}{8}$

Qn	
6(i)	$\mathrm{f}(x) = 2x^2 + 4x + k = 2(x+1)^2 + k - 2$ For f^{-1} to exist, f must be one-one. Largest value of $a = -1$
6(ii)	Let $y = \mathrm{f}(x)$ $y = 2(x+1)^2 + k - 2$ $x = -1 \pm \sqrt{\frac{1}{2}(y-k+2)}$ Since $x \in [-5,-1)$, $x < -1$ Hence $x = -1 - \sqrt{\frac{1}{2}(y-k+2)}$ For $-5 \le x < a$, $k-2 < \mathrm{f}(x) \le 30+k$, $\mathrm{f}(x) = -1 - \sqrt{\frac{1}{2}(x-k+2)}$, $D_{\mathrm{f}^{-1}} = (k-2, 30+k]$
6(iii)	 Number of solutions to $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$ is 0.

Qn	
7(i)	$V=\frac{1}{3}\pi x^2(3a-x) \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t}=(2\pi ax-\pi x^2)\frac{\mathrm{d}x}{\mathrm{d}t}$ Since $\frac{\mathrm{d}V}{\mathrm{d}t}=-\pi k\sqrt{x}$, $(2\pi ax-\pi x^2)\frac{\mathrm{d}x}{\mathrm{d}t}=-\pi k\sqrt{x}$ $(2ax-x^2)\frac{\mathrm{d}x}{\mathrm{d}t}=-k\sqrt{x}$
7(ii)	$\int \frac{2ax-x^2}{\sqrt{x}}\,\mathrm{d}x=\int -k\,\mathrm{d}t$ $\Rightarrow \int 2a\sqrt{x}-x^{\frac{3}{2}}\,\mathrm{d}x=-kt+c$ $\Rightarrow \frac{4}{3}ax^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}=-kt+c$ $\Rightarrow t=\frac{1}{k}\left[c-\frac{4}{3}ax^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}\right]$
7(iii)	If the tank is initially full, $x=2a$, thus $c=\frac{4}{3}a(2a)^{\frac{3}{2}}-\frac{2}{5}(2a)^{\frac{5}{2}}=\frac{16}{15}a^2\sqrt{2a}$ Thus $T_1=\frac{c}{k}=\frac{16}{5k}a^2\sqrt{2a}$ If the tank is initially half full, $x=a$, thus $c=\frac{4}{3}a(a)^{\frac{3}{2}}-\frac{2}{5}(a)^{\frac{5}{2}}=\frac{14}{15}a^2\sqrt{a}$ Thus $T_2=\frac{c}{k}=\frac{14}{5k}a^2\sqrt{a}$ Thus $\frac{T_1}{T_2}=\frac{16a^2\sqrt{2a}}{14a^2\sqrt{a}}=\frac{8\sqrt{2}}{7}$ Required ratio is $8\sqrt{2}:7$

Qn	
8(i)	$y^2+xy=4$ ___(1) Differentiate w.r.t. x, $2y\frac{\mathrm{d}y}{\mathrm{d}x}+y+x\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $(2y+x)\frac{\mathrm{d}y}{\mathrm{d}x}=-y$ ___(2) $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y}{2y+x}$ When $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y}{2y+x}=-\frac{1}{5}$ $5y=2y+x$ $x=3y$ Substitute $x=3y$ in (1), $y^2+3y^2=4$ $y^2=1$ Hence $y=1$ ($\because y>0$) Coordinates of the point are $(3,1)$
8(ii)	$y^2+z^2=10y$ ___(3) Differentiate (3) with respect to y, $2y+2z\frac{\mathrm{d}z}{\mathrm{d}y}=10$ $y+z\frac{\mathrm{d}z}{\mathrm{d}y}=5$ $\frac{\mathrm{d}z}{\mathrm{d}y}=\frac{5-y}{z}$ $\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\mathrm{d}z}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}$ $=\frac{5-y}{z}\left(-\frac{y}{2y+x}\right)\frac{\mathrm{d}x}{\mathrm{d}t}$

Qn	

At $x=3$, $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{5}$ from (i).

From (3), $1^2+z^2=10$

$$z=3 \quad (\because z>0)$$

Hence $\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{5-1}{3}\left(-\frac{1}{5}\right)\frac{1}{2}=-\frac{2}{15}$

<u>Alternatively,</u>

$y^2+z^2=10y$ ___(3)

Differentiate (3) with respect to y,

$$2y+2z\frac{\mathrm{d}z}{\mathrm{d}y}=10$$

$$y+z\frac{\mathrm{d}z}{\mathrm{d}y}=5$$

$$\frac{\mathrm{d}z}{\mathrm{d}y}=\frac{5-y}{z}$$

From (2), $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}$

$$=-\frac{y}{2y+x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

At $x=3$, $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{5}$ from (i).

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{5}\cdot\frac{1}{2}=-\frac{1}{10}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\mathrm{d}z}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$=\frac{5-y}{z}\frac{\mathrm{d}y}{\mathrm{d}t}$$

From (3), $1^2+z^2=10$

$$z=3 \quad (\because z>0)$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{5-1}{3}\left(-\frac{1}{10}\right)=-\frac{2}{15}$$

Qn	
9(a)	$\lvert z\rvert - w^* = -3-\sqrt{2}\mathrm{i}$ $\Rightarrow w^* = \lvert z\rvert + 3 + \sqrt{2}\mathrm{i}$ and $w = \lvert z\rvert + 3 - \sqrt{2}\mathrm{i}$ Sub into $w^* + w + 5z = 1 + 20\mathrm{i}$, Let $z = x + y\mathrm{i}$, where x and y are real. $2\sqrt{x^2+y^2} + 5x + 5y\mathrm{i} = -5 + 20\mathrm{i}$ Comparing real and imaginary components, $2\sqrt{x^2+y^2} + 5x = -5$, $5y = 20 \Rightarrow y = 4$ $2\sqrt{x^2+16} + 5x = -5$ $2\sqrt{x^2+16} = -5x-5$ $4\left(x^2+16\right) = 25x^2 + 50x + 25$ $21x^2 + 50x - 39 = 0$ $x = \frac{13}{21}$ or $x = -3$ (reject $x = \frac{13}{21} \because 2\sqrt{x^2+y^2} + 5x = -5$) $z = -3 + 4\mathrm{i}$, $w = 8 - \sqrt{2}\mathrm{i}$
9(b) (i)	$\mathrm{i}(8\mathrm{i})^3 + (8-2\mathrm{i})(8\mathrm{i})^2 + a(8\mathrm{i}) + 40 = 0$ $512 - 64(8-2\mathrm{i}) + 8a\mathrm{i} + 40 = 0$ $a\mathrm{i} = -5 - 16\mathrm{i} \Rightarrow a = -16 + 5\mathrm{i}$
9(b) (ii)	$(z-8\mathrm{i})\left(Az^2 + Bz + C\right) = 0$ Comparing coefficient for z^3, $A = \mathrm{i}$ Comparing coefficient for constant, $C = 5\mathrm{i}$ Comparing coefficient for z^2, $B - 8\mathrm{i}A = 8 - 2\mathrm{i}$ $B = -2\mathrm{i}$

Qn	
	$(z-8\mathrm{i})(\mathrm{i}z^2-2\mathrm{i}z+5\mathrm{i})=0$ $(z-8\mathrm{i})(z^2-2z+5)=0$ The other roots are $z=\dfrac{2\pm\sqrt{4-20}}{2}$ $=1\pm 2\mathrm{i}$
9(b) (iii)	Replacing z with iz, $\mathrm{i}z=8\mathrm{i}$ or $\mathrm{i}z=1\pm 2\mathrm{i}$ $z=8$ $\quad z=\pm 2-\mathrm{i}$ Therefore 1 real root.

Qn	
10(i)	$\begin{aligned}\sum_{r=1}^{n} r^2(2r-1) &= \sum_{r=1}^{n}\left(2r^3-r^2\right)\\ &= \frac{2}{4}n^2(n+1)^2-\frac{n}{6}(n+1)(2n+1)\\ &= \frac{1}{6}n(n+1)\left[3n(n+1)-(2n+1)\right]\\ &= \frac{1}{6}n(n+1)\left(3n^2+n-1\right)\end{aligned}$
10(ii)	$\begin{aligned}\sum_{r=1}^{n} r^2(r-1) &= \sum_{r=1}^{n}\left(r^3-r^2\right)\\ &= \frac{1}{4}n^2(n+1)^2-\frac{n}{6}(n+1)(2n+1)\\ &= \frac{1}{12}n(n+1)\left[3n(n+1)-2(2n+1)\right]\\ &= \frac{1}{12}n(n+1)\left(3n^2-n-2\right)\\ &= \frac{1}{12}n(n+1)(3n+2)(n-1)\end{aligned}$ $\begin{aligned}\sum_{r=2}^{n-1} r(r+1)^2 &= \sum_{k-1=2}^{k-1=n-1}(k-1)k^2\\ &= \sum_{k=3}^{k=n} k^2(k-1)\\ &= \sum_{k=1}^{k=n} k^2(k-1)-\sum_{k=1}^{k=2} k^2(k-1)\\ &= \frac{1}{12}n(n+1)(n-1)(3n+2)-\frac{1}{12}(2)(3)(1)(8)\\ &= \frac{1}{12}n(n+1)(n-1)(3n+2)-4\end{aligned}$

Qn	
10(iii)	**(iii)** $4(25)-5(36)-\cdots-59(3600)$ $=4(25)+5(36)+\cdots+59(3600)-2\left[5(36)+7(64)\cdots+59(3600)\right]$ $=\sum_{r=5}^{60} r^2(r-1)-2\sum_{r=3}^{30}(2r)^2(2r-1)$ $=\sum_{r=1}^{60} r^2(r-1)-\sum_{r=1}^{4} r^2(r-1)-2\sum_{r=1}^{30}(2r)^2(2r-1)+2\sum_{r=1}^{2}(2r)^2(2r-1)$ $=\frac{1}{12}(60)(61)(59)(182)-\frac{1}{12}(4)(5)(3)(14)$ $\quad-\frac{4}{3}(30)(31)(2729)+\frac{4}{3}(2)(3)(13)$ $=-108836$

Qn	
11(i)	Denote the position of the boy by X. Let $\angle OXA=\alpha$ and $\angle OXB=\beta$. Then $\theta=\beta-\alpha$ and $\tan\theta=\tan(\beta-\alpha)=\dfrac{\tan\beta-\tan\alpha}{1+\tan\beta\tan\alpha}$ $=\dfrac{\frac{b}{x}-\frac{a}{x}}{1+\frac{b}{x}\cdot\frac{a}{x}}$ $=\dfrac{\left(\frac{b-a}{x}\right)x^2}{\left(1+\frac{ab}{x^2}\right)x^2}=(b-a)\dfrac{x}{x^2+ab}$ Alternatively: Applying sine rule, $\dfrac{\sin\theta}{b-a}=\dfrac{\sin B}{\sqrt{x^2+a^2}} \quad\Rightarrow\quad \sin\theta=\dfrac{(b-a)}{\sqrt{x^2+a^2}}\sin B$ $=\dfrac{(b-a)}{\sqrt{x^2+a^2}}\cdot\dfrac{x}{\sqrt{x^2+b^2}}$ Applying cosine rule, $(b-a)^2=(x^2+a^2)+(x^2+b^2)-2\sqrt{x^2+a^2}\sqrt{x^2+b^2}\cos\theta$ $\Rightarrow \cos\theta=\dfrac{(x^2+a^2)+(x^2+b^2)-(b-a)^2}{2\sqrt{x^2+a^2}\sqrt{x^2+b^2}}$ $=\dfrac{x^2+ab}{\sqrt{x^2+a^2}\sqrt{x^2+b^2}}$ Hence $\tan\theta=\dfrac{(b-a)}{\sqrt{x^2+a^2}}\cdot\dfrac{x}{\sqrt{x^2+b^2}}\Big/\dfrac{x^2+ab}{\sqrt{x^2+a^2}\sqrt{x^2+b^2}}$ $=(b-a)\dfrac{x}{x^2+ab}$

Qn	
(ii)	Differentiate $\tan\theta = \dfrac{5x}{x^2+300}$ with respect to x: $$\sec^2\theta \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{5\left(x^2+300\right)-5x\cdot 2x}{\left(x^2+300\right)^2}$$ $$= \frac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2}$$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0 \Rightarrow x^2 = 300$ $x = \sqrt{300}$ or 17.3 (3s.f.) Alternatively, Differentiate $\tan\theta\left(x^2+300\right) = 5x$ with respect to x: $$\sec^2\theta \frac{\mathrm{d}\theta}{\mathrm{d}x}\left(x^2+300\right) + \tan\theta(2x) = 5$$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0 \Rightarrow \tan\theta(2x) = 5$ Substitute $\tan\theta = \dfrac{5}{2x}$ into $\tan\theta = \dfrac{5x}{x^2+300}$: $$\frac{5}{2x} = \frac{5x}{x^2+300}$$ $$x^2+300 = 2x^2$$ $$x^2 = 300$$ $$x = \sqrt{300}$$ $$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2\sec^2\theta}.$$

Qn	
	Since $\left(x^2+300\right)^2 \sec^2\theta > 0$ for any x and θ, it suffices to test the sign of $-x^2+300$.

x	$\left(\sqrt{300}\right)^-$	$\sqrt{300}$	$\left(\sqrt{300}\right)^+$
$\dfrac{\mathrm{d}\theta}{\mathrm{d}x}=\dfrac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2\sec^2\theta}$	>0	=0	<0

Hence θ is maximum

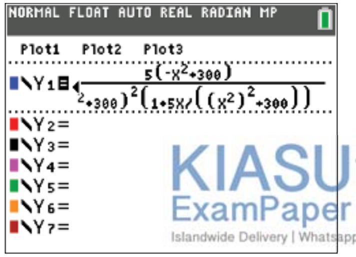
Alternatively, apply second derivative test:

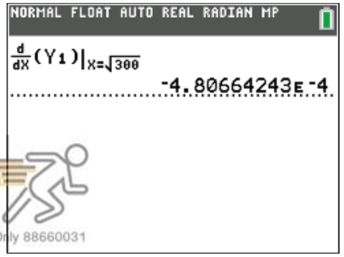
Using GC:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2\sec^2\theta}$$

$$=\frac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2\left(1+\tan^2\theta\right)}$$

$$=\frac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2\left(1+\frac{5x}{x^2+300}\right)}$$





$$\left.\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\right|_{x=\sqrt{300}}=-4.81\times10^{-4}<0$$. Hence θ is maximum.

Or:

Qn

Differentiate $\sec^2\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}x}=\dfrac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2}$ with respect to x:

$$2\sec\theta\left(\sec\theta\tan\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)\frac{\mathrm{d}\theta}{\mathrm{d}x}+\sec^2\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}$$

$$=\frac{5(-2x)\left(x^2+300\right)^2-5\left(-x^2+300\right)\cdot 2\cdot 2x\left(x^2+300\right)}{\left(x^2+300\right)^4}$$

$$=\frac{-10x\left(x^2+300\right)\left[\left(x^2+300\right)+2\left(-x^2+300\right)\right]}{\left(x^2+300\right)^4}$$

$$=\frac{-10x\left(x^2+300\right)\left[-x^2+900\right]}{\left(x^2+300\right)^4}$$

At $x=\sqrt{300}$, $\dfrac{\mathrm{d}\theta}{\mathrm{d}x}=0$, $-x^2+900=-300+900=600$

$\dfrac{-10x\left(x^2+300\right)(600)}{\left(x^2+300\right)^4}<0$, and $\sec^2\theta>0$,

Hence $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}x^2}<0$ and θ is maximum.

(iii)

Since $b=2a$, $\tan\theta=(2a-a)\dfrac{x}{x^2+a(2a)}$

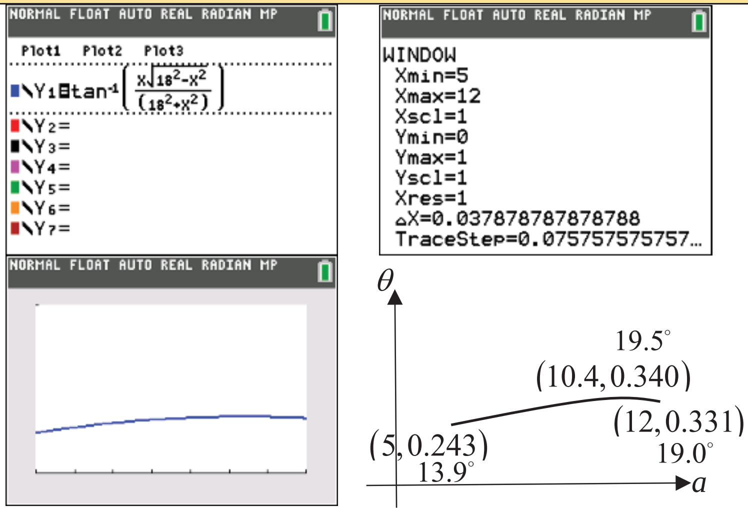
$$\tan\theta=\frac{ax}{x^2+2a^2}$$

$$x^2+a^2=18^2$$
$$x^2=18^2-a^2$$

Hence $\tan\theta=\dfrac{a\sqrt{18^2-a^2}}{18^2-a^2+2a^2}$

$$\theta=\tan^{-1}\left(\frac{a\sqrt{18^2-a^2}}{18^2+a^2}\right)$$

Qn



Largest possible $\theta = 0.340$ rad (3 s.f.) or $\theta = 19.5^\circ$ (1 d.p.)

(iv)

Differentiate $h = -\left(\frac{1}{10}k + 2\right)^2 + 6$ with respect to k:

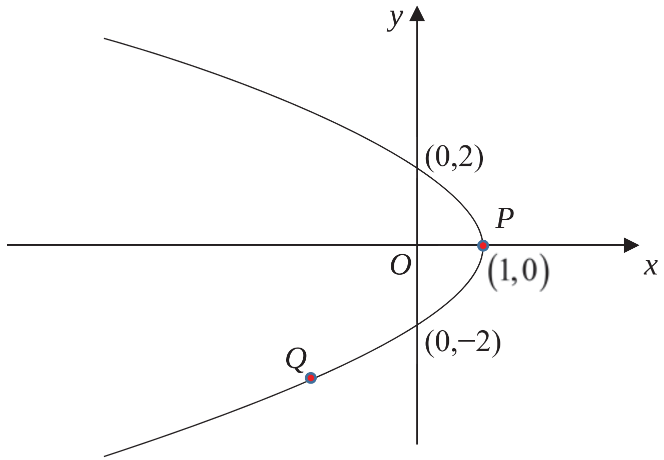
$$\frac{dh}{dk} = -\frac{2}{10}\left(\frac{1}{10}k + 2\right)$$

At the instant when the ball crosses the goal line, $k = 0$

$$\frac{dh}{dk} = -\frac{2}{5}$$

$$\tan\phi = -\frac{2}{5}$$

$\phi = -0.381$ (3 s.f.) or $\theta \approx -21.8^\circ$ (1 d.p.)

Qn	
12(i)	
12(ii)	$y^2 = 4(1-x)$ Differentiate wrt x: $2y\frac{\mathrm{d}y}{\mathrm{d}x} = -4$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{y}$
12(iii)	At P, $x=1$, $y=0$; at Q, $x=-3$, $y=-4$. Thus equation of line PQ is $\frac{y}{x-1} = \frac{0-(-4)}{1-(-3)} \quad \Rightarrow \quad y = x-1$

<table>
<tr><th>Qn</th><th></th></tr>
<tr><td>12(iv)</td><td>
Along arc QP, $y=-2\sqrt{1-x}$.

$$W_C=\int_{-3}^{1}\left(x^2+xy^2\cdot\left(\frac{-2}{y}\right)\right)\mathrm{d}x$$
$$=\int_{-3}^{1}\left(x^2-2xy\right)\mathrm{d}x$$
$$=\int_{-3}^{1}\left(x^2+4x(1-x)^{\frac{1}{2}}\right)\mathrm{d}x$$

<table>
<tr><td>
Method 1

$$W_C=\left[\frac{x^3}{3}\right]_{-3}^{1}-\left[\frac{8}{3}x(1-x)^{\frac{3}{2}}\right]_{-3}^{1}$$
$$+\int_{-3}^{1}\frac{8}{3}(1-x)^{\frac{3}{2}}\,\mathrm{d}x$$
$$=\frac{28}{3}-64-\left[\frac{16}{15}(1-x)^{\frac{5}{2}}\right]_{-3}^{1}$$
$$=-\frac{164}{3}+\frac{512}{15}=-\frac{308}{15}$$
</td><td>
Method 2

$$W_C=\int_{-3}^{1}\left(x^2+4x(1-x)^{\frac{1}{2}}\right)\mathrm{d}x$$
$$=\int_{-3}^{1}\left(x^2-4(1-x)(1-x)^{\frac{1}{2}}+4(1-x)^{\frac{1}{2}}\right)\mathrm{d}x$$
$$=\left[\frac{x^3}{3}+\frac{8}{5}(1-x)^{\frac{5}{2}}-\frac{8}{3}(1-x)^{\frac{3}{2}}\right]_{-3}^{1}$$
$$=\frac{28}{3}-\frac{256}{5}+\frac{64}{3}=-\frac{308}{15}$$
</td></tr>
</table>
</td></tr>
<tr><td>12(v)</td><td>

$$W_L=\int_{-3}^{1}\left(x^2+xy^2\right)\mathrm{d}x=\int_{-3}^{1}\left(x^2+x(x-1)^2\right)\mathrm{d}x$$
$$=-33.33$$
</td></tr>
<tr><td>12(vi)</td><td>Since the work done for the two paths are different, the force field F is not conservative.</td></tr>
</table>

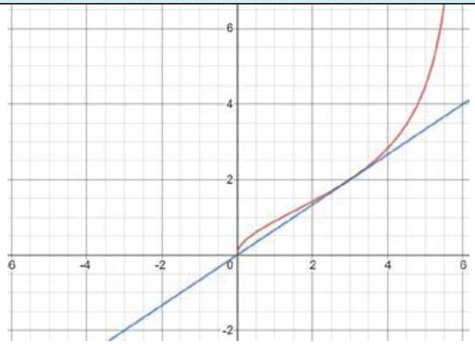
Qn	
1(i)	$\dfrac{1}{(n-1)!}-\dfrac{3}{n!}+\dfrac{2}{(n+1)!}=\dfrac{n(n+1)-3(n+1)+2}{(n+1)!}$ $=\dfrac{n^2+n-3n-3+2}{(n+1)!}=\dfrac{n^2-2n-1}{(n+1)!}$ Hence $A=1, B=-2, C=-1$
1(ii)	$\displaystyle\sum_{n=1}^{N}\frac{n^2-2n-1}{5(n+1)!}=\frac{1}{5}\sum_{n=1}^{N}\frac{n^2-2n-1}{(n+1)!}=\frac{1}{5}\sum_{n=1}^{N}\left[\frac{1}{(n-1)!}-\frac{3}{n!}+\frac{2}{(n+1)!}\right]$ $=\dfrac{1}{5}\left[\begin{array}{c}\frac{1}{0!}-\frac{3}{1!}+\cancel{\frac{2}{2!}}\\+\frac{1}{1!}-\cancel{\frac{3}{2!}}+\cancel{\frac{2}{3!}}\\+\cancel{\frac{1}{2!}}-\cancel{\frac{3}{3!}}+\cancel{\frac{2}{4!}}\\+\vdots\\+\cancel{\frac{1}{(N-3)!}}-\cancel{\frac{3}{(N-2)!}}+\cancel{\frac{2}{(N-1)!}}\\+\cancel{\frac{1}{(N-2)!}}-\cancel{\frac{3}{(N-1)!}}+\frac{2}{N!}\\+\cancel{\frac{1}{(N-1)!}}-\frac{3}{N!}+\frac{2}{(N+1)!}\end{array}\right]$ $=\dfrac{1}{5}\left[\dfrac{1}{0!}-\dfrac{3}{1!}+\dfrac{1}{1!}+\dfrac{2}{N!}+\dfrac{2}{(N+1)!}-\dfrac{3}{N!}\right]=\dfrac{1}{5}\left(\dfrac{2}{(N+1)!}-\dfrac{1}{N!}-1\right)$
1(iii)	$\displaystyle\sum_{n=1}^{\infty}\frac{n^2-2n-1}{5(n+1)!}=\lim_{N\to\infty}\sum_{n=1}^{N}\frac{n^2-2n-1}{5(n+1)!}=\lim_{N\to\infty}\left[\frac{1}{5}\left(\frac{2}{(N+1)!}-\frac{1}{N!}-1\right)\right]$ Since $\dfrac{1}{(N+1)!}\to 0$ & $\dfrac{1}{N!}\to 0$ when $N\to\infty$, $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-2n-1}{5(n+1)!}$ converges to $-\dfrac{1}{5}$

Qn	
2(i)	$x = 6t^2 \qquad y = \dfrac{2t}{\sqrt{1-t^2}}$ $\dfrac{dx}{dt} = 12t \qquad \dfrac{dy}{dt} = \dfrac{2\sqrt{1-t^2} - 2t\left(\frac{-2t}{2\sqrt{1-t^2}}\right)}{1-t^2}$ $= \dfrac{2\left(1-t^2\right) + 2t^2}{\left(1-t^2\right)\sqrt{1-t^2}}$ $= \dfrac{2}{\left(1-t^2\right)^{3/2}}$ $\dfrac{dy}{dx} = \dfrac{1}{6t\left(1-t^2\right)^{3/2}}$ The tangent to the curve C has equation $y = \dfrac{1}{6t\left(1-t^2\right)^{3/2}}x$ for some t. $\dfrac{2t}{\sqrt{1-t^2}} = \dfrac{1}{6t\left(1-t^2\right)^{3/2}} \cdot 6t^2 \quad (t \neq 0)$ $2\left(1-t^2\right) = 1$ $t^2 = \dfrac{1}{2}$ $t = \dfrac{1}{\sqrt{2}}$ since $0 < t < 1$ Hence the tangent line has equation $y = \dfrac{1}{6\frac{1}{\sqrt{2}}\left(1-\frac{1}{2}\right)^{3/2}}x$ $y = \dfrac{2}{3}x$ Alternative method:

Qn	
	Cartesian equation of curve C: Sub $t=\sqrt{\frac{x}{6}}$ into $y=\frac{2t}{\sqrt{1-t^2}}$ to get $y=\frac{2\sqrt{\frac{x}{6}}}{\sqrt{1-\frac{x}{6}}}=\frac{2\sqrt{x}}{\sqrt{6-x}}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cdot\frac{1}{2\sqrt{x}}\cdot\sqrt{6-x}-2\sqrt{x}\cdot\frac{-1}{2\sqrt{6-x}}}{6-x}$ $=\frac{6}{(6-x)^{3/2}\sqrt{x}}$ The required tangent line passes through the point $\left(6t^2,\frac{2t}{\sqrt{1-t^2}}\right)$ for some x. $y=\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\vert_{x=6t^2}x$ $\frac{2t}{\sqrt{1-t^2}}=\frac{6}{\left(6-6t^2\right)^{3/2}\sqrt{6t^2}}\cdot 6t^2 \quad (t\neq 0)$ $2=\frac{1}{\left(1-t^2\right)}$ $2\left(1-t^2\right)=1$ $t^2=\frac{1}{2}$ $t=\frac{1}{\sqrt{2}}$ since $0\leq t<1$ Hence the tangent line has equation $y=\frac{1}{6\frac{1}{\sqrt{2}}\left(1-\frac{1}{2}\right)^{3/2}}x$ $y=\frac{2}{3}x$

Qn	

2(ii)



The point of intersection, A, of the tangent line and the curve corresponds to $t=\dfrac{1}{\sqrt{2}}$.

Coordinates of point A are $(3,2)$.

$$\text{Area of } R=\int_0^3 y\ \mathrm{d}x-\frac{1}{2}(3)(2)$$

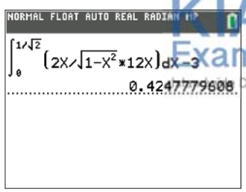
$$=\int_0^{\frac{1}{\sqrt{2}}}\frac{2t}{\sqrt{1-t^2}}\cdot 12t\mathrm{d}t-\frac{1}{2}(3)(2)$$

$$=0.425 \text{ (3 s.f.)}$$

<u>Alternatively,</u>

$$\text{Area of } R=\int_0^3 y-\frac{2}{3}x\ \mathrm{d}x=\int_0^{\frac{1}{\sqrt{2}}}\frac{2t}{\sqrt{1-t^2}}\cdot 12t\mathrm{d}t-\int_0^3\frac{2}{3}x\ \mathrm{d}x$$

$$=0.425 \text{ (3 s.f.)}$$



Qn	
2(iii)	Sub $t=\sqrt{\frac{x}{6}}$ into $y=\frac{2t}{\sqrt{1-t^2}}$ to get $y=\frac{2\sqrt{\frac{x}{6}}}{\sqrt{1-\frac{x}{6}}}=\frac{2\sqrt{x}}{\sqrt{6-x}}$
2(iv)	Volume, $V=\pi\int_0^3 y^2\ \mathrm{d}x-\frac{1}{3}\pi\left(2^2\right)(3)$ $=\pi\int_0^3 \frac{4x}{6-x}\ \mathrm{d}x-4\pi$ $=\pi\int_0^3 -4+\frac{24}{6-x}\ \mathrm{d}x-4\pi$ $=\pi\left[-4x-24\ln\lvert 6-x\rvert\right]_0^3-4\pi$ $=\pi\left[-12-24\ln 3+24\ln 6\right]-4\pi$ $=\pi\left[24\ln 2\right]-16\pi$ $=\left(24\ln 2-16\right)\pi$ <u>Alternatively</u>, Volume, $V=\pi\int_0^3\left(\frac{2\sqrt{x}}{\sqrt{6-x}}\right)^2-\left(\frac{2}{3}x\right)^2\ \mathrm{d}x$ $=\pi\int_0^3 \frac{4x}{6-x}-\frac{4}{9}x^2\ \mathrm{d}x$ $=\pi\int_0^3 -4+\frac{24}{6-x}-\frac{4}{9}x^2\ \mathrm{d}x$ $=\pi\left[-4x-24\ln\lvert 6-x\rvert-\frac{4}{27}x^3\right]_0^3$ $=\pi\left[\left(-12-24\ln 3-4\right)+24\ln 6\right]$ $=\left(24\ln 2-16\right)\pi$

Qn	
3(i)	Let S_n be the total distance travelled by the ball just before the n-th bounce. Thus $S_n = 10 + 2(10e) + 2(10e^2) + \cdots + 2(10e^{n-1})$ $= 20 + 20e + 20e^2 + \cdots + 20e^{n-1} - 10$ $= \dfrac{20(1-e^n)}{1-e} - 10$ $= \dfrac{10(1+e-2e^n)}{1-e}$
3(ii)	Let d_k be the maximum height of the ball after the k-th bounce. Thus $d_k = 10e^k$. Hence $t_k = 0.90305\sqrt{d_k}$. Thus for $k \in \mathbb{Z}^+$, $\dfrac{t_{k+1}}{t_k} = \dfrac{0.90305\sqrt{d_{k+1}}}{0.90305\sqrt{d_k}}$ $= \dfrac{\sqrt{10e^{k+1}}}{\sqrt{10e^k}} = \sqrt{e}$ Hence t_n is a geometric sequence with common ratio $\sqrt{e}$.
3(iii)	As $n \to \infty$, the ball will come to rest. Thus total distance travelled is $S = \lim_{n\to\infty}\left(\dfrac{10(1+e-2e^n)}{1-e}\right)$ $= \dfrac{10(1+e)}{1-e}$ Total time taken $= 0.5(0.90305)\sqrt{10} + \sum_{n=1}^{\infty} t_n$ $= 1.4278 + \dfrac{0.90305\sqrt{10e}}{1-\sqrt{e}}$ $= 1.43 + \dfrac{2.86\sqrt{e}}{1-\sqrt{e}}$

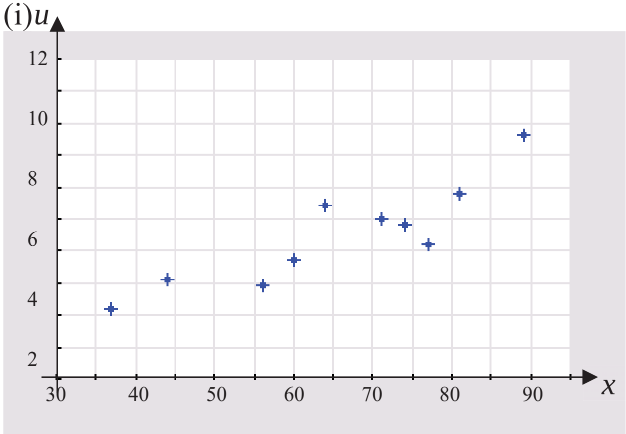
Qn	
4(i)	$\mathbf{n}_{\pi_1} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ a \end{pmatrix} = \begin{pmatrix} 2 \\ -(2a+1) \\ 4 \end{pmatrix}$ $\mathbf{d}_l \cdot \mathbf{n}_{\pi_1} = \begin{pmatrix} 4a \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2a-1 \\ 4 \end{pmatrix}$ $= 8a - 8a - 4 + 4 = 0$ Since $\mathbf{d}_l \perp \mathbf{n}_{\pi_1}$, then l is parallel to π_1
4(ii)	Equation of π_1: $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -2a-1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2a-1 \\ 4 \end{pmatrix} = 10$ Since l is parallel to π_1, we want l to lie inside π_1. $\begin{pmatrix} 3 \\ 0 \\ a \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2a-1 \\ 4 \end{pmatrix} = 10$ $6 + 4a = 10 \quad \Rightarrow \quad a = 1$
4(iii)	Since B lies on the line, required vector is the vector $\overrightarrow{FB}$, where F is the foot of perpendicular from A to π_1. $\overrightarrow{OF} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + k\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ for some $k \in \mathbb{R}$. $\left[\begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + k\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 10$ $-19 + 29k = 10 \quad \Rightarrow \quad k = 1$

Qn	
	$\overrightarrow{OF}=\begin{pmatrix}1\\0\\2\end{pmatrix}$ $\overrightarrow{FB}=\begin{pmatrix}3\\0\\1\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}$
4(iv)	Let π_2 be the required plane. Point C is the reflection of A in π_1. $\overrightarrow{OC}=\begin{pmatrix}-1\\3\\-2\end{pmatrix}+2\begin{pmatrix}2\\-3\\4\end{pmatrix}=\begin{pmatrix}3\\-3\\6\end{pmatrix}$, $\overrightarrow{BC}=\begin{pmatrix}3\\-3\\6\end{pmatrix}-\begin{pmatrix}3\\0\\1\end{pmatrix}=\begin{pmatrix}0\\-3\\5\end{pmatrix}$ $\mathbf{n}_{\pi_2}=\begin{pmatrix}4\\4\\1\end{pmatrix}\times\begin{pmatrix}0\\-3\\5\end{pmatrix}=\begin{pmatrix}23\\-20\\-12\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}23\\-20\\-12\end{pmatrix}=\begin{pmatrix}3\\0\\1\end{pmatrix}\cdot\begin{pmatrix}23\\-20\\-12\end{pmatrix}=57$. Thus $23x-20y-12z=57$
4(v)	Maximum value of $\angle ADC = 2\times\angle CDF$ $=2\times\cos^{-1}\dfrac{\begin{pmatrix}2\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}23\\-20\\-12\end{pmatrix}}{\left\|\begin{pmatrix}2\\-3\\4\end{pmatrix}\right\|\left\|\begin{pmatrix}23\\-20\\-12\end{pmatrix}\right\|}$ $=2\times\cos^{-1}\dfrac{58}{\sqrt{29}\sqrt{1073}}\approx 2(70.804)$ $=141.6°$ (1dp)

Qn	
5(i)	Number of ways = $\dfrac{9!}{2!2!3!} = 15120$
5(ii)	Number of ways = $\dfrac{7!}{3!} = 840$
5(iii)	Number of ways $= \dfrac{5!}{2!} \cdot {}^{6}C_{3}$ $= 1200$
5(iv)	Let the event D be such that the D's are together, the event E be such that the E's are together and S be such that the S's are together. $n(D \cup E \cup S) = n(D) + n(E) + n(S) - n(D \cap E)$ $- n(E \cap S) - n(D \cap S) + n(D \cap E \cap S)$ $= \dfrac{8!}{2!3!} + \dfrac{7!}{2!2!} + \dfrac{8!}{2!3!} - \dfrac{6!}{2!} - \dfrac{6!}{2!} - \dfrac{7!}{3!} + 5!$ $= 6540$ Number of ways $= n(D' \cap E' \cap S')$ $= n(S) - n(D \cup E \cup S)$ $= 15120 - 6540$ $= 8580$

Qn	
6(i)	Let X denotes the number of 1-year old flares that fail to fire successfully, out of the 100, $X \sim \mathrm{B}(100,\ 0.005)$ $\mathrm{P}(X \le 2) = 0.985897 \approx 0.986$
6(ii)	Let Y denotes the number of boxes with a hundred 1-year old flares with at most 2 that fail to fire, out of 50 boxes, ie $Y \sim \mathrm{B}(50,\ 0.985897)$ $\mathrm{P}(Y \le 48) = 0.156856 \approx 0.157$
6(iii)	Let T denotes the number of 10-year old flares that fire successfully, out of the 6, $T \sim \mathrm{B}(6,\ 0.75)$ (a) Required prob = $(1-0.970)\times \mathrm{P}(T \ge 4)$ $= 0.03\times\left(1-P(T\le 3)\right)$ $= 0.0249$ (b) P(at least 4 of the 7 flares fire successfully) $= 0.024917 + 0.970\times \mathrm{P}(T \ge 3)$ $= 0.024917 + 0.970\times\left(1-\mathrm{P}(T\le 2)\right)$ $= 0.958$

Qn	
7(i)	Let X be the rv denoting the amount of time taken by a cashier to deal with a randomly chosen customer, ie $X \sim \mathrm{N}(150,\ 45^2)$. $\mathrm{P}(X > 180) = 0.25249 \approx 0.252$
7(ii)	Assume that the time taken to deal with each customer is independent of the other, ie $X_1 + X_2 \sim \mathrm{N}(2\times 150, 2\times 45^2)$ $\mathrm{P}(X_1 + X_2 < 200) = 0.058051 \approx 0.0581$
7(iii)	Let Y be the rv denoting the amount of time taken by a the second cashier to deal with a randomly chosen customer, ie $Y \sim \mathrm{N}(150,\ 45^2)$. $X_1 + X_2 + X_3 + X_4 \sim \mathrm{N}(4\times 150, 4\times 45^2)$ and $Y_1 + Y_2 + Y_3 \sim \mathrm{N}(3\times 150, 3\times 45^2)$ $\mathrm{P}(X_1 + X_2 + X_3 + X_4 < Y_1 + Y_2 + Y_3) = \mathrm{P}(X_1 + X_2 + X_3 + X_4 - (Y_1 + Y_2 + Y_3) < 0)$ Using $X_1 + X_2 + X_3 + X_4 - (Y_1 + Y_2 + Y_3) \sim N(150, 7\times 45^2)$ $\mathrm{P}(X_1 + X_2 + X_3 + X_4 - (Y_1 + Y_2 + Y_3) < 0) = 0.10386 \approx 0.104$ KIASU ExamPaper Islandwide Delivery | Whatsapp Only 88660031

Qn	
8(i)	(i) 
8(ii)	Using GC, $r = 0.884$ for the model $u = ax + b$ $u = ae^{bx} \Rightarrow \ln u = bx + \ln a$ Using GC, $r =$ 0.906 for the model $u = ae^{bx}$ Since the value of r is closer to 1 for the 2nd model, $u = ae^{bx}$ is a better model. $\ln u = 0.013633x + 0.94964$ $u = e^{0.013633x+0.94964}$ $u = 2.58e^{0.0136x} = 2.6e^{0.014x}$
8(iii)	$7 = 2.58e^{0.0136x} \Rightarrow x = \dfrac{\ln\left(\dfrac{7}{2.58}\right)}{0.0136} = 73.391 \approx 73$ A patient with urea serum is 7 mmol per litre is approximately 73 years old. Since $r = 0.906$ is close to 1 and 7 is within the data range of urea serum, estimate is reliable.
8(iv) (a)	The product moment correlation coefficient in part **(ii)** will not be changed if the units for the urea serum is given in mmol per decilitre.
8(iv) (b)	$u = 0.258e^{0.0136x}$

Qn	
9(i)	$\mathrm{P}(X=2)=\frac{18}{18}\frac{2}{17}\frac{15}{16}\frac{3!}{2!}$ $=\frac{45}{136}$ $\mathrm{P}(X=0)=\frac{18}{18}\frac{15}{17}\frac{12}{16}$ $=\frac{45}{68}$ $\mathrm{P}(X=3)=\frac{18}{18}\frac{2}{17}\frac{1}{16}$ $=\frac{1}{136}$
9(ii)	$\mathrm{E}(X)=\frac{93}{136}$ $\mathrm{E}(X^2)=0\times\frac{45}{68}+2^2\times\frac{45}{136}+3^2\times\frac{1}{136}=\frac{189}{136}$ $\mathrm{Var}(X)=\frac{189}{136}-\left(\frac{93}{136}\right)^2$ ≈ 0.922
9(iii)	Since $n=40$ is large, by Central Limit Theorem, $\overline{X}\sim N\left(\frac{93}{136},\frac{0.922}{40}\right)$ approximately $P(\overline{X}>1)=0.0186$

Qn	
9(iv)	Expected winnings $= -\frac{45}{68}a + \frac{45}{136}(a+10) + \frac{1}{136}(a+10)$ $-\frac{11}{34}a + \frac{115}{34} > 0$ $a < \frac{115}{11}$ $a < 10.\dot{4}\dot{5}$ The possible amounts will be $1 \le a \le 10$ and $a \in \mathbb{Z}$.

Qn	
10(i)	Let X be the thickness of the coating on a randomly chosen computer device. Let μ be the mean thickness of the coating of a computer device. Assume that the standard deviation of the coating of a computer device remains unchanged. To test : $\begin{array}{l} H_0 : \mu = 100 \\ H_1 : \mu \neq 100 \end{array}$ Level of Significance: 5% Under H_0, since sample size $n = 50$ is large, by Central Limit Theorem, $Z = \dfrac{\overline{X} - 100}{10/\sqrt{50}} \sim N(0,1)$ approx. Reject H_0 if $p - value \leq 0.05$. Calculations: $\overline{x} = 103.4$ $p - value = 0.0162$ Conclusion: Since $p - value < 0.05$, we reject H_0 and conclude that there is significant evidence at 5% level of significance that the process is not in control.
10(ii)	Reject H_0 is $\lvert z_{calc} \rvert \geq 1.960$ For H_0 to be rejected, $\left\lvert \dfrac{\overline{x} - 100}{10/\sqrt{50}} \right\rvert \geq 1.95996$ $\Rightarrow \overline{x} \leq 100 - 1.95996\left(\dfrac{10}{\sqrt{50}}\right)$ or $\overline{x} \geq 100 + 1.95996\left(\dfrac{10}{\sqrt{50}}\right)$ $\Rightarrow \overline{x} \leq 97.228$ or $\overline{x} \geq 102.772$ Thus the required range of values of $\overline{x}$ is $0 < \overline{x} \leq 97.2$ or $\overline{x} \geq 102.8$.

Qn	
10(iii)	$\bar{y}=\dfrac{4164}{40}=104.1$ $\Sigma(y-100)=4164-4000=164$ $s^2=\dfrac{1}{39}\left[\Sigma(y-100)^2-\dfrac{\left(\Sigma(y-100)\right)^2}{40}\right]$ $=\dfrac{1}{39}\left[9447-\dfrac{164^2}{40}\right]$ $=\dfrac{43873}{195}=224.9897$
10(iv)	The standard deviation may have changed due to the wear out of mechanical parts as well.
10(v)	To test : $H_0:\mu=100$ $H_1:\mu\neq 100$ Level of Significance: 4% Under H_0, since sample size $n=40$ is large, by Central Limit Theorem, $Z=\dfrac{\bar{Y}-100}{S/\sqrt{40}}\sim N(0,1)$ approx. Reject H_0 if $p-value\leq 0.04$. Calculations: $\bar{x}=104.1, s^2=224.9897$ $p-value=0.0839$ Conclusion: Since $p-value>0.04$, we do not reject H_0 and conclude that there is insignificant evidence at 4% level of significance that the process is not in control.

RAFFLES INSTITUTION
2019 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 19

MATHEMATICS 9758/01

PAPER 1

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

FOR EXAMINER'S USE					
Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**
Q7	**Q8**	**Q9**	**Q10**	**Q11**	**Total**

This document consists of **6** printed pages.

RAFFLES INSTITUTION
Mathematics Department

1 A curve C has equation

$$y = \frac{a}{x^3} + bx + c\,,$$

where a, b and c are constants. It is given that C has a stationary point $(-1.2, 6.6)$ and it also passes through the point $(2.1, -4.5)$.

(i) Find the values of a, b and c, giving your answers correct to 1 decimal place. [4]

(ii) One asymptote of C is the line with equation $x = 0$. Write down the equation of the other asymptote of C. [1]

2 Two variables u and v are connected by the equation $\frac{1}{u} + \frac{1}{v} = \frac{1}{20}$. Given that u and v both vary with time t, find an equation connecting $\frac{\mathrm{d}u}{\mathrm{d}t}, \frac{\mathrm{d}v}{\mathrm{d}t}$, u and v. Given also that u is decreasing at a rate of 2 units per second, calculate the rate of increase of v when $u = 60$ units.

[4]

3

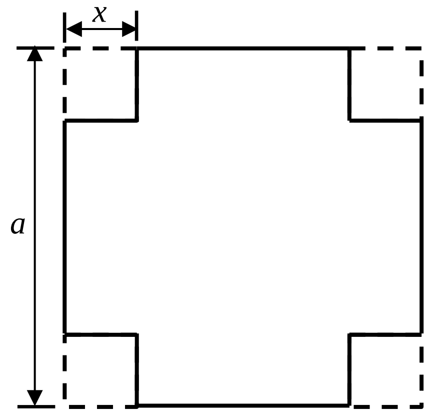


Fig. 1

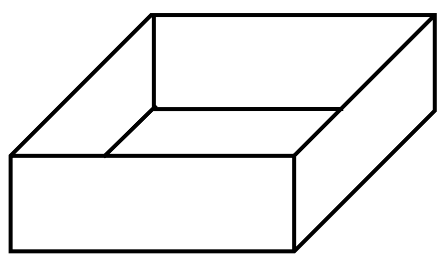


Fig. 2

Fig. 1 shows a square sheet of metal with side a cm. A square x cm by x cm is cut from each corner. The sides are then bent upwards to form an open box as shown in Fig. 2. Use differentiation to find, in terms of a, the maximum volume of the box, proving that it is a maximum. [6]

4 A curve C has parametric equations

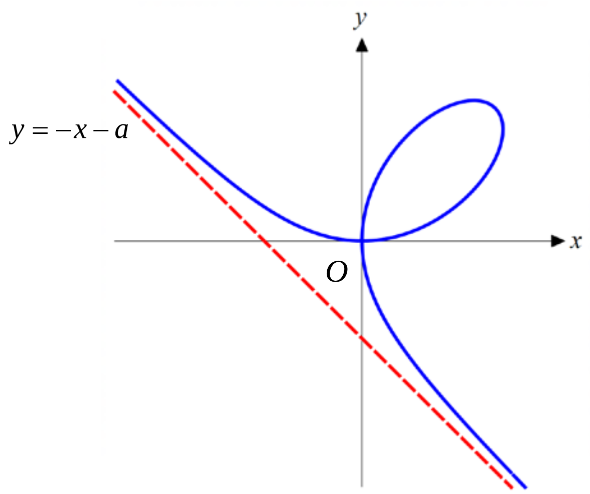
$$x=(1+t)^2, \qquad y=2(1-t)^2.$$

(i) Find the coordinates of the point A where the tangent to C is parallel to the x-axis. [4]

(ii) The line $y=-x+d$ intersects C at the point A and another point B. Find the exact coordinates of B. [4]

(iii) Find the area of the triangle formed by A, B and the origin. [1]

5



The diagram shows the graph of Folium of Descartes with cartesian equation

$$x^3+y^3=3axy,$$

where a is a positive constant. The curve passes through the origin, and has an oblique asymptote with equation $y=-x-a$.

(i) Given that (0, 0) is a stationary point on the curve, find, in terms of a, the coordinates of the other stationary point. [5]

(ii) Sketch the graph of

$$|x|^3+y^3=3a|x|y,$$

including the equations of any asymptotes, coordinates of the stationary points and the point where the graph crosses the x-axis. [3]

6 **(a)** Given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, find an expression for $\sum_{r=n+1}^{2n} \left(r(r-1)\right)$, simplifying your answer. [3]

(b) **(i)** Use the method of differences to find $\sum_{r=1}^{n}\left(\frac{(r+1)-\mathrm{e}r}{\mathrm{e}^r}\right)$. [3]

(ii) Hence find $\sum_{r=2}^{n}\left(\frac{\mathrm{e}+r-\mathrm{e}r}{\mathrm{e}^{r-1}}\right)$. [2]

7 **(a)** The complex numbers $3+2\mathrm{i}$, z, $4-6\mathrm{i}$ are the first three terms in a geometric progression. Without using a calculator, find the two possible values of z. [4]

(b) **(i)** The complex number w is such that $w = a + \mathrm{i}b$, where a and b are non-zero real numbers. The complex conjugate of w is denoted by w^*. Given that $\frac{w^2}{w^*}$ is a real number, find the possible values of w in terms of a only. [4]

(ii) Hence, find the exact possible arguments of w if a is positive. [2]

8 **(a)** Find the exact value of m such that

$$\int_0^3 \frac{1}{9+x^2}\,\mathrm{d}x = \int_0^{\frac{1}{2m}} \frac{1}{\sqrt{1-m^2x^2}}\,\mathrm{d}x.$$ [5]

(b) **(i)** Use the substitution $u = \sin 2x$ to show that

$$\int_0^{\frac{\pi}{4}} \sin^3 2x \cos^3 2x \,\mathrm{d}x = \frac{1}{2}\int_0^1 \left(u^3 - u^5\right)\,\mathrm{d}u.$$ [4]

(ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \sin^3 2x \cos^3 2x\,\mathrm{d}x$. [2]

9 **(i)** Write down $\int \frac{1}{a^2 - v^2}\,\mathrm{d}v$, where a is a positive constant. [1]

(ii) In the motion of an object through a certain medium, the medium furnishes a resisting force proportional to the square of the velocity of the moving object. Suppose that a body falls vertically through the medium, the model used to describe the velocity, v ms^{-1} of the body at time t seconds after release from rest is given by the differential equation

$$\frac{a^2}{10}\frac{\mathrm{d}v}{\mathrm{d}t} = a^2 - v^2,$$

where a is a positive constant.

(a) Show that $v = a\left(\dfrac{\mathrm{e}^{\frac{20t}{a}} - 1}{\mathrm{e}^{\frac{20t}{a}} + 1}\right)$. [8]

(b) The rate of change of the displacement, x metres, of the body from the point of release is the velocity of the body. Given that $a = 2$, find the value of x when $t = 1$, giving your answer correct to 3 decimal places. [3]

10 The curve C_1 has equation $x^2 + y^2 = 25, \quad y \le 0$.

The curve C_2 has equation $y = x + 3 + \dfrac{24}{x-3}$.

(i) Verify that $(-3, -4)$ lies on both C_1 and C_2. [1]

(ii) Sketch C_1 and C_2 on the same diagram, stating the coordinates of any stationary points, points of intersection with the axes and the equations of any asymptotes. [4]

The region bounded by C_1 and C_2 is R.

(iii) Find the exact volume of solid obtained when R is rotated through 2π radians about the x-axis. [6]

The region bounded by C_1, the x-axis and the vertical asymptote of C_2, where $x > 3$, is S.

(iv) Write down the equation of the curve obtained when C_1 is translated by 3 units in the negative x-direction.

Hence, or otherwise, find the volume of solid obtained when S is rotated through 2π radians about the vertical asymptote of C_2. [4]

11 Path integration is a predominant mode of navigation strategy used by many animals to return home by the *shortest* possible route during a food foraging journey. In path integration, animals continuously compute a homebound global vector relative to their starting position by integrating the angles steered and distances travelled during the entire foraging run. Once a food item has been found, the animal commences its homing run by using the homebound global vector, which was acquired during the outbound run.

(a) A Honeybee's hive is located at the origin *O*. The Honeybee travels 6 units in the direction $-\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ before moving 15 units in the direction $3\mathbf{i}-4\mathbf{k}$. The Honeybee is now at point *A*.

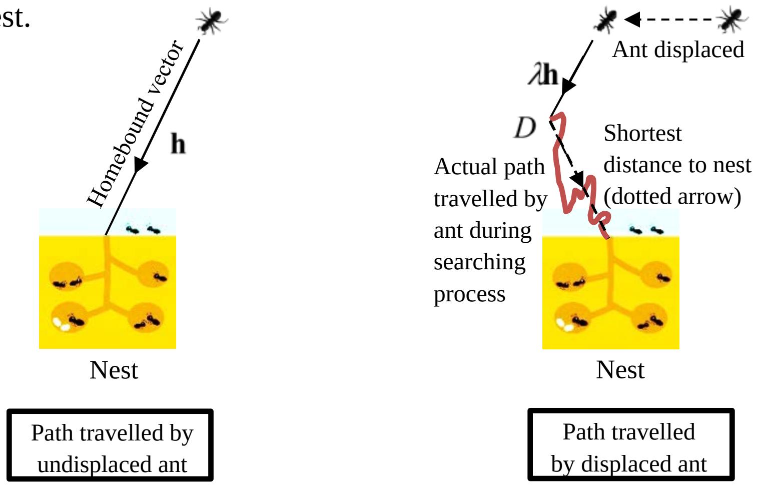
(i) Show that the homebound global vector $\overrightarrow{AO}$ is $-7\mathbf{i}-4\mathbf{j}+16\mathbf{k}$. Hence find the exact distance the Honeybee is from its hive. [3]

(ii) Explain why path integration may fail. [1]

A row of flowers is planted along the line $\dfrac{x-3}{5}=y+2,\ z=2$.

(iii) The Honeybee will take the shortest distance from point *A* to the row of flowers. Find the position vector of the point along the row of flowers which the Honeybee will fly to. [4]

(b) To further improve their chances of returning home, apart from relying on the path integration technique, animals depend on visual landmarks to provide directional information. When an ant is displaced to distant locations where familiar visual landmarks are absent, its initial path is guided solely by the homebound global vector, $\mathbf{h}$, until it reaches a point *D* and begins a search for their nest (see diagram). During the searching process, the distance travelled by the ant is 2.4 times the shortest distance back to the nest.



Let an ant's nest be located at the origin *O*. The ant has completed its foraging journey and is at a point with position vector $4\mathbf{i}+3\mathbf{j}$. A boy picks up the ant and displaces it 4 units in the direction $-\mathbf{i}$. Given $\lambda(-4\mathbf{i}-3\mathbf{j})$ as the initial path taken by the ant before it begins a search for its nest, find the value of λ which gives the minimum total distance travelled by the ant back to the nest. [4]

[It is not necessary to verify the nature of the minimum point in this part.]

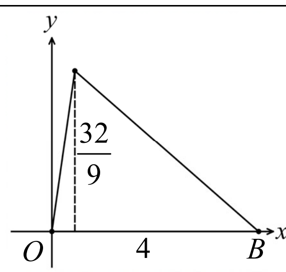
RAFFLES INSTITUTION
2019 YEAR 6 PRELIMINARY EXAMINATION

MATHEMATICS 9758/01 Suggested Solutions

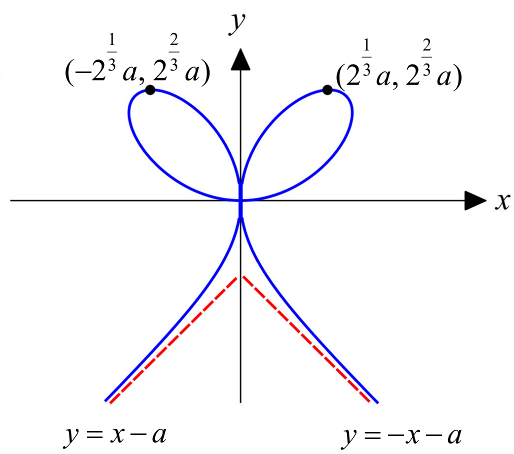
SOLUTION		COMMENTS
1(i) **[4]**	$y=\frac{a}{x^3}+bx+c$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3a}{x^4}+b$ At $x=-1.2$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, we have $-\frac{3a}{(-1.2)^4}+b=0$ ...(1) At $(-1.2,6.6)$, we have $\frac{a}{(-1.2)^3}-1.2b+c=6.6$ ...(2) At $(2.1,-4.5)$, we have $\frac{a}{(2.1)^3}+2.1b+c=-4.5$ ...(3) Using GC to solve (1), (2) and (3): $a=-2.03260\approx-2.0$, $b=-2.94068\approx-2.9$, $c=1.89491\approx1.9$	Generally well done for most students, except for a small number. These are the points to note: 1. Some students could not get the 3 equations as they didn't realise that the point $(-1.2, 6.6)$ could result in 2 equations instead of only one. 2. There were a number of students who could get the 3 equations but they end up with the wrong solutions. Do be careful when keying the equations into the GC! 3. A number of students differentiated wrongly: Eg. $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3a}{x^2}+b$.
(ii) **[1]**	$y=-2.9x+1.9$ (1 d.p.)	From equation of C, the other asymptote is $y=bx+c$. Most students were able to obtain a mark for this.

SOLUTION		COMMENTS
2 **[4]**	$\frac{1}{u}+\frac{1}{v}=\frac{1}{20}$ Differentiating w.r.t t, $-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t}=0$ --------------- (1) When $u=60$, $\frac{1}{60}+\frac{1}{v}=\frac{1}{20}$ $\frac{1}{v}=\frac{1}{20}-\frac{1}{60}$ $\frac{1}{v}=\frac{1}{30}$ $v=30$ Substituting $v=30$ and $\frac{\mathrm{d}u}{\mathrm{d}t}=-2$ into (1), $-\frac{1}{(60)^2}(-2)-\frac{1}{(30)^2}\frac{\mathrm{d}v}{\mathrm{d}t}=0$ $\frac{1}{1800}-\frac{1}{900}\frac{\mathrm{d}v}{\mathrm{d}t}=0$ $\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{900}{1800}$ $=\frac{1}{2}$ $\therefore$ Rate of increase of $v=\frac{1}{2}$ units/s	Probably 70% of students managed to do this. Some points to note: 1. Students are strongly advised to use implicit differentiation instead of making u or v the subject and applying quotient/product rule which often ends up with long and complicated expressions. 2. Some students mixed up differentiation and integration Eg. $\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{u}\right)=\ln\|u\|$. 3. Students are again reminded of the need to write clearly and properly as they mixed up u and v and hence obtained the wrong answer eventually.

SOLUTION		COMMENTS
3 **[6]**	$V=(a-2x)^2x=a^2x-4ax^2+4x^3$ $\frac{\mathrm{d}V}{\mathrm{d}x}=a^2-8ax+12x^2=0$ $(6x-a)(2x-a)=0$ $x=\frac{a}{6}$ or $x=\frac{a}{2}$ (rejected $\because V=0$) $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}=-8a+24x=-4a<0$ when $x=\frac{a}{6}$ Maximum $V=\left(a-2\left(\frac{a}{6}\right)\right)^2\left(\frac{a}{6}\right)=\frac{2a^3}{27}$ cm^3.	This question is generally well done. Some points to note are • Some students did not justify why the value of $x=\frac{a}{2}$ was rejected. Many students checked that this gives a minimum value using the second derivative test, a good strategy. • Many students solve for *x* by completing the square (of which a number made careless mistakes) when a direct factorization is much faster. Good for some of them to relearn this method. • Students who used the first derivative test to check for the nature of the root need to be reminded that the expression for dV/dx need to be in factorized form. • A small number of students were not aware that the *a* in the question is a constant and proceed to differentiate wrt *a*. Such students also tend to bring in expressions involving surface area showing that they are not sure of what the question is asking.

SOLUTION		COMMENTS
4 **(i)** **[4]**	$x=(1+t)^2, \quad y=2(1-t)^2.$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=2(1+t), \quad \dfrac{\mathrm{d}y}{\mathrm{d}t}=-4(1-t)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-4(1-t)}{2(1+t)}=\dfrac{-2(1-t)}{1+t}$ Tangent parallel to x-axis, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-2(1-t)}{1+t}=0 \Rightarrow t=1$ Thus, the coordinates of point A is $(4,0)$.	This question is well done. The main mistakes for **(i)** are • giving $\dfrac{\mathrm{d}y}{\mathrm{d}t}=4(1-t)$ • the gradient of the tangent parallel to the x-axis is undefined.
(ii) **[4]**	Let the coordinates of B be $\left((1+b)^2, 2(1-b)^2\right)$, where $b \in \mathbb{R}$. $y=-x+d$ passes through $A(4,0) \Rightarrow d=4$ Sub $B: 2(1-b)^2=-(1+b)^2+4$ $2-4b+2b^2=-1-2b-b^2+4$ $3b^2-2b-1=0$ $(3b+1)(b-1)=0$ $b=-\dfrac{1}{3} \quad$ or $\quad b=1$ (point A) Thus, the coordinates of B is $\left(\dfrac{4}{9},\dfrac{32}{9}\right)$.	This part of the question asked for the **exact** coordinates, which means that the use of the GC is **not** allowed.
(iii) **[1]**	Area of triangle OAB $=\dfrac{1}{2}(4)\left(\dfrac{32}{9}\right)$ $=\dfrac{64}{9}$  	This part is well done.

	SOLUTION	COMMENTS
5 **(i)** **[5]**	$x^3+y^3=3axy$ Differentiate with respect to x, $3x^2+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=3a\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)$ When $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $3x^2=3ay \Rightarrow y=\frac{x^2}{a}$ Substitute $y=\frac{x^2}{a}$ into the given curve $x^3+y^3=3axy$, $x^3+\frac{x^6}{a^3}=3x^3$ $x^6-2a^3x^3=0$ $x^3\left(x^3-2a^3\right)=0$ $x=0$ or $x=2^{\frac{1}{3}}a$ and, $y=0$ or $y=2^{\frac{2}{3}}a$ The required coordinates are $(0,0)$ (shown) and $\left(2^{\frac{1}{3}}a, 2^{\frac{2}{3}}a\right)$. For the following working, note that we need $y^2 \neq ax$ to get $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. $3x^2+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=3a\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)$ $\left(y^2-ax\right)\frac{\mathrm{d}y}{\mathrm{d}x}=ay-x^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{ay-x^2}{y^2-ax}$ if $y^2 \neq ax$ ...(*) Letting (*) = 0 and solving will also give (0,0) as an answer, but would need to be rejected as $y^2 \neq ax$.	Differentiation was well done. A few students made the mistakes: - did not apply chain rule - did not apply product rule Many did not substitute $y=\frac{x^2}{a}$ back into the given curve. Note a stationary point **lies on the curve** and has 0 gradient. It does **NOT** lie on the asymptote $y=-x-a$ or any other lines such as $y=x$. Many made mistakes when manipulating the indices. Some students did not simply their answers. In addition, please leave the answers in exact form when the question explicitly states that the coordinates are to be expressed "in terms of a".

<table>
<tr>
<td>(ii)
[3]</td>
<td>$x^3+y^3=3axy \xrightarrow{\text{Replace } x \text{ by } |x|} |x|^3+y^3=3a|x|y$
</td>
<td>Some common mistakes:
- incomplete graph
- graph not symmetrical about y-axis
- (0, 0) was not drawn as stationary point with zero gradient
- stationary points are the points with "maximum" y values, NOT the points furthest away from the origin.
- label the asymptotes wrongly
- curve not approaching the asymptotes as $x\to\infty$
- not labelling/labelling the required coordinates wrongly</td>
</tr>
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SOLUTION		COMMENTS
6 (a) [3]	$\sum_{r=n+1}^{2n}\left(r(r-1)\right)$ $=\sum_{r=n+1}^{2n}\left(r^2-r\right)$ $=\sum_{r=n+1}^{2n}r^2-\sum_{r=n+1}^{2n}r$ $=\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2-\frac{n}{2}(n+1+2n)$ $=\frac{1}{6}(2n)(2n+1)(4n+1)-\frac{1}{6}(n)(n+1)(2n+1)-\frac{1}{2}n(3n+1)$ $=\frac{1}{6}n\left[2(2n+1)(4n+1)-(n+1)(2n+1)-3(3n+1)\right]$ $=\frac{1}{6}n\left(14n^2-2\right)$ $=\frac{1}{3}n\left(7n^2-1\right)$ Alternatively, $\sum_{r=n+1}^{2n}\left(r^2-r\right)$ $=\sum_{r=n+1}^{2n}r^2-\sum_{r=n+1}^{2n}r$ $=\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2-\left(\sum_{r=1}^{2n}r-\sum_{r=1}^{n}r\right)$ $=\frac{1}{6}(2n)(2n+1)(4n+1)-\frac{1}{6}(n)(n+1)(2n+1)-\frac{1}{2}(2n)(2n+1)$ $+\frac{1}{2}n(n+1)$ $=\frac{1}{6}n\left[2(2n+1)(4n+1)-(n+1)(2n+1)-6(2n+1)+3(n+1)\right]$ $=\frac{1}{6}n\left(14n^2-2\right)$ $=\frac{1}{3}n\left(7n^2-1\right)$	This part was well done. However, quite a number of students lost 1 mark for not simplifying their answer. Another common mistake seen was miscalculation of the number of terms in the sum of AP , $\sum_{r=n+1}^{2n} r$.
(b) (i) [3]	$\sum_{r=1}^{n}\left(\frac{(r+1)-\mathrm{e}\,r}{\mathrm{e}^r}\right)=\sum_{r=1}^{n}\left(\frac{(r+1)}{\mathrm{e}^r}-\frac{\mathrm{e}\,r}{\mathrm{e}^r}\right)$ $=\sum_{r=1}^{n}\left(\frac{(r+1)}{\mathrm{e}^r}-\frac{r}{\mathrm{e}^{r-1}}\right)$	This part was well done. Some common mistakes seen were: • Missing out the $-$ sign in -1

	$\begin{aligned} &= \cancel{\frac{2}{e^1}} - \frac{1}{e^0} \\ &\quad + \cancel{\frac{3}{e^2}} - \cancel{\frac{2}{e^1}} \\ &\quad + \cancel{\frac{4}{e^3}} - \cancel{\frac{3}{e^2}} \\ &\qquad \vdots \\ &\quad + \cancel{\frac{n}{e^{n-1}}} - \cancel{\frac{n-1}{e^{n-2}}} \\ &\quad + \frac{n+1}{e^n} - \cancel{\frac{n}{e^{n-1}}} \\ &= \frac{n+1}{e^n} - \frac{1}{e^0} \\ &= \frac{n+1}{e^n} - 1 \end{aligned}$	• Writing the last term wrongly as $\frac{n+1}{e^{n+1}}$
(b) (ii) [2]	$\begin{aligned} \sum_{r=2}^{n}\left(\frac{e+r-er}{e^{r-1}}\right) &= \sum_{r+1=2}^{n}\left(\frac{e+(r+1)-e(r+1)}{e^{(r+1)-1}}\right) \\ &= \sum_{r=1}^{n-1}\left(\frac{(r+1)-er}{e^r}\right) \\ &= \frac{n}{e^{n-1}}-1 \end{aligned}$	Take note of the "Hence" method required in this part. Many students failed to show proper working of using answer to **(b)(i)**. There were also a lot of mistakes in the change of lower and upper index of the summation.

SOLUTION		**COMMENTS**
7 (a) [4]	$\frac{z}{3+2i}=\frac{4-6i}{z}$ $\Rightarrow z^2=(4-6i)(3+2i)=24-10i$ Let $z=a+bi, \quad a,b\in\mathbb{R}$ $(a+bi)^2=24-10i$ $\Rightarrow a^2-b^2+2abi=24-10i$ Comparing Real and Imaginary parts, $a^2-b^2=24$ $2ab=-10 \Rightarrow ab=-5$ $\Rightarrow \frac{25}{b^2}-b^2=24$	Students need to understand that when a question states "without using a calculator", they need to show all working no matter how trival it seemed to them. Many lose marks for failing to show the proper factorization of

	$\Rightarrow b^4+24b^2-25=0$ ------- (*) $\Rightarrow (b^2+25)(b^2-1)=0$ $\Rightarrow b=\pm 1$ $\Rightarrow a=\mp 5$ The two possible complex numbers are $-5+\mathrm{i}$ or $5-\mathrm{i}$.	the quartic equation (*).
(b) (i) [4]	$\dfrac{w^2}{w^*}=\dfrac{(a+\mathrm{i}b)^2}{a-\mathrm{i}b}$ $=\dfrac{(a+\mathrm{i}b)^3}{a^2+b^2}$ $=\dfrac{a^3+3\mathrm{i}a^2b-3ab^2-\mathrm{i}b^3}{a^2+b^2}$ $=\dfrac{(a^3-3ab^2)+\mathrm{i}(3a^2b-b^3)}{a^2+b^2}$ $\dfrac{w^2}{w^*}$ is purely real $\mathrm{Im}\left(\dfrac{w^2}{w^*}\right)=0 \Rightarrow \dfrac{3a^2b-b^3}{a^2+b^2}=0$ $\Rightarrow b(3a^2-b^2)=0$ $\Rightarrow b=0$ (rejected, since $b\neq 0$) or $b=\pm a\sqrt{3}$ Thus, $w=a+\mathrm{i}a\sqrt{3}$ or $w=a-\mathrm{i}a\sqrt{3}$. **<u>Alternative method 1:</u>** $\arg\left(\dfrac{w^2}{w^*}\right)=k\pi$, where $k\in\mathbb{Z}$ $3\arg w=k\pi$ $\arg w=\dfrac{k\pi}{3}$ Since a and b are non-zero, $\tan^{-1}\left(\left\|\dfrac{b}{a}\right\|\right)=\dfrac{\pi}{3}$ $\|b\|=\|a\|\sqrt{3}$ $b=a\sqrt{3}$ or $-a\sqrt{3}$ Thus, $w=a+\mathrm{i}a\sqrt{3}$ or $w=a-\mathrm{i}a\sqrt{3}$.	Generally well done for students who use the first method, except for a few who forgot to reject $b=0$. Students who use the argument method (Alternative Method 1) mostly wrote $\arg w=\tan^{-1}\left(\dfrac{b}{a}\right)$ Students who wrote this were not penalized in this exam, but please do note

	Alternative method 2: Let $\dfrac{w^2}{w^*} = k$, where $k \in \mathbb{R}$ $\dfrac{(a+\mathrm{i}b)^2}{a-\mathrm{i}b} = k$ $a^2 - b^2 + \mathrm{i}(2ab) = ka - \mathrm{i}(kb)$ Comparing real and imaginary parts, $a^2 - b^2 = ka$ ----------- (1) $2ab = -kb$ ------------ (2) From (2), $k = -2a$ Substitute into (1): $a^2 - b^2 = -2a^2$ $b^2 = 3a^2$ $b = a\sqrt{3}$ or $-a\sqrt{3}$	that this is not correct.
(b) **(ii)** **[2]**	Basic angle = $\tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right) = \dfrac{\pi}{3}$ Thus, $\arg(w) = \dfrac{\pi}{3}$ or $\arg(w) = -\dfrac{\pi}{3}$.	

<table>
<tr><th colspan="2">SOLUTION</th><th>COMMENTS</th></tr>
<tr><td>8(a)
[5]</td><td>$\int_0^3 \frac{1}{9+x^2}\,\mathrm{d}x = \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$
$=\frac{1}{3}\left(\frac{\pi}{4}\right)$
$=\frac{\pi}{12}.$

$\int_0^{\frac{1}{2m}} \frac{1}{\sqrt{1-m^2x^2}}\,\mathrm{d}x = \frac{1}{m}\int_0^{\frac{1}{2m}} \frac{m}{\sqrt{1-(mx)^2}}\,\mathrm{d}x$
$=\frac{1}{m}\left[\sin^{-1}(mx)\right]_0^{\frac{1}{2m}}$
$=\frac{1}{m}\left[\sin^{-1}\left(\frac{1}{2}\right)-0\right]$
$=\frac{\pi}{6m}.$

So $\int_0^{\frac{1}{2m}} \frac{1}{\sqrt{1-m^2x^2}}\,\mathrm{d}x = \int_0^3 \frac{1}{9+x^2}\,\mathrm{d}x$
$\frac{\pi}{6m}=\frac{\pi}{12}$
$m=2$</td><td>Common mistakes:
a) Exclude the $\frac{1}{m}$ in the $\frac{1}{m}\sin^{-1}(mx)$;
b) Wrong value for $\tan^{-1}\left(\frac{3}{3}\right)$ or $\sin^{-1}\left(\frac{1}{2}\right)$.
Some students resorted to using calculator to compute $\frac{\sin^{-1}\left(\frac{1}{2}\right)}{\tan^{-1}\left(\frac{3}{3}\right)}$, which illustrated that they do not have knowledge of the special angles. Note that since question asks for exact value of m, calculator should not be used in the computation.</td></tr>
<tr><td>(b)
[6]</td><td>$u=\sin 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=2\cos 2x$
When $x=0,\ u=0$.
When $x=\frac{\pi}{4},\ u=1$.

$\int_0^{\frac{\pi}{4}} \sin^3 2x\cos^3 2x\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \sin^3 2x\cos^2 2x\cos 2x\mathrm{d}x$
$=\frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin^3 2x)\left(1-\sin^2 2x\right)(2\cos 2x)\mathrm{d}x$
$=\frac{1}{2}\int_0^1 u^3\left(1-u^2\right)\,\mathrm{d}u$
$=\frac{1}{2}\int_0^1 \left(u^3-u^5\right)\,\mathrm{d}u$</td><td>Some students did this: $\frac{\mathrm{d}u}{\mathrm{d}x}=-2\cos 2x$ which is less than desirable.

Very common presentation error: $\frac{1}{2}\int_0^1 (u^3)\left(\cos^3 2x\right)\frac{\mathrm{d}u}{\cos 2x}$ and the variations of mixing x and u in the integrand, or even integrating wrt to u but the limits are still in x.

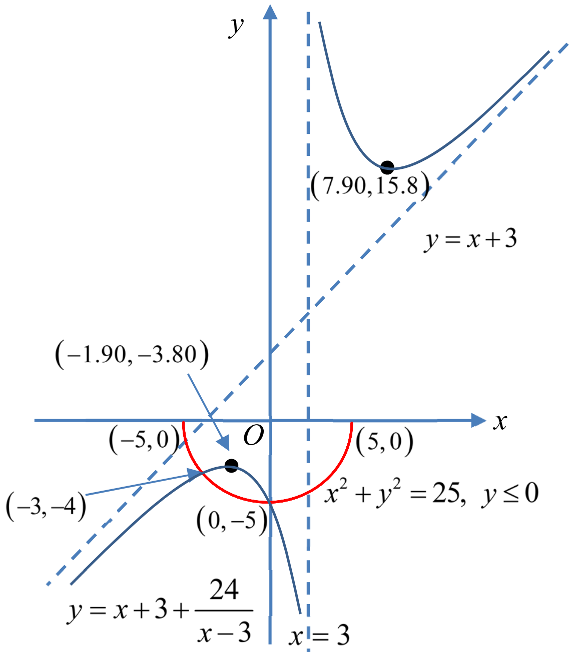
Some students failed to use the “hence” for the last numerical</td></tr>
</table>

	$\int_0^{\frac{\pi}{4}} \sin^3 2x \cos^3 2x \ \mathrm{d}x = \frac{1}{2}\int_0^1 \left(u^3 - u^5\right) \ \mathrm{d}u$ $= \frac{1}{2}\left[\frac{u^4}{4} - \frac{u^6}{6}\right]_0^1$ $= \frac{1}{2}\left[\frac{1}{4} - \frac{1}{6}\right]$ $= \frac{1}{24}$	computation and reworked from the start.

SOLUTION		**COMMENTS**
9(i) **[1]**	$\int \frac{1}{a^2 - v^2} \ \mathrm{d}v = \frac{1}{2a}\ln\left\lvert\frac{a+v}{a-v}\right\rvert + c$	This part was very poorly done. Most students either missed out on the modulus sign or the +c.
(ii) **(a)** **[8]**	$\frac{a^2}{10}\frac{\mathrm{d}v}{\mathrm{d}t} = a^2 - v^2$ $\frac{1}{a^2 - v^2}\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{10}{a^2}$ $\int \frac{1}{a^2 - v^2} \ \mathrm{d}v = \int \frac{10}{a^2}\mathrm{d}t$ $\frac{1}{2a}\ln\left\lvert\frac{a+v}{a-v}\right\rvert = \frac{10}{a^2}t + C$, where C is an arbitrary constant $\ln\left\lvert\frac{a+v}{a-v}\right\rvert = \frac{20t}{a} + D$, where $D = 2aC$ $\frac{a+v}{a-v} = A\mathrm{e}^{\frac{20t}{a}}$, where $A = \pm \mathrm{e}^D$ Given that $v = 0$ when $t = 0$, $A = 1$.	Most students were able to do this part relatively well. A few points to take note 1. There should be a modulus in the natural log. 2. The modulus will be taken care of when we take $A = \pm \mathrm{e}^D$ and then substitute initial conditions to calculate for A. 3. Some students substituted the initial conditions too early to calculate D, obtaining D=0. This is insufficient because upon simplifying the equation by removing the modulus sign, you will still get $\pm$. In this case you will need to substitute the initial conditions again to decide that the positive form is to be retained. Otherwise, students generally have no problems expanding

	$\frac{a+v}{a-v}=\mathrm{e}^{\frac{20t}{a}}$ $a+v=(a-v)\mathrm{e}^{\frac{20t}{a}}$ $\left(1+\mathrm{e}^{\frac{20t}{a}}\right)v=a\left(\mathrm{e}^{\frac{20t}{a}}-1\right)$ $v=a\left(\frac{\mathrm{e}^{\frac{20t}{a}}-1}{\mathrm{e}^{\frac{20t}{a}}+1}\right)$ (shown)	and rearranging the terms to obtain the final answer.
(ii) **[3]**	Since $a=2$, $v=2\left(\frac{\mathrm{e}^{10t}-1}{\mathrm{e}^{10t}+1}\right)$. $x=2\int_0^1\frac{\mathrm{e}^{10t}-1}{\mathrm{e}^{10t}+1}\,\mathrm{d}t$ $=1.723$ metres (3 d.p.)	In this part of the question, many students wrongly substituted $t=1$ to find the value of v at that point. Do note that we are looking for x, not v. Many students did the integration manually without using GC, which led to many mistakes. Some examples: 1. Wrong method of integration. 2. Applying indefinite integration and then forgetting or wrongly calculating the value of the arbitrary constant c. Some students left their answer in 3 significant figures instead of 3 decimal places.

SOLUTION		COMMENTS
10 (i) [1]	For $x^2+y^2=25, \quad y\le 0$ $(-3)^2+(-4)^2=25.$ For $y=x+3+\dfrac{24}{x-3}$, $x=-3,\ y=-3+3+\dfrac{24}{-3-3}=-4.$	This part is done well.

(ii) **[4]**	 Asymptotes : $y = x + 3$ and $x = 3$	Graphs were generally sketched well. The only errors seen were related to inappropriate points i.e. (5, 0) being further to the right of (7.90, 15.8) or the curve C_2 not being sketched completely. A significant number of candidates used differentiation to compute the turning points when the graphing calculator could have been used.
(iii) **[6]**	Volume of solid obtained when R is rotated through 2π radians about the x-axis $= \pi \int_{-3}^{0} \left(25 - x^2\right) - \left(x + 3 + \frac{24}{x-3}\right)^2 \, dx$ $= \pi \int_{-3}^{0} \left[25 - x^2 - (x+3)^2 - 48\left(\frac{x+3}{x-3}\right) - \frac{24^2}{(x-3)^2} \right] dx$ $= \pi \int_{-3}^{0} \left[25 - x^2 - (x+3)^2 - 48\left(1 + \frac{6}{x-3}\right) - \frac{24^2}{(x-3)^2} \right] dx$ $= \pi \left[25x - \frac{x^3}{3} - \frac{(x+3)^3}{3} - 48\left(x + 6\ln\lvert x-3 \rvert\right) + \frac{24^2}{(x-3)} \right]_{-3}^{0}$ $= \pi \left(-\frac{27}{3} - 48(6\ln 3) + \frac{24^2}{-3} - 25(-3) + \frac{(-3)^3}{3} + 48(-3 + 6\ln 6) - \frac{24^2}{-6} \right)$ $= \pi\left(-9 - 192 + 75 - 9 - 144 + 96 - 288\ln 3 + 288\ln 6\right)$ $= \pi\left(288\ln 2 - 183\right)$	The instruction "Find the exact volume" was ignored by some candidates who had set up the integrals correctly but went on to use the calculator to compute the volume. It was necessary to realize that the points of intersection (–3, –4) and (0, –5) were already verified and found in parts **(i)** and **(ii)** respectively. Note $x + 3 + \frac{24}{x-3} \neq \frac{x^2 - 15}{x-3}$.

(iv) **[4]**	Translating the graph of $x^2+y^2=25$ 3 units in the negative x-direction, we have $(x+3)^2+y^2=25$. $(x+3)^2+y^2=25$ $(x+3)^2=25-y^2$ $x=-3\pm\sqrt{25-y^2}$ $x=-3+\sqrt{25-y^2}$ for $x\ge 3$ Volume of solid obtained when S is rotated through 2π radians about the line $x=3$ $=\pi\int_{-4}^{0}\left(-3+\sqrt{25-y^2}\right)^2 \, \mathrm{d}y$ $=28.6$ (3 s.f.)	Most candidates were able to write down the equation for the translated graph but many went on to use the original equation and make y the subject to find the volume generated using integration of the region rotated about the x-axis when the question stated "rotated … about the vertical asymptote" which meant that the y-axis would be a more likely option.

SOLUTION		COMMENTS
11(i) **[3]**	$\overrightarrow{OA}=6\left[\frac{1}{3}\begin{pmatrix}-1\\2\\-2\end{pmatrix}\right]+15\left[\frac{1}{5}\begin{pmatrix}3\\0\\-4\end{pmatrix}\right]=\begin{pmatrix}-2\\4\\-4\end{pmatrix}+\begin{pmatrix}9\\0\\-12\end{pmatrix}=\begin{pmatrix}7\\4\\-16\end{pmatrix}$ Homebound global vector = $\overrightarrow{AO}=\begin{pmatrix}-7\\-4\\16\end{pmatrix}$ (shown) Distance from the hive $=\left\lvert\begin{pmatrix}-7\\-4\\16\end{pmatrix}\right\rvert=\sqrt{7^2+4^2+16^2}=\sqrt{321}$	A significant minority of students forgot to scale the directions to unit vectors. Some students missed this part.
(ii) **[1]**	The homebound vector may be blocked by obstacles along the path.	Any reasonable response is acceptable. However, anything blaming the poor bee is not, and neither are general statements that something can go wrong. Only a few students realized that (b) actually provided a plausible answer here.
(iii) **[4]**	Equation of line: $l:\mathbf{r}=\begin{pmatrix}3\\-2\\2\end{pmatrix}+\mu\begin{pmatrix}5\\1\\0\end{pmatrix},\mu\in\mathbb{R}$ Let N be the foot of perpendicular from A to the line. $\overrightarrow{ON}=\begin{pmatrix}3\\-2\\2\end{pmatrix}+\mu\begin{pmatrix}5\\1\\0\end{pmatrix}$, for some $\mu\in\mathbb{R}$ $\overrightarrow{AN}=\begin{pmatrix}3\\-2\\2\end{pmatrix}+\mu\begin{pmatrix}5\\1\\0\end{pmatrix}-\begin{pmatrix}7\\4\\-16\end{pmatrix}=\begin{pmatrix}-4+5\mu\\-6+\mu\\18\end{pmatrix}$ Since $\overrightarrow{AN}$ is perpendicular to the line,	This part is generally handled proficiently by most students.

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$$\overrightarrow{AN}\cdot\begin{pmatrix}5\\1\\0\end{pmatrix}=0$$

$$\begin{pmatrix}-4+5\mu\\-6+\mu\\18\end{pmatrix}\cdot\begin{pmatrix}5\\1\\0\end{pmatrix}=0$$

$$\Rightarrow -20+25\mu-6+\mu=0$$

$$\Rightarrow \mu=1$$

$$\Rightarrow \overrightarrow{ON}=\begin{pmatrix}8\\-1\\2\end{pmatrix}$$

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<td>

(b)
[4]

</td>
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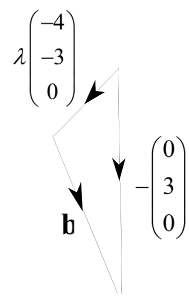
Let **b** be the direction vector back to the nest from any point along the homebound vector.



$$\lambda\begin{pmatrix}-4\\-3\\0\end{pmatrix}+\mathbf{b}=-\begin{pmatrix}0\\3\\0\end{pmatrix}$$

$$\mathbf{b}=\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}-\begin{pmatrix}0\\3\\0\end{pmatrix}=\begin{pmatrix}4\lambda\\3\lambda-3\\0\end{pmatrix}$$

Distance to travel from the position vector $3\mathbf{j}$ back to the origin,
D = Distance travelled via homebound vector + Distance travelled via searching process

$$=\left|\begin{pmatrix}-4\lambda\\-3\lambda\\0\end{pmatrix}\right|+2.4\left|\begin{pmatrix}4\lambda\\3\lambda-3\\0\end{pmatrix}\right|$$

$$=\sqrt{16\lambda^2+9\lambda^2}+\frac{12}{5}\sqrt{16\lambda^2+(3\lambda-3)^2}$$

$$=5\lambda+\frac{12}{5}\sqrt{25\lambda^2-18\lambda+9}$$

Differentiating with respect to λ,

$$\frac{\mathrm{d}D}{\mathrm{d}\lambda}=5+\frac{6}{5}\left(16\lambda^2+(3\lambda-3)^2\right)^{-1/2}\left(32\lambda+6(3\lambda-3)\right)$$

$$5+\frac{6}{5}\left(16\lambda^2+(3\lambda-3)^2\right)^{-1/2}\left(32\lambda+6(3\lambda-3)\right)=0$$

From G.C.,
$\lambda\approx 0.140$

OR

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Students' success in this part mostly relies on interpreting the question correctly and quality of responses various greatly.

Many students who got here squared to remove the radical. This is not only unnecessary, but also introduced a spurious solution that needs to be rejected.

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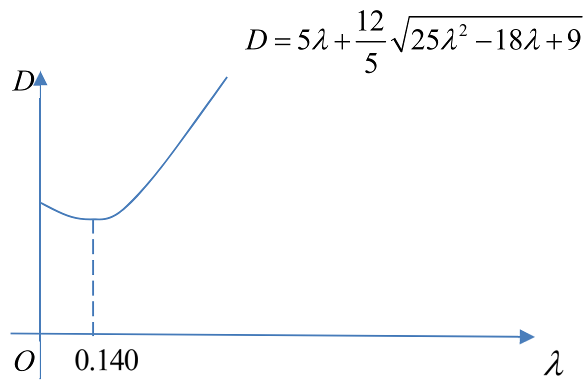
	 From G.C., $\lambda \approx 0.140$	

RAFFLES INSTITUTION
2019 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME	
CLASS	19

MATHEMATICS **9758/02**

PAPER 2

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

FOR EXAMINER'S USE							
SECTION A: PURE MATHEMATICS							TOTAL
Q1	Q2	Q3	Q4	Total			
SECTION B: PROBABILITY AND STATISTICS							
Q5	Q6	Q7	Q8	Q9	Q10	Total	
							100

This document consists of **7** printed pages.

RAFFLES INSTITUTION
Mathematics Department

Section A: Pure Mathematics [40 Marks]

1 The function f is defined as follows.

$$\mathrm{f} : x \mapsto \frac{1}{x^2}, \text{ for } x \in \mathbb{R},\ x \neq 0.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. [1]

(ii) If the domain of f is further restricted to $x > k$, state with a reason the least value of k for which the function f^{-1} exists. [2]

In the rest of this question, the domain of f is $x \in \mathbb{R},\ x \neq 0$, as originally defined.

A function h is said to be an odd function if $\mathrm{h}(-x) = -\mathrm{h}(x)$ for all x in the domain of h.
The function g is defined as follows.

$$\mathrm{g} : x \mapsto \frac{2}{3^x - 1} + m, \text{ for } x \in \mathbb{R},\ x \neq 0.$$

(iii) Given that g is an odd function, find the value of m.

(iv) Using the value of m found in part **(iii)**, find the range of fg.

2 **(a)** The curve $y = \mathrm{f}(x)$ passes through the point (0, 81) and has gradient given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{1}{3}y - 15x\right)^{\frac{1}{3}}.$$

Find the first three non-zero terms in the Maclaurin series for y. [4

(b) Let $\mathrm{g}(x) = \dfrac{4 - 3x + x^2}{(1+x)(1-x)^2}$.

(i) Express $\mathrm{g}(x)$ in the form $\dfrac{A}{1+x} + \dfrac{1}{1-x} + \dfrac{B}{(1-x)^2}$, where A and B are constants to be determined.

The expansion of $\mathrm{g}(x)$, in ascending powers of x, is

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_r x^r + \ldots.$$

(ii) Find the values of c_0, c_1, and c_2 and show that $c_3 = 3$.

(iii) Express c_r in terms of r .

3 Referred to an origin O, the position vectors of three non-collinear points A, B and C are **a**, **b** and **c** respectively. The coordinates of A, B and C are $(-2,4,2)$, $(1,3,1)$ and $(0,1,2)$ respectively.

(i) Find $(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{b})$. [1]

(ii) Hence

(a) find the exact area of triangle ABC,

(b) show that the cartesian equation of the plane ABC is $3x+2y+7z=16$.

(iii) The line l_1 has equation $\mathbf{r}=\begin{pmatrix}1\\1\\6\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$ for $\lambda\in\mathbb{R}$.

(a) Show that l_1 is parallel to the plane ABC but does not lie on the plane ABC.

(b) Find the distance between l_1 and plane ABC.

(iv) The line l_2 passes through B and is perpendicular to the xy-plane. Find the acute angle between l_2 and its reflection in the plane ABC, showing your working clearly.

4 **(a)** The non-zero numbers a, b and c are the first, third and fifth terms of an arithmetic series respectively.

(i) Write down an expression for b in terms of a and c. [1

(ii) Write down an expression for the common difference, d, of this arithmetic series in terms of a and b. [1

(iii) Hence show that the sum of the first ten terms can be expressed as

$$\frac{5}{4}(9c-a).$$ [2]

(iv) If a, b and c are also the fourth, third and first terms, respectively of a geometric series, find the common ratio of this series in terms of a and b and hence show that

$$(2b-a)a^2=b^3.$$

(b) The nth term of a geometric series is $\left(2\sin^2\alpha\right)^{n-1}$.

(i) Find all the values of α, where $0\le\alpha\le 2\pi$, such that the series is convergent.

(ii) For the values of α found in part **(i)**, find the sum to infinity, simplifying your ans

Section B: Probability and Statistics [60 Marks]

5 **(a)** The probability that a hospital patient has a particular disease is p. A test for the disease has a probability of 0.99 of giving a positive result when the patient has the disease and a probability of 0.95 of giving a negative result when the patient does not have the disease. A patient is given the test.

For a general value of p, the probability that a randomly chosen patient has the disease given that the result of the test is positive is denoted by $\mathrm{f}(p)$.

Find an expression for $\mathrm{f}(p)$ and show that f is an increasing function for $0 < p < 1$. Explain what this statement means in the context of this question. [5

(b) For events A and B, it is given that $\mathrm{P}(A) = 0.7$ and $\mathrm{P}(B) = k$.

(i) Given that A and B are independent events, find $\mathrm{P}(A \cap B')$ in terms of k.

(ii) Given instead that A and B are mutually exclusive events, state the range of values of k.

Find $\mathrm{P}(B \mid A')$ in terms of k.

6 Five objects a, b, c, d and e are to be placed in five containers A, B, C, D and E, with one in each container. An object is said to be correctly placed if it is placed in the container of the same letter (e.g. a in A) but incorrectly placed if it is placed in any of the other four containers. Find

(i) the number of ways the objects can be placed in the containers so that a is correctly placed and b is incorrectly placed, [2]

(ii) the number of ways the objects can be placed in the containers so that both a and b are incorrectly placed, [3]

(iii) the number of ways the objects can be placed in the containers so that there are at least 2 correct placings. [3]

7 Based on past records, at government polyclinics, on average each medical consultation lasts 15 minutes with a standard deviation of 10 minutes.

(i) Give a reason why a normal distribution, with this mean and standard deviation, would not give a good approximation to the distribution of consultation duration. [2

From recent patient and doctor feedback, a polyclinic administrator claims that the average consultation duration at government polyclinic is taking longer. State suitable null and alternative hypotheses to test this claim.

(ii) Given that the null hypothesis will be rejected if the sample mean from a random sample of size 30 is at least 18, find the smallest level of significance of the test. State clearly any assumptions required to determine this value.

The administrator suspects the average consultation duration at private clinics is actually less than 15 minutes. A survey is carried out by recording the consultation duration, y, in minutes, from 80 patients as they enter and leave a consultation room at a private clinic. The results are summarized by

$$\sum(y-15) = -50, \ \sum(y-15)^2 = 555.$$

(iii) Calculate unbiased estimates of the population mean and variance of the consultation duration at private clinics. Determine whether there is sufficient evidence at the 5% level of significance to support the administrator's claim.

8 A biased cubical die has its faces marked with the numbers 1, 3, 5, 7, 11 and 13. The random variable X is defined as the score obtained when the die is thrown, with probabilities given by

$$\mathrm{P}(X = r) = kr, \ r = 1, 3, 5, 7, 11, 13,$$

where k is a constant.

(i) Show that $\mathrm{P}(X = 3) = \dfrac{3}{40}$. [3]

(ii) Find the exact value of Var(X).

The die is thrown 15 times and the random variable R denotes the number of times that a score less than 10 is observed.

(iii) Find $\mathrm{P}(R \geq 5)$.

(iv) Find the probability that the last throw is the 8^{th} time that a score less than 10 is observed.

9 Rroar Tyre Company develops Brand R tyres. The working lifespan in kilometres of a Brand R tyre is a random variable with the distribution $N(64000, 8000^2)$.

(i) Find the probability that a randomly selected Brand R tyre has a working lifespan of at least 70000 km. [1

(ii) Rroar Tyre Company wishes to advertise that 98% of their Brand R tyres have working lifespans of more than t kilometres. Determine the value of t, correct to the nearest kilometre. [2

Ssoar Tyre Company, a rival company, develops Brand S tyres. The working lifespan in kilometres of a Brand S tyre is a random variable with the distribution $N(68000, \alpha^2)$.

(iii) A man selects 50 Brand S tyres at random. Given that $\alpha = 7500$, find the probability that the average of their working lifespans exceeds 70000 km.

(iv) Using $\alpha = 8000$, find the probability that the sum of the working lifespans of 3 randomly chosen Brand R tyres is less than 3 times the working lifespan of a randomly chosen Brand S tyre.

(v) Find the range of α, correct to the nearest kilometre, if there is a higher percentage of Brand S tyres than Brand R tyres lasting more than 50000 km.

(vi) State clearly an assumption needed for your calculations in parts **(iii)**, **(iv)** and **(v)**.

10 **(a)** With the aid of suitable diagrams, describe the differences between the least square linear regression line of y on x and that of x on y. [2]

(b) The government of the Dragon Island Country is doing a study on its population growth in order to implement suitable policies to support the aging population of the country. The population sizes, y millions in x years after Year 2000, are as follows.

x	9	10	11	12	14	15	16	17	18
Population size, y	5.05	5.21	5.32	5.41	5.51	5.56	5.60	5.65	5.67

(i) Draw a scatter diagram of these values, labelling the axes clearly. Use your diagram to explain whether the relationship between x and y is likely to be well modelled by an equation of the form $y = ax + b$, where a and b are constants.

(ii) Find, correct to 6 decimal places, the value of the product moment correlation coefficient between

(a) $\ln x$ and y,

(b) x^2 and y.

(iii) Use your answers to part **(ii)** to explain which of $y = a + b\ln x$ or $y = a + bx^2$ is the better model.

(iv) It is required to estimate the year in which the population size first exceed 6.5 millions. Use the model that you identified in part **(iii)** to find the equation of a suitable regression line, and use your equation to find the required estimate.

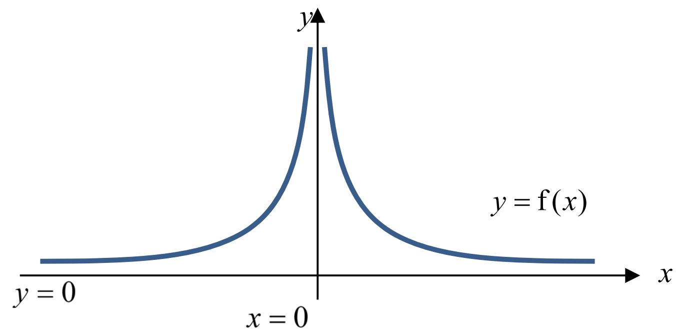
Comment on the reliability of this estimate.

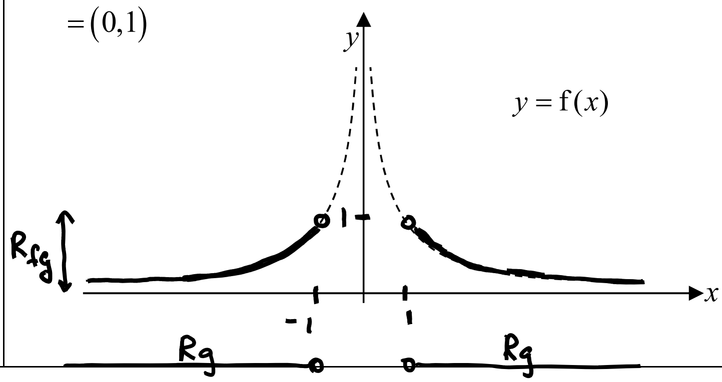
(v) As the population size in Year 2013 is not available, a government statistician uses the regression line in part **(iv)** to estimate the population size in 2013. Find this estimate.

(vi) It was later found that in Year 2013, the population size was in fact 5.31 millions. Comment on this figure with reference to the estimate found in part **(v)**, providing a possible reason in context.

RAFFLES INSTITUTION
2019 YEAR 6 PRELIMINARY EXAMINATION

MATHEMATICS 9758/02 Suggested Solution

	SOLUTION	COMMENTS
1(i) **[1]**		Students should be careful in showing asymptotic behavior and in the labeling of the two asymptotes $x=0$ (y-axis), $y=0$ (x-axis) .
1(ii) **[2]**	Least value of $k = 0$. If the domain of f is restricted to $x>0$, then for any horizontal line $y=h,\ \ h\in\mathbb{R}^+$, it cuts the graph of f at one and only one point, therefore f is a 1-1 function. Hence funcion f^{-1} exists.	For the horizontal line test to show that f is 1-1, many did not state that it is for any $y=h$, $h\in\mathbb{R}^+$ (which is the range of f) that cuts the graph of f at one and only one point.
1(iii) **[2]**	Consider $\mathrm{g}(-x)=-\mathrm{g}(x)$, with $x=1$. $$\frac{2}{3^{-1}-1}+m=-\left(\frac{2}{3-1}+m\right)$$ $$2m=3-1$$ $$m=1$$ NOTE : We may attempt this part with any specific value of x (in the domain of g).	Many did not see that since it is given that g is an odd function, then $\mathrm{g}(-x)=-\mathrm{g}(x)$ holds for any x in the domain. For those who tried to solve it for a general value of x, some did not simplify m to $m=\frac{-1}{3^x-1}-\frac{1}{3^{-x}-1}$ $=\frac{-1}{3^x-1}-\frac{3^x}{1-3^x}$. $=\frac{3^x-1}{3^x-1}=1$
1(iv) **[2]**	$\mathrm{R_g}=(-\infty\ ,-1)\cup\ (1,\infty)$.	$\mathrm{R_g}$ is found by looking at the asymptotic behavior of g as

	$\mathrm{R_{fg}} = \mathrm{R_f}$ with domain restricted to $(-\infty,-1)\cup(1,\infty)$ $=(0,1)$  	$x \to \pm\infty$, which gives the y-values of g as $y<-1$ or $y>1$. Note that equivalent set notations for $(-\infty,-1)\cup(1,\infty)$ are $\mathbb{R}\setminus[-1,1]$, or $(-\infty,\infty)\setminus[-1,1]$. Note also that $\cup$ should not be written as "or", "and", "$\cap$", ",".

SOLUTION		**COMMENTS**
2(a) **[4]**	When $x=0$, $\dfrac{dy}{dx}=\left(\dfrac{81}{3}-15(0)\right)^{\frac{1}{3}}=3$ $\dfrac{d^2y}{dx^2}=\dfrac{1}{3}\left(\dfrac{1}{3}y-15x\right)^{-\frac{2}{3}}\left(\dfrac{1}{3}\dfrac{dy}{dx}-15\right)$ When $x=0$, $\dfrac{d^2y}{dx^2}=\dfrac{1}{3}\left(\dfrac{81}{3}-15(0)\right)^{-\frac{2}{3}}\left(\dfrac{1}{3}(3)-15\right)=-\dfrac{14}{27}$ Hence $y=81+3x-\dfrac{7}{27}x^2+\ldots$	Students should not present the following: $\dfrac{dy}{dx}=\left(\dfrac{y}{3}-15x\right)^{\frac{1}{3}}$ $=\left(\dfrac{81}{3}-15(0)\right)^{\frac{1}{3}}\cdots(*)$ (*) is only true when $\dfrac{dy}{dx}$ is evaluated at (0, 81). The presentation of $y=81+3x-\dfrac{7}{27}x^2+\ldots$ is not ideal for a significant number of students. Students should use $\approx$ or include $+\ldots$ in their answer.
2(b) **(i)** **[2]**	$\dfrac{4-3x+x^2}{(1+x)(1-x)^2}=\dfrac{A}{1+x}+\dfrac{1}{1-x}+\dfrac{B}{(1-x)^2}$ Therefore, $4-3x+x^2=A(1-x)^2+(1+x)(1-x)+B(1+x)$ When $x=1$: $2=2B \Rightarrow B=1$ When $x=-1$: $8=4A \Rightarrow A=2$	Quite a number of students tried to solve partial fractions in the form $\dfrac{A}{1+x}+\dfrac{C}{1-x}+\dfrac{B}{(1-x)^2}$, which involves solving 3 unknowns and is not necessary at all. A handful of students used

		$\frac{A}{1+x}+\frac{B}{1-x}+\frac{C}{(1-x)^2}$, i.e. the same unknowns as the ones given in the question, which is not accepted.
2(b) (ii) [3]	$\mathrm{g}(x)=\frac{4-3x+x^2}{(1+x)(1-x)^2}$ $=\frac{2}{1+x}+\frac{1}{1-x}+\frac{1}{(1-x)^2}$ $=2(1-x+x^2-x^3+...)+(1+x+x^2+x^3+...)$ $+(1+2x+3x^2+4x^3+...)$ $=4+x+6x^2+3x^3+...$ Hence $c_0=4, c_1=1, c_2=6,$ and $c_3=3$.	A good number of students differentiated $\mathrm{g}(x)$ repeatedly and obtained the following: $c_0=\mathrm{g}(0), c_1=\mathrm{g}'(0),$ $c_2=\frac{\mathrm{g}''(0)}{2!}$ and $c_3=\frac{\mathrm{g}'''(0)}{3!}$. They will arrive at the same answer, however, the working seems significantly longer. Please note that $(1+x)^{-1}$ and $(1-x)^{-1}$ could be obtained easily from MF26. Also note that, $\frac{1}{(1-x)^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right)$.
2(b) (iii) [1]	From above, $c_r=2(-1)^r+1+(r+1)=2(-1)^r+r+2$.	Students should note that c_r only refers to the coefficient of the x^r term.

<table>
<tr><th colspan="2">SOLUTION</th><th>COMMENTS</th></tr>
<tr><td colspan="3">General Comments:

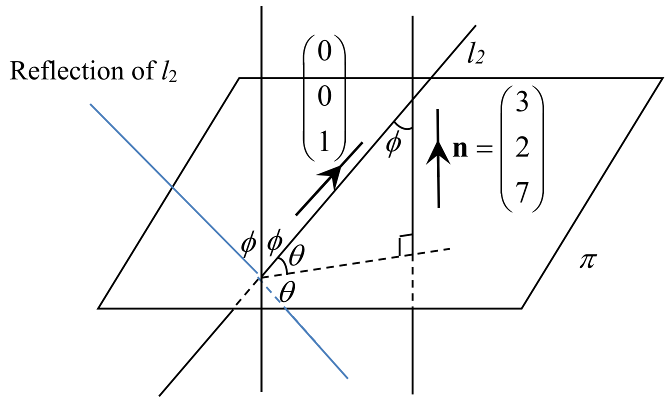
The question is well done in general, but many parts (ii)(b), (iii)(a), (iv) require detailed and clear explanations as they are show questions or the question explicitly required clear working. For example, in such show questions, scalar products should be expanded in full.

For (iv), there are many correct numerical answers but not all obtained full credit. For students who obtained $2\cos^{-1}\left(\frac{7}{\sqrt{62}}\right)=54.5^\circ$, the angle in question must be clearly defined or labelled in a diagram for full credit to be awarded.</td></tr>
<tr><td>3(i)
[1]</td><td>$(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{b})=\begin{pmatrix}-3\\1\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\2\\7\end{pmatrix}$.</td><td>When the position vectors/coordinates are given, there is no need to evaluate using properties of cross product and end up doing more work.</td></tr>
<tr><td>3(ii)
(a)
[1]</td><td>Hence exact area of triangle ABC is $\frac{1}{2}\left\|\begin{pmatrix}3\\2\\7\end{pmatrix}\right\|=\frac{\sqrt{62}}{2}$.</td><td>Do not forget the $\frac{1}{2}$.</td></tr>
<tr><td>3(ii)
(b)
[1]</td><td>Hence the equation of the plane ABC is given by
$\mathbf{r}\cdot\begin{pmatrix}3\\2\\7\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\2\\7\end{pmatrix}=2+14=16$.
Hence the cartesian equation of the plane is $3x+2y+7z=16$.</td><td>Choose a point on the plane with more 0's in the coordinates– the corresponding scalar product is easier to evaluate.</td></tr>
<tr><td>3(iii)
(a)
[3]</td><td>$\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\2\\7\end{pmatrix}=3+4-7=0$
Since the line is perpendicular to the normal vector of the plane, the line and the plane are parallel.

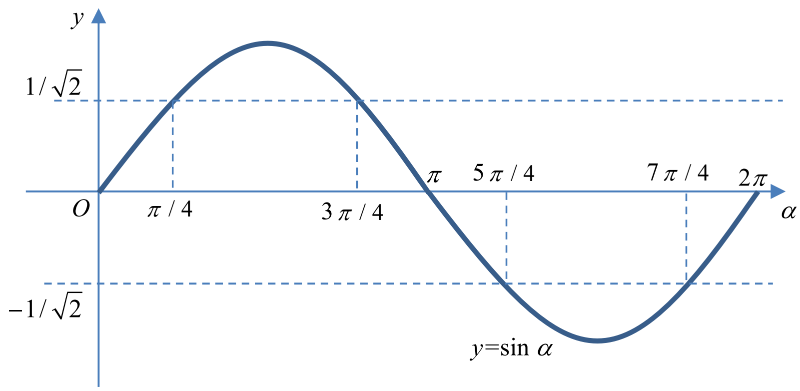
Since $\begin{pmatrix}1\\1\\6\end{pmatrix}\cdot\begin{pmatrix}3\\2\\7\end{pmatrix}=3+2+42=47\neq 16$, a point on the line does not lie on the plane, and since the line and the plane are parallel, the line does not lie on the plane.</td><td>The scalar product being 0 only shows that the line and the normal vector of the plane are perpendicular. This relation has to be clearly stated.

$\begin{pmatrix}1\\1\\6\end{pmatrix}\cdot\begin{pmatrix}3\\2\\7\end{pmatrix}\neq 16$ only shows the point does not lie on the plane.</td></tr>
</table>

	Alternatively, $\left[\begin{pmatrix}1\\1\\6\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}\right]\cdot\begin{pmatrix}3\\2\\7\end{pmatrix}=47\neq 16$ for all real λ so there is no intersection between the line and the plane. This means the line and the plane have to be parallel, and the line does not lie on the plane.	You need to show that the line and plane are parallel first before concluding that the the line is not contained in the plane.
3(iii) (b) [2]	The distance between the line and the plane is $$\frac{\left|\left[\begin{pmatrix}1\\1\\6\end{pmatrix}-\begin{pmatrix}0\\1\\2\end{pmatrix}\right]\cdot\begin{pmatrix}3\\2\\7\end{pmatrix}\right|}{\sqrt{7^2+3^2+2^2}}=\frac{31}{\sqrt{62}}=\sqrt{\frac{31}{2}}=\frac{\sqrt{62}}{2}.$$ **Alternate method** (more tedious and not encouraged since question did not ask for foot of perpendicular) Let F be the foot of perpendicular from $P(1, 1, 6)$ to the plane. Then $\overrightarrow{OF}=\begin{pmatrix}1\\1\\6\end{pmatrix}+\mu\begin{pmatrix}3\\2\\7\end{pmatrix}$ for some $\mu\in\mathbb{R}$. Then since F lies on the plane, $\left[\begin{pmatrix}1\\1\\6\end{pmatrix}+\mu\begin{pmatrix}3\\2\\7\end{pmatrix}\right]\bullet\begin{pmatrix}3\\2\\7\end{pmatrix}=16\Rightarrow\mu=-\frac{1}{2}$. Therefore the distance is $\left\|\overrightarrow{PF}\right\|=\left\|\overrightarrow{OF}-\overrightarrow{OP}\right\|=\left\|\mu\begin{pmatrix}3\\2\\7\end{pmatrix}\right\|=\frac{1}{2}\sqrt{62}.$	Please be careful when transferring the values to your work. Many '6' morphed to '0' and the scalar products were not carefully evaluated. There are many students who went to find the foot of perpendicular from (1, 1, 6) to the plane. This is unnecessary and often showed conceptual errors such as letting $\overrightarrow{OF}$ be $\overrightarrow{PF}$.
3(iv) [3]	The angle θ between the line and the plane is given by $$\sin\theta=\frac{\left|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}3\\2\\7\end{pmatrix}\right|}{\sqrt{62}}=\frac{7}{\sqrt{62}}\Rightarrow\theta=62.748^\circ.$$ Hence the angle between the line and its reflection in the plane is $2(62.748)=125.496^\circ$ and the required acute angle is thus $180^\circ-125.496^\circ=54.5^\circ$ (1 d.p) **Alternatively**, The angle ϕ between the line and the normal of the plane is given by	As mentioned earlier in the comments, you will need clearly defined angles or diagrams with angles indicated to gain full credit for this question even if the numerical answer is correct. Common errors include angle between line and plane is $\cos^{-1}\left(\frac{7}{\sqrt{62}}\right)$

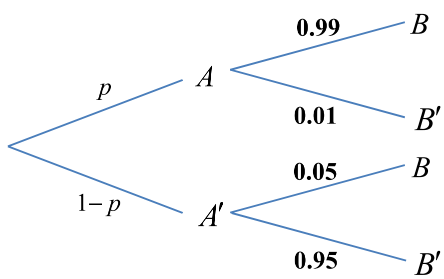
	$\cos\phi = \dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}3\\2\\7\end{pmatrix}\right\|}{\sqrt{62}} = \dfrac{7}{\sqrt{62}} \Rightarrow \phi = 27.252^\circ$ The angle between the line and its reflection in the plane ABC is thus 2 times of the above angle, and since $2(27.252^\circ) = 54.5^\circ < 90^\circ$, this is the required angle.  	, when it should be $\sin^{-1}\left(\dfrac{7}{\sqrt{62}}\right)$ or $90^\circ - \cos^{-1}\left(\dfrac{7}{\sqrt{62}}\right)$. Again, there were a handful of students who went to calculate the foot of perpendicular from a point on the line, before finding the reflection of the point about the plane. This is totally inefficient and unnecessary, and often resulted in algebraic and conceptual errors mentioned in (iii).

SOLUTION		COMMENTS
4(a) (i) [1]	$b = \dfrac{a+c}{2}$	Many students did this in a long-winded manner. It may be noted that a, b, c are also consecutive terms of an AP. Hence, $b-a=c-b$, and the answer can be quoted right away.
4(a) (ii) [1]	Common difference, $d = \dfrac{b-a}{2}$	
4(a) (iii) [2]	Sum of first 10 terms $= \dfrac{10}{2}\left(2a+(10-1)\left(\dfrac{b-a}{2}\right)\right)$ $= 5\left(2a+\dfrac{9}{2}\left(\dfrac{a+c}{2}-a\right)\right)$ $= \dfrac{5}{4}(9c-a)$	Generally well done. Reminders ♦ To write 9 and a more distinctly different. ♦ Clear steps are expected (as the result is given in the question).
4(a) (iv) [2]	Let r be the common ratio. $r = \dfrac{4^{\text{th}}\text{ term}}{3^{\text{rd}}\text{ term}} = \dfrac{a}{b}$. Also, 3^{rd} term $= cr^2 = b$. $\Rightarrow (2b-a)\left(\dfrac{a}{b}\right)^2 = b$ $\Rightarrow \quad (2b-a)a^2 = b^3$	There are so many approaches to do this part. If your solution is more than 4 or 5 lines, please take a look at this suggested solution.
4(b) (i) [4]	Common ratio, $r = 2\sin^2\alpha$ For the series to be convergent, $\lvert r\rvert < 1$. That is, $\lvert 2\sin^2\alpha\rvert < 1$. $\Rightarrow \quad \lvert\sin\alpha\rvert < \dfrac{1}{\sqrt{2}}$ $\Rightarrow \quad -\dfrac{1}{\sqrt{2}} < \sin\alpha < \dfrac{1}{\sqrt{2}}$	Many students went on to consider u_n / u_{n-1} to find r. For this question, you don't have to, really. Other common **mistakes** : ♦ $\sin\alpha < \dfrac{1}{\sqrt{2}}$ (No lower bound.)

	 However, for this particular question, $r = 0$ makes the first term 0^0, which is not defined. Therefore, $2\sin^2\alpha \neq 0$. $\Rightarrow \quad \sin\alpha \neq 0$ $\Rightarrow \quad \alpha \neq 0,\ \pi,\ 2\pi$ Hence, $0 < \alpha < \frac{\pi}{4}$ or $\frac{3\pi}{4} < \alpha < \pi$ or $\pi < \alpha < \frac{5\pi}{4}$ or $\frac{7\pi}{4} < \alpha < 2\pi$.	♦ $0 < \sin\alpha < \frac{1}{\sqrt{2}}$ (Claiming $\sin\alpha$ cannot take negative values.) ♦ $\sin\alpha < \pm\frac{1}{\sqrt{2}}$ (This makes no sense.) ♦ $-\frac{1}{4} < \sin\alpha < \frac{1}{4}$ Many struggled to solve the inequality involving trigo expression. Advice : Sketch the graphs out! Very few students considered $r \neq 0$.
4(b) (ii) [2]	Sum to infinity $= \frac{1}{1-2\sin^2\alpha}$ $= \frac{1}{\cos 2\alpha}$ $= \sec 2\alpha$	Reminders : ♦ To write α and *a* more distinctly different. ♦ $\frac{1}{\cos 2\alpha} \neq \cos^{-1} 2\alpha$

<table>
<tr><th colspan="2">SOLUTION</th><th>COMMENTS</th></tr>
<tr><td>5(a)
[5]</td><td>

Let A be the event that a patient has a particular disease, and B be the event that a test for the disease gives a positive result.



$$\mathrm{f}(p) = \mathrm{P}(A|B) = \frac{0.99p}{0.99p + 0.05(1-p)}$$
$$= \frac{99p}{94p+5}$$

$$\mathrm{f}'(p) = \frac{99(94p+5) - 99p(94)}{(94p+5)^2}$$
$$= \frac{495}{(94p+5)^2} > 0, \quad \text{since } (94p+5)^2 > 0 \text{ for } 0 < p < 1.$$

Hence, f is an increasing function for $0 < p < 1$.

The greater the prevalence of the disease in the population (hence the higher the possibility of being infected), the higher the chance that a positive test means that the patient actually has the disease.

Examples of answers which ***<u>were not accepted</u>***:

As the probability of a patient being infected with a particular disease increases,

i) probability of patient being tested positive increases (No reference to f(p) being conditional probability)

ii) probability of a patient being diagnosed correctly increases (not clear whether the given condition refers to patients who have tested positive or negative or patients who have disease or no disease)

</td><td>

A tree diagram is best suited for this type of question involving different conditional probabilities.

Using the GC to sketch the graph of f is <u>not an acceptable method</u> to show f is an increasing function. Also, f(p) >0 does not imply f is an increasing function.

The terms p and f(p) and their relationship has to be explained in the <u>context of the question.</u>

</td></tr>
<tr><td>5(b)
(i)
[1]</td><td>

Since A and B are independent events, A and B' are also independent events.

$$\mathrm{P}(A \cap B') = \mathrm{P}(A)P(B') = \frac{7(1-k)}{10}$$

</td><td></td></tr>
</table>

5(b) (ii) [2]	Since A and B are mutually exclusive events $B \subseteq A'$, $0 \le k = \mathrm{P}(B) \le \mathrm{P}(A') = 0.3$ $\mathrm{P}(B \mid A') = \dfrac{\mathrm{P}(B \cap A')}{\mathrm{P}(A')} = \dfrac{\mathrm{P}(B)}{\mathrm{P}(A')} = \dfrac{k}{0.3} = \dfrac{10}{3}k$.	You are encouraged to draw your own Venn diagram to visualize the relation between A and B. The condition that A and B are independent is only applicable in (bi) and should not be assumed in (bii) (see the word " instead" used in the question for bii).

SOLUTION		COMMENTS
6(i) [2]	We require $a \to A$, $b \to C$ or D or E and no restriction on c, d and e. Number of ways $= 1 \times 3 \times 3! = 18$	Students who did not get the final answer but presented the correct "thinking process" were given a mark here. As such, for P&C questions, do provide some explanation instead of just writing a string of numbers.
6(ii) [3]	<u>**Method 1**</u> : Number of ways without restriction = 5! = 120 **Case 1** : Exactly one of a or b placed correctly. Number of ways = $(\underbrace{1 \times 3 \times 3!}_{\text{(from part (i))}}) \times 2 = 36$ **Case 2** : both a and b placed correctly. Number of ways $= 1 \times 1 \times 3! = 6$ Total number of ways = 120 – 36 – 6 = 78 <u>**Method 2**</u> : Number of ways without restriction = 5! = 120 Number of ways $a \to A$ $= 1 \times 4! = 24$ Number of ways $b \to B$ $= 1 \times 4! = 24$ Number of ways $a \to A$ and $b \to B$ $= 1 \times 1 \times 3! = 6$ Total number of ways = 120 – 2(24) + 6 = 78	There are at least 10 ways to tackle this part. 4 have been out-lined here. Method 1 is "direct" as it makes use of the answer in (i). Quite a number of students were not careful and thought the complement was just Case 2. We can see (ii) as $n(A' \cap B') = n(\Omega) - n(A \cup B)$ $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

	Method 3 : <table><tr><th>Cases</th><th>Number of ways</th></tr><tr><td>$a \to B$, $b \to A, C, D, E$</td><td>$1\times4\times3!=24$</td></tr><tr><td>$a \to C$, $b \to A, D, E$</td><td>$1\times3\times3!=18$</td></tr><tr><td>$a \to D$, $b \to A, C, E$</td><td>18 (as above)</td></tr><tr><td>$a \to E$, $b \to A, C, D$,</td><td>18 (as above)</td></tr></table> Total number of ways $=24+18+18+18=78$ **Method 4** : **Case 1** : a is placed in B Number of ways = $1\times4!=24$ **Case 2** : a is not placed in B Number of ways = ${}^3C_1\times{}^3C_1\times3!=54$ Total number of ways $=24+54=78$	A number of students did not realize that the 1st two cases are not mutually exclusive. Method 3 is straightforward, although seemingly long. A simple table can help to make the solution less cumbersome however. When listing cases, it is important to ensure that you have considered all possibilities, and check if the cases are mutually exclusive. Notice that Method 4 is similar to method 3. It considers cases 2 to 4 in method 3 as one case.
6(iii) **[3]**	**Case 1:** 2 correct, 3 incorrect placings Number of ways to choose 2 objects to be placed correctly $={}^5C_2=10$ Suppose $a\to A$ and $b\to B$. Then $c\to D$, $d\to E$, $e\to C$ or $c\to E$, $d\to C$, $e\to D$ Number of ways = $10\times2=20$ **Case 2:** 3 correct, 2 incorrect placings Number of ways to choose 3 objects to be placed correctly $={}^5C_3=10$ Suppose $a\to A$, $b\to B$ and $c\to C$. Then $d\to E$, $e\to D$ Number of ways = $10\times1=10$ **Case 3:** 5 correct placings Number of ways = 1 Total number of ways = 20 + 10 + 1 = 31	Note that it is not possible to have 4 correct placings and 1 incorrect placing. Many students thought that the answer is ${}^5C_2\times3!$. This is actually the number of ways of getting at least 2 correct placings with repetitions. Try listing out the cases to convince yourself. Although there are only 2 cases to consider if using the complement method, it is not so straightforward. The answer is $5!-{}^5C_1\times{}^3C_1\times3-{}^4C_1\left(2+3\times3\right)$. Try figuring it out if you are keen.

<table>
<tr><th colspan="2">SOLUTION</th><th>COMMENTS</th></tr>
<tr><td>7(i)
[2]</td><td>Let X be the time in minutes of a medical consultation.

If $X \sim \mathrm{N}(15, 10^2)$, then
$\mathrm{P}(X < 0) = 0.066807$ (5 s.f.) $= 0.0668$ (3 s.f.)

This means that about 7 patients out of every 100 take a "negative" amount of time which is not possible.</td><td>Most students did not mention that time cannot be negative.
Proper justification needs to be given in order to get full credit.</td></tr>
<tr><td>[1]</td><td>$\mathrm{H}_0: \ \mu = 15$ vs $\mathrm{H}_1: \ \mu > 15$
where μ denotes the population mean consultation time in minutes.</td><td></td></tr>
<tr><td>7(ii)
[3]</td><td>Under H_0, since $n = 30$ is large,
$\bar{X} \sim \mathrm{N}\left(15, \frac{10^2}{30}\right)$ approximately by Central Limit Theorem.

We also assume that the population standard deviation is unchanged.

$\mathrm{P}(\bar{X} > 18) = 0.050174$ (5 s.f.) $= 0.0502$ (3 s.f.)
The smallest level of significance is 5.02%</td><td>Most students were able to pen down the sample mean distribution with correct use of CLT.
Students must remember that to reject the null hypothesis, the level of significance must be greater or equals to the p-value.</td></tr>
<tr><td>7(iii)
[4]</td><td>$\sum(y-15) = -50$, $\sum(y-15)^2 = 555$.
An unbiased estimate of the population mean is,
$\bar{y} = 15 + \frac{-50}{80} = \frac{115}{8}$
An unbiased estimate of the population variance is,
$s^2 = \frac{1}{79}\left(555 - \frac{(50)^2}{80}\right) = \frac{2095}{316}$

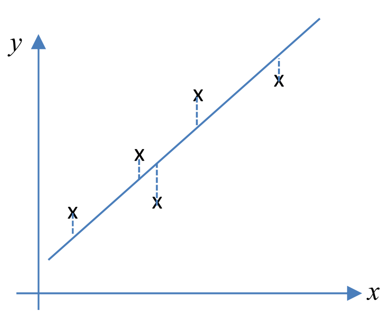
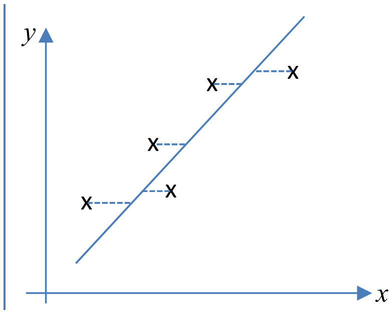
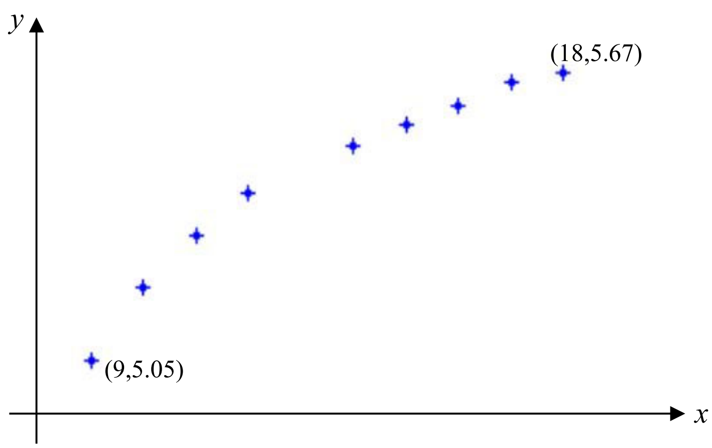
$\mathrm{H}_0: \ \mu = 15$ vs $\mathrm{H}_1: \ \mu < 15$
Under H_0, since $n = 80$ is large,
$\bar{Y} \sim \mathrm{N}\left(15, \frac{\frac{2095}{316}}{80}\right)$ approximately by Central Limit Theorem.

Since p-value $= \mathrm{P}\left(\bar{Y} < \frac{115}{8}\right) = 0.0149623748 = 0.0150$ (3 sf)
< 0.05.
We reject H_0, and conclude that there is sufficient evidence, at the 5% significance level, to support the administrator's claim that the average consultation time at private clinics is less than 15 minutes.</td><td>This part was very well done. Students who have studied this topic were able to find the unbiased estimate of the population mean and variance, set up the null and alternate hypothesis, correct sample mean distribution with CLT, and lastly, correct p-value. A handle of students were penalized for their tardy presentation and incomplete conclusion.</td></tr>
</table>

SOLUTION		COMMENTS
8(i) **[3]**	Sum of all probabilities = 1, thus, $$k+3k+5k+7k+11k+13k=1$$ $$40k=1$$ $$\therefore k=\frac{1}{40}$$ $\mathrm{P}(X=3)=3k=\dfrac{3}{40}$.	This is a 3-mark show question. You need to show your detailed working.
8(ii) **[3]**	$$\mathrm{E}(X)=k+3(3k)+5(5k)+7(7k)+11(11k)+13(13k)$$ $$=374k$$ $$=\frac{374}{40}$$ $$\mathrm{E}(X^2)=k(1+3^3+5^3+7^3+11^3+13^3)$$ $$=4024k$$ $$=\frac{4024}{40}$$ $$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$$ $$=\frac{4024}{40}-\left(\frac{374}{40}\right)^2$$ $$=\frac{5271}{400}$$	GC can give you the answers for this part. However, Question asked for exact value. You have to show the working on how you get the values of E(X) and E(X^2).
8(iii) **[2]**	$R \sim \mathrm{B}(15,0.4)$ $$P(R\ge 5)=1-P(R\le 4)$$ $$=0.783$$	R is defined in the question. You NEED to state the distribution.
8(iv) **[2]**	Let W be the random variable denoting the number of times that a score less than 10 is observed in 14 throws. Then $W \sim \mathrm{B}(14, 0.4)$. Required probability = $\mathrm{P}(W=7)\times 0.4=0.0630$ (Explanation: "W = 7" represent 7 out of the first 14 throws have score less than 10 follow by the 15th throw having score less than 10) Alternatively (without defining new random variable, BUT using the same argument AND the correct formula), Required probability = ${}^{14}C_7\,(0.6)^7\,(0.4)^7\,(0.4)=0.0630$	Remember to define your random variable AND state its distribution.

	SOLUTION	COMMENTS
9(i) **[1]**	Let R denote the working lifespan of Brand R tyre. $R \sim \mathrm{N}(64000, 8000^2)$ Required probability = $\mathrm{P}(R > 70000) = 0.227$ (to 3 s.f.).	In general, the letter used for a new random variable should always be defined with the distribution explicitly written down, same for part (iii) This part was well done as it is only 1 mark so not penalized for presentation
9(ii) **[2]**	Consider $\mathrm{P}(R > t) = 0.98$. From G.C., $t = 47570$ (to nearest whole number).	It is a 2 mark question so student should have a simple line stating what they are working with, or a normal distribution graph with region shaded, before writing down the value of t
9(iii) **[2]**	Let S denote the working lifespan of Brand S tyre. $\bar{S} \sim \mathrm{N}\left(68000, \frac{7500^2}{50}\right)$. Required probability $= \mathrm{P}(\bar{S} > 70000)$ $= 0.0297$ (correct to 3 s.f.).	Some students chose to do by total sum of 50 tyres instead which is fine too. Do note that the question already gave the lifespan to be normally distributed so there is no reason to bring in C.L.T.
9(iv) **[3]**	$(R_1 + R_2 + R_3) - 3S \sim \mathrm{N}(-12000, 12(8000^2))$ Required probability $= \mathrm{P}((R_1 + R_2 + R_3) < 3S)$ $= \mathrm{P}((R_1 + R_2 + R_3) - 3S < 0)$ $= 0.667$ (to 3 s.f.)	Generally well done, usual careless mistake of not taking the square root of the variance when entering into the GC.
9(v) **[3]**	Consider $\mathrm{P}(S > 50000) > \mathrm{P}(R > 50000)$ $\mathrm{P}\left(Z > \frac{50000 - 68000}{\alpha}\right) > 0.9599408865$ $\mathrm{P}\left(Z > \frac{-18000}{\alpha}\right) > 0.9599408865$ $\frac{-18000}{\alpha} < -1.750000503$ $\therefore \quad 0 < \alpha < 10286$ (to nearest whole number)	Easy 1 mark for writing down this inquality. Subsequently, students who do not sketch the normal curve would have to be careful to get the

	[or $0 < \alpha \leq 10285$]	correct direction for the inequality sign Unfortunately, many students could not get the last mark for this question, mostly either not stating the lower bound or stating wrongly (≤ instead of <)
9(vi) [1]	Assume that the working lifespan of **all** the tyres are independent.	This is another part where many students lost mark by not being clear enough. It is much simpler to say "all the tyres" but some students solution only suggested that Brand S needs to be independent with Brand R

SOLUTION		COMMENTS
10(a) **(i)** **[2]**	Regression line y on x :  Regression line x on y :  The least squares regression line of y on x is the line with the minimum sum of the squared vertical distances for each point to the line while the the least squares regression line x on y is the line with the minimum sum of the squared horizontal distances from each point to the line.	Both the least squares regression line is related to the minimum sum of the squared distances and NOT the sum of the distances.
10(b) **(i)** **[2]**	 The scatter diagram shows that the rate of increase of y decreases as x increases. So the variables do not have a linear relation and hence the relationship is not likely to be well modelled by an equation of the form $y = a + bx$, where a and b are constants.	The points which defines the ranges of the values of x and y must be labelled. The scales of the axes should be carefully selected to "space out" the points so that the trend can be observed correctly. The scatter diagram should be used to explain the relationship and not using the product moment coefficient.
10(b) **(ii)** **[2]**	**(a)** product moment correlation coefficient between $\ln x$ and y , $r_1 = 0.986554$ (correct to 6 d.p.) **(b)** product moment correlation coefficient between x^2 and y , $r_2 = 0.945806$ (correct to 6 d.p.)	Note that the question has required the answers to be corrected to 6 decimal places and not 3 significant figures.

10(b) (iii) [1]	Since $\|r_2\| < \|r_1\| < 1$, the product moment correlation coefficient between ln x and y (= 0.986553) is closer to 1, hence $y = a + b\ln x$ is the better model.	It is must be noted that the better model is chosen based on the product moment coefficient being closer to 1 (or -1 in other cases) and NOT just being larger.
10(b) (iv) [3])	The equation of the suitable regression line is $y = 3.237876 + 0.854181\ln x$ $y = 3.24 + 0.854\ln x$ (3 s.f.) When y = 6.5, x = 45.55899 The population size first exceed 6.5 millions in the year 2045. Since the population size of 6.5 millions is not in the data range (from 5.05 to 5.67), the estimate is an extrapolation and so it is not reliable.	The regression line of ln x on y should NOT be used even though a similar answer is obtained. Need to observe carefully the required year is 2045 and NOT 2046 based on the context. It is not sufficient to mention that it is an extrapolation. It is necessary to state the given population size is not in the given data range.
10(b) (v) [1]	When x = 13, $y = 3.237876 + 0.854181\ln 13 = 5.428807$ The required estimate is 5.43 millions	It is necessary to state the estimate as 5.43 millions and not just 5.43.
10(b) (vi) [1]	5.31 millions is lower than 5.43 millions calculated in part (v) (and it does not follow the rising trend of the data given - outlier). There might be some form of deadly diseases spreading in part(s) of the communities in the country that caused the drop in actual figure that year. OR There might be a natural disaster in the country that caused the drop in actual figure that year.	Need to take note that the difference of estimated and actual population sizes is more than100,000 and the actual size is smaller than the previous year, so a decrease in birth rate is not enough to justify the difference.

RIVER VALLEY HIGH SCHOOL
2019 JC2 Preliminary Examination
Higher 2

NAME	

CLASS					

INDEX NUMBER		

MATHEMATICS 9758/01

Paper 1

19 September 2019

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your class, index number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
You are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

Calculator Model:

This document consists of 5 printed pages.

1 Find the range of values of x that satisfy

$$\frac{2x}{x+5} < \frac{1}{x-1}.$$ [4]

Hence deduce the range of values of x that satisfy

$$\frac{2|x|}{|x|+5} - \frac{1}{|x|-1} < 0.$$ [2]

2. A curve $y = \mathrm{f}(x)$ passes through the point (0 , 2) and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{2y}$.

(i) Obtain the Maclaurin series of y in ascending powers of x, up to and including the term in x^2.
Write down the equation of the tangent to the curve $y = \mathrm{f}(x)$ at $x = 0$. [4]

(ii) Show that $y = \sqrt{4 + \sin x}$. [2]

(iii) Using your results in part (ii), and assuming x to be sufficiently small for terms in x^3 and higher powers to be ignored, obtain the binomial series of y. [3]

3. The curve C has equation

$$y = \frac{2x-a}{2x^2-b},$$

where a and b are positive integers.

C passes through the point $\left(\frac{9}{2}, 0\right)$ and the equation of an asymptote of C is $x = 3$.

(i) Show that $a = 9$ and $b = 18$. [2]

(ii) Sketch C, stating clearly the equations of asymptotes, the x-coordinates of turning points and axial intercepts. [4]

(iii) Hence, giving your reasons, deduce the range of values of h such that the graph of $\frac{(x-8)^2}{h^2} + \frac{y^2}{100} = 1$, where h is a positive integer, intersects C at exactly 6 distinct points. [2]

4. A function is said to be self-inverse when $\mathrm{f} = \mathrm{f}^{-1}$ for all x in the domain of f.

Given that the function f is defined by

$$\mathrm{f} : x \mapsto \frac{ax+b}{cx-a}, \text{ for } x \in \mathbb{R}, x \neq \frac{a}{c},$$

where a, b and c are positive constants.

(i) Show that f is self-inverse. [2]

(ii) Using the result of part (i), deduce $\mathrm{f}^2(x)$ and state the range of f^2. [2]

(iii) Solve the equation $\mathrm{f}^{-1}(x) = x$, leaving your answers in the exact form. [3]

For the rest of the question, let $a = 2$, $b = 5$ and $c = 3$.
The function g is defined by $\mathrm{g} : x \mapsto \mathrm{e}^x + 2$ for $x \in \mathbb{R}$.

(iv) Show that the composite function fg exists, justifying your answer clearly. [1]

(v) By considering the graph of f, or otherwise, find the exact range of fg. [2]

5 Consider the following definitions:

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \quad \text{and} \quad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

They are known as *hyberbolic functions*. They are used in modeling suspended bridges and equations of motion related to skydiving.

(i) Find an expression for $\tanh x$ in terms of e^{-2x}. [1]

(ii) Given that

$$f(n+1) - f(n) = \sinh x\left[\operatorname{sech}\left(n+\frac{1}{2}\right)x\right]\left[\operatorname{sech}\left(n-\frac{1}{2}\right)x\right],$$

where $f(n) = \tanh\left(n-\frac{1}{2}\right)x$ and $x \neq 0$, find an expression for

$$S_N = \sum_{n=1}^{N}\left[\operatorname{sech}\left(n+\frac{1}{2}\right)x\right]\left[\operatorname{sech}\left(n-\frac{1}{2}\right)x\right]$$

in the form $\left(\operatorname{cosech} x\right)\left(\tanh Ax - \tanh\frac{1}{2}x\right)$ where A is a constant to be determined. [3]

(iii) Explain why S_∞ exists. Deduce an expression for S_∞ in the form $\left(\operatorname{cosech} x\right)\left(P - \tanh Qx\right)$, where P and Q are constants to be determined. [3]

6 (a) Find the roots of the equation $(1+i)z^2 - z + (2-2i) = 0$, giving your answers in the form $x + iy$, where x and y are exact real numbers. [4]

(b) Given that $f(z) = z^4 - 2z^3 + z^2 + az + b$, where a, $b \in \mathbb{R}$, and that 1 + 2i satisfies the equation f(z) = 0, find the values of a and b and the other roots. [5]

Hence solve the equation $w^4 - 2iw^3 - w^2 - 8iw - 20 = 0$, showing your workings clearly. [2]

7 The area A of the region bounded by $y = \sin^6 x$, the x-axis, $x = 0$ and $x = \dfrac{\pi}{2}$ is to be found.

(i) The area A can be approximated by dividing the region into 5 vertical strips of rectangles of equal width.
Show that $0.10625\pi < A < 0.20625\pi$. [3]

(ii) It is given that $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$, where $n \geq 0$.
By writing $\sin^{2n} x = (\sin x)(\sin^{2n-1} x)$, show by integration by parts that

$$I_n = \frac{2n-1}{2n} I_{n-1}, \text{ for } n \geq 1.$$

Deduce that $I_2 = \dfrac{3}{16}\pi$.
Using the value of I_2, obtain the exact value of A. [7]

8. Two planes have equations given by

$$p_1 : x + 5y + 4z = 4,$$
$$p_2 : x - y + z = 4.$$

(i) Explain why p_1 and p_2 intersect in a line ℓ and determine the cartesian equation of ℓ. [3]

(ii) The plane p_3 contains the line ℓ and the point $Q(5, 3, -6)$, find a cartesian equation of p_3.
Describe the geometrical relationship between these 3 planes and ℓ. [5]

(iii) The line m passes through the points $S(0, 2, 0)$ and $T(4, 0, 0)$. Find the position vector of the foot of perpendicular of S on p_2.
Hence find a vector equation of the line of reflection of m in p_2. [5]

9 On 1 January 2019, Mr Tan started a savings account with a bank with an initial deposit of \$5000. The bank offered compound interest of 1% computed based on the amount of money in this account at the end of each month. Due to personal financial needs, Mr Tan withdrew \$100 from the savings account at the beginning of each month starting from the second month, i.e. 1 February 2019.

(i) Find the amount of money in Mr Tan's account at the end of March 2019. [3]

(ii) Deduce that the amount of money in Mr Tan's account at the end of the n^{th} month is given by $\$100\left(101 - 50\left(1.01^n\right)\right)$. [3]

(iii) Determine the month and year Mr Tan will deplete the savings in his account if he continues to withdraw money from this account. [2]

On 1 January 2019, Mrs Tan also started a savings account with the same bank with an initial deposit of \$3000. The bank offered her interest of \$10 for the first month and increment of \$5 for each subsequent month. For example, in the first 3 months, her interest was \$10 for January, \$15 for February and \$20 for March. In addition, Mrs Tan further deposits \$50 into her savings account every mid-month starting from 15 January 2019.

(iv) Determine the month and year Mrs Tan will first have more money in her account than that of Mr Tan by the end of month. [5]

10 Since the 1990s, a group of scientists has started conservation work to prevent the extinction of a rare species of flying fox in the wild Western Australia forests. The number of such species, in thousands, observed in the forests at time t years after the start of the conservation is denoted by N. It is known that the death rate of the flying foxes is proportional to the number of flying foxes present and that the birth rate of the flying foxes has been maintained constant at 2000 per year. There were only 1000 flying foxes in the forests at the start of the conservation and the rate of increase of the number of flying foxes was 1500 per year when there were 3000 flying foxes.

(i) Form a differential equation relating N and t and show that it can be reduced to

$$6\frac{\mathrm{d}N}{\mathrm{d}t} = 12 - N.$$ [2]

(ii) Solve the differential equation to find N in terms of t. [5]

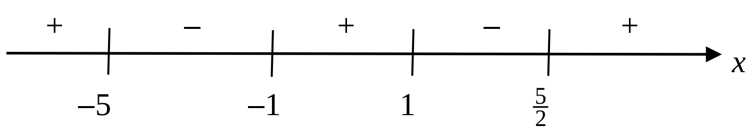
(iii) Deduce the minimum number of years in integer, needed for the number of flying foxes to exceed 6000. [2]

(iv) Sketch a graph to show how the number of flying foxes in the forests varies with time and suggest the long term behavior of the flying fox population. [3]

(v) Suggest a possible limitation of the model in part (i). [1]

END OF PAPER

2019 RVHS H2 Maths Prelim P1 Solutions

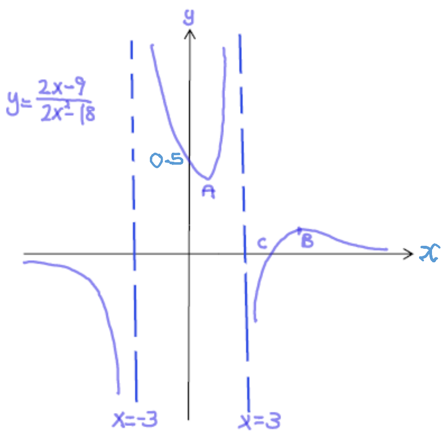
1	Solution [5] Inequality	
	$\dfrac{2x}{x+5}<\dfrac{1}{x-1} \Rightarrow \dfrac{2x}{x+5}-\dfrac{1}{x-1}<0$ --------------- (1) Then, we have $\dfrac{2x^2-2x-x-5}{(x+5)(x-1)}<0$ $\dfrac{2x^2-3x-5}{(x+5)(x-1)}<0$ $\dfrac{(2x-5)(x+1)}{(x+5)(x-1)}<0$ Applying number line test:  Therefore, the solution to inequality (1) is $-5<x<-1 \quad \text{or} \quad 1<x<\dfrac{5}{2}$.	
	Next, we replace ‘ x ’ by ‘ $\lvert x\rvert$ ’ in inequality (1) to obtain $\dfrac{2\lvert x\rvert}{\lvert x\rvert+5}-\dfrac{1}{\lvert x\rvert-1}<0$ -------------- (2) Thus, the solution to inequality (2) correspondingly is $-5<\lvert x\rvert<-1 \quad \text{or} \quad 1<\lvert x\rvert<\dfrac{5}{2}$ i.e. $-5<\lvert x\rvert<-1$ (NA) or $1<\lvert x\rvert<\dfrac{5}{2}$ Thus, the solution to inequality (2) is $-\dfrac{5}{2}<x<-1 \quad \text{or} \quad 1<x<\dfrac{5}{2}$.	

<table>
<tr><td>2</td><td>Solution [8] Maclaurin's Series</td><td></td></tr>
<tr><td></td><td>(i)
$\frac{dy}{dx}=\frac{\cos x}{2y}$
$2y\frac{dy}{dx}=\cos x$
Diff Implicitly w.r.t x,
$2y\frac{d^2y}{dx^2}+2\frac{dy}{dx}\frac{dy}{dx}=-\sin x$
$2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=-\sin x$
When $x = 0$, $y = 2$ (GIVEN)
Hence $\frac{dy}{dx}=\frac{1}{4}$
$(2)(2)\frac{d^2y}{dx^2}+2\left(\frac{1}{4}\right)^2=0$
$\frac{d^2y}{dx^2}=-\frac{1}{32}$

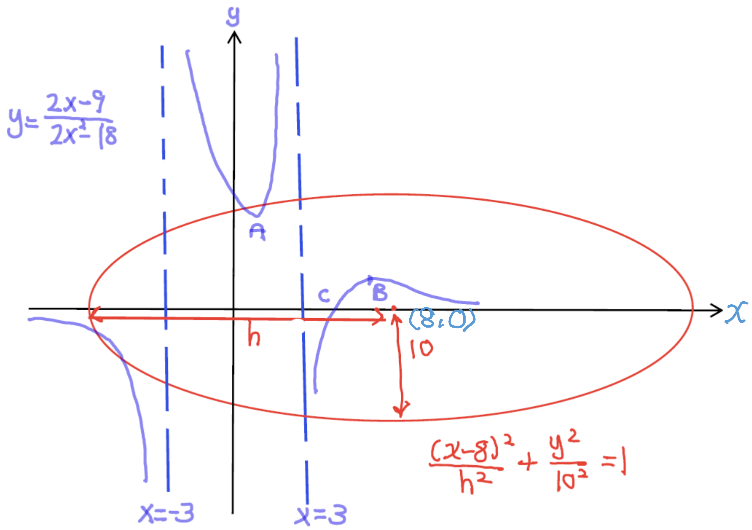
Using the Maclaurin's formula,
$y=2+x(\frac{1}{4})+\frac{x^2}{2!}(-\frac{1}{32})+...$
$\approx 2+\frac{x}{4}-\frac{x^2}{64}$ (up to the x^2term)

Equation of tangent at $x=0$:
$y=2+\frac{1}{4}x$</td><td></td></tr>
<tr><td></td><td>(ii)
$\frac{dy}{dx}=\frac{\cos x}{2y}$
$\int 2y\ dy=\int \cos x\ dx$
$y^2=\sin x+C$
Subst $(0,2)$, then $C=4$
$y=\pm\sqrt{4+\sin x}$
Since $x=0$ and $y=2$,
$\therefore y=\sqrt{4+\sin x}$</td><td></td></tr>
</table>

	(iii) $y=\sqrt{4+\sin x}$ $=(4+\sin x)^{\frac{1}{2}}$ $\approx(4+x)^{\frac{1}{2}}$ (since x is small) $=2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$ $=2\left(1+\frac{1}{2}.\frac{x}{4}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)^2}{2!}+...\right)$ $\approx 2+\frac{x}{4}-\frac{x^2}{64}$	

3	Solution [8] Curve Sketching	
	(i) Since $x = 3$ is an asymptote, $2(3)^2 - b = 0$ $b = 18$ C passes through $\left(\frac{9}{2}, 0\right)$ implies $2\left(\frac{9}{2}\right) - a = 0$ $a = 9$	
	(ii)  A(1.15, 0.436) B(7.85, 0.0637) C(4.50, 0.00) TO find stationary points, set $\frac{dy}{dx} = 0$ And work through the maths to do it. There should be 2 stationary points. This is actually the most important part. The rest is the axes intercepts (let x=0 and then let y=0), identify the asymptotes. And the shape is important – especially the part on moving as close to the asymptote as possible.	

(iii)



$\frac{(x-8)^2}{h^2}+\frac{y^2}{10^2}=1$ is the ellipse with centre at $(8,0)$, with axes of length h and 10.

Horizontal width need to be at least $8+4=12$ units.

Therefore $h \geq 12$, for 6 distinct points of intersections.

4	Solution [10] Functions	
	(i) Let $y=\dfrac{ax+b}{cx-a}, x\in\mathbb{R}, x\neq\dfrac{a}{c}$ $y(cx-a)=ax+b$ $x(cy-a)=ay+b$ $x=\dfrac{ay+b}{cy-a}$ Replacing y by x, $\therefore \mathrm{f}^{-1}(x)=\dfrac{ax+b}{cx-a}, x\in\mathbb{R}, x\neq\dfrac{a}{c}$ Since $\mathrm{f}(x)=\mathrm{f}^{-1}(x) = \dfrac{ax+b}{cx-a}$, f is self-inverse. (SHOWN)	
	(ii) $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ Composing function f on both sides, $\mathrm{ff}(x)=\mathrm{ff}^{-1}(x)$ $\mathrm{f}^2(x)=x$ $\mathrm{D}_{\mathrm{f}^2}=\mathbb{R}\setminus\left\{\dfrac{a}{c}\right\}, \quad \mathrm{R}_{\mathrm{f}^2}=\mathbb{R}\setminus\left\{\dfrac{a}{c}\right\}$ or present as $\mathrm{R}_{\mathrm{f}^2}=\left(-\infty,\dfrac{a}{c}\right)\cup\left(\dfrac{a}{c},\infty\right)$	
	(iii) $\mathrm{f}^{-1}(x)=x$ $\dfrac{ax+b}{cx-a}=x$ $ax+b=cx^2-ax$ $cx^2-2ax-b=0$ $x^2-\dfrac{2a}{c}x-\dfrac{b}{c}=0$ $\left(x-\dfrac{a}{c}\right)^2=\dfrac{b}{c}+\left(\dfrac{a}{c}\right)^2$	

	$x-\dfrac{a}{c}=\pm\sqrt{\dfrac{bc+a^2}{c^2}}$ $x=\dfrac{a}{c}+\sqrt{\dfrac{bc+a^2}{c^2}}$ or $\dfrac{a}{c}-\sqrt{\dfrac{bc+a^2}{c^2}}$ $=\dfrac{a+\sqrt{bc+a^2}}{c}$ or $\dfrac{a-\sqrt{bc+a^2}}{c}$	
	(iv) Now, a =2, b = 5 and c = 3 $\mathrm{f}(x)=\dfrac{2x+5}{3x-2}, x\in\mathbb{R}, x\neq\dfrac{2}{3}$ $\mathrm{g}(x)=\mathrm{e}^x+2, x\in\mathbb{R}$ FACT: For fg to exist , need $\mathrm{R_g}\subseteq\mathrm{D_f}$ to hold , $\mathrm{R_g}=(2,\infty)\subseteq\mathbb{R}\setminus\left\{\dfrac{2}{3}\right\}=\mathrm{D_f}$ fg does exist.	
	(v) $D_g=\mathbb{R}\overset{g}{\mapsto}R_g=(2,\infty)\overset{\mathrm{f}}{\mapsto}R_{\mathrm{fg}}=\left(\dfrac{2}{3},\dfrac{9}{4}\right)$ Therefore Range of fg is $\left(\dfrac{2}{3},\dfrac{9}{4}\right)$	

5	Solution [7] MOD	
	(i) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $= \dfrac{e^x(1-e^{-2x})}{e^x(1+e^{-2x})}$ $= \dfrac{1-e^{-2x}}{1+e^{-2x}}$	
	(ii) $f(n+1) - f(n) = [\sinh x][\operatorname{sech}\left(n+\frac{1}{2}\right)x][\operatorname{sech}\left(n-\frac{1}{2}\right)x]$ $\sum_{n=1}^{N} f(n+1) - f(n) = (\sinh x)\sum_{n=1}^{N}[\operatorname{sech}\left(n+\frac{1}{2}\right)x][\operatorname{sech}\left(n-\frac{1}{2}\right)x]$ $\sum_{n=1}^{N}[\operatorname{sech}\left(n+\frac{1}{2}\right)x][\operatorname{sech}\left(n-\frac{1}{2}\right)x] = \frac{1}{\sinh x}\sum_{n=1}^{N} f(n+1) - f(n)$ $S_n = \dfrac{1}{\sinh x}\sum_{n=1}^{N} f(n+1) - f(n)$ $= \dfrac{1}{\sinh x}\left[\begin{array}{ccc} \cancel{f(2)} & - & f(1) \\ \cancel{f(3)} & - & \cancel{f(2)} \\ \cancel{f(4)} & - & \cancel{f(3)} \\ & \ldots & \\ \cancel{f(N)} & - & \cancel{f(N-1)} \\ f(N+1) & - & \cancel{f(N)} \end{array}\right]$ $= \dfrac{1}{\sinh x}\left(f(N+1) - f(1)\right)$ $= (\operatorname{cosech} x)\left(\tanh(N+\frac{1}{2})x - \tanh(\frac{1}{2})x\right)$ $\therefore\ A = N + \dfrac{1}{2}$	
	(iii) S_∞ $= \lim_{N\to\infty}\sum_{n=1}^{N}\left[\operatorname{sech}\left(n+\frac{1}{2}x\right)\right]\left[\operatorname{sech}\left(n-\frac{1}{2}x\right)\right]$ $= \lim_{N\to\infty}(\operatorname{cosech} x)\left[\tanh\left(N+\frac{1}{2}\right)x - \tanh\left(\frac{1}{2}\right)x\right]$	

	Consider $\lim_{N\to\infty} \tanh\left(N+\frac{1}{2}\right)x$ $= \lim_{N\to\infty} \dfrac{1-e^{-2\left(N+\frac{1}{2}\right)x}}{1+e^{-2\left(N+\frac{1}{2}\right)x}}$ $=1$ Since $\tanh\left(N+\frac{1}{2}\right)x \to 1$ as $N\to\infty$, therefore S_∞ exists. S_∞ $=\left(\text{cosech}\, x\right)\left[1-\tanh\left(\frac{1}{2}\right)x\right]$ $P=1$, $Q=\frac{1}{2}$	

6	Solution [11] Complex Numbers		
	(a) $(1+\mathrm{i})z^2 - z + (2-2\mathrm{i}) = 0$ $z = \dfrac{1 \pm \sqrt{1-4(1+\mathrm{i})(2-2\mathrm{i})}}{2+2\mathrm{i}}$ $= \dfrac{1 \pm \sqrt{1-8(1+\mathrm{i})(1-\mathrm{i})}}{2+2\mathrm{i}}$ $= \dfrac{1 \pm \sqrt{1-8(2)}}{2+2\mathrm{i}}$ $= \dfrac{1 \pm \sqrt{15}\mathrm{i}}{2+2\mathrm{i}} \quad ----(*)$ $= \dfrac{1 + \sqrt{15}\mathrm{i}}{2+2\mathrm{i}} \quad or \; \dfrac{1 - \sqrt{15}\mathrm{i}}{2+2\mathrm{i}}$ $= \dfrac{(1+\sqrt{15}\mathrm{i})(2-2\mathrm{i})}{(2+2\mathrm{i})(2-2\mathrm{i})} \quad or \quad \dfrac{(1-\sqrt{15}\mathrm{i})(2-2\mathrm{i})}{(2+2\mathrm{i})(2-2\mathrm{i})}$ $= \dfrac{(1+\sqrt{15}) + \mathrm{i}(-1+\sqrt{15})}{4} \quad or \quad \dfrac{(1-\sqrt{15}) + \mathrm{i}(-1-\sqrt{15})}{4}$	
	(ii) $z^4 - 2z^3 + z^2 + az + b = 0$ ---- (*) Sub $z = 1+2\mathrm{i}$ into (*), $(1+2\mathrm{i})^4 - 2(1+2\mathrm{i})^3 + (1+2\mathrm{i})^2 + a(1+2\mathrm{i}) + b = 0$ $(-7-24\mathrm{i}) - 2(-11-2\mathrm{i}) + (-3+4\mathrm{i}) + a(1+2\mathrm{i}) + b = 0$ $(12+a+b) + (2a-16)\mathrm{i} = 0$ Comparing the real and imaginary coefficients: $2a-16 = 0$ ---- (1) $12+a+b = 0$ ---- (2) Solving (1) & (2): $a = 8,\ b = -20$ Therefore $z^4 - 2z^3 + z^2 + 8z - 20 = 0$ Using GC: $z = 1+2\mathrm{i},\ 1-2\mathrm{i},\ 2,\ -2$	
	(ii) Alternative Method $\mathrm{f}(z) = z^4 - 2z^3 + z^2 + 8z - 20$ Since 1 + 2i is a root of f(z)=0, and the polynomial have all	

	real coefficients, this imply 1 – 2i is also a root. Hence $z^4-2z^3+z^2+az+b=(z-(1+2\text{i}))(z-(1-2\text{i}))(z^2+pz+q)$ $=((z-1)-2\text{i})((z-1)+2\text{i})(z^2+pz+q)$ $=((z-1)^2+4)(z^2+pz+q)$ $=(z^2-2z+5)(z^2+pz+q)$ Equating coeff of z^3 in f(z): $-2=p-2$. Hence $p=0$ Equating coeff of z^2 in f(z): $1=q-2p+5$. Hence $q=-4$ Now $z^4-2z^3+z^2+az+b=(z^2-2z+5)(z^2-4)$ Equating coeff of z in f(z): $a=8$ Equating constant in f(z): $b=-20$ **Hence** $a=8$ and $b=-20$ $f(z)=0$ $(z^2-2z+5)(z^2-4)=0$ Hence the other 2 roots are ± 2	
	$z^4-2z^3+z^2+az+b=0$ ----(1) Let $z=-\text{i}w$ $w^4-2\text{i}w^3-w^2-8\text{i}w-20=0$ ----(2) Therefore $w=\frac{1}{-\text{i}}z \Rightarrow w=\text{i}z$ $w=\text{i}(1+2\text{i}),\ \text{i}(1-2\text{i}),\ \text{i}(2),\ \text{i}(-2)$ $w=\text{i}-2,\ \text{i}+2,\ 2\text{i},\ -2\text{i}$	

Question 7 [10] Integration		
(i)	When the left height of a rectangle is used, area of region $=\frac{\pi}{10}\left(0+\sin^6\frac{\pi}{10}+\sin^6\frac{2\pi}{10}+\sin^6\frac{3\pi}{10}+\sin^6\frac{4\pi}{10}\right)$ $=\frac{\pi}{10}(1.0625)=0.10625\pi$ When the right height of a rectangle is used, area of region $=\frac{\pi}{10}\left(\sin^6\frac{\pi}{10}+\sin^6\frac{2\pi}{10}+\sin^6\frac{3\pi}{10}+\sin^6\frac{4\pi}{10}+\sin^6\frac{5\pi}{10}\right)$ $=\frac{\pi}{10}(2.0625)=0.20625\pi$ Thus, $0.10625\pi < A < 0.20625\pi$.	
(ii)	$I_n=\int_0^{\frac{\pi}{2}}\sin^{2n}x\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{2}}\sin x\sin^{2n-1}x\,\mathrm{d}x$ $=\left[-\cos x\sin^{2n-1}x\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x(2n-1)\sin^{2n-2}x\cos x\,\mathrm{d}x$ $=(2n-1)\int_0^{\frac{\pi}{2}}\cos^2 x\sin^{2n-2}x\ \mathrm{d}x$ $=(2n-1)\int_0^{\frac{\pi}{2}}(1-\sin^2 x)\sin^{2n-2}x\ \mathrm{d}x$ $=(2n-1)\int_0^{\frac{\pi}{2}}\sin^{2n-2}x-\sin^{2n}x\,\mathrm{d}x$ $=(2n-1)(I_{n-1}-I_n)$ $\Rightarrow I_n+(2n-1)I_n=(2n-1)I_{n-1}$ $\Rightarrow I_n=\frac{2n-1}{2n}I_{n-1}$	

	I_2 $=\frac{3}{4}I_1$ $=\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)I_0$ $=\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{2}}1\,\mathrm{d}x,$ $=\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$ $=\frac{3}{16}\pi$ (Shown) Area of region I_3 $=\frac{5}{6}I_2$ $=\left(\frac{5}{6}\right)\left(\frac{3}{16}\pi\right)$ $=\frac{5}{32}\pi$	

8	Solution [13] Vectors	
	(i) $p_1:\ \mathbf{r}\bullet\begin{pmatrix}1\\5\\4\end{pmatrix}=4,\quad p_2:\mathbf{r}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=4$ Since the two normal vectors are not parallel to each other, the 2 planes are not parallel and hence intersecting. From GC, $x=4-\frac{3}{2}\lambda\ \ -----(1)$ $y=-\frac{1}{2}\lambda\ \ -----(2)$ $z=\ \ \lambda\ \ \ \ -----(3)$ Hence $\ell:\ \mathbf{r}=\begin{pmatrix}4\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}$ where $\lambda\in\mathbb{R}$ $\frac{x-4}{3}=y=-\frac{z}{2}$	
	(ii) p_3 contains $\ell:\ \mathbf{r}=\begin{pmatrix}4\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}$ and point $Q(5,3,-6)$. Let T denote the point $(4,0,0)$. $\overrightarrow{QT}=\begin{pmatrix}4\\0\\0\end{pmatrix}-\begin{pmatrix}5\\3\\-6\end{pmatrix}=\begin{pmatrix}-1\\-3\\6\end{pmatrix}$ and $\begin{pmatrix}3\\1\\-2\end{pmatrix}$ are 2 direction vectors parallel to p_3. Consider $\begin{pmatrix}-1\\-3\\6\end{pmatrix}\times\begin{pmatrix}3\\1\\-2\end{pmatrix}=\begin{pmatrix}0\\16\\8\end{pmatrix}=8\begin{pmatrix}0\\2\\1\end{pmatrix}$ Therefore $\mathbf{n}=\begin{pmatrix}0\\2\\1\end{pmatrix}$ is a normal vector to p_3.	

	$\mathbf{r}\cdot\mathbf{n}=\begin{pmatrix}4\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\2\\1\end{pmatrix}=0$ $p_3\ :\ \mathbf{r}\cdot\begin{pmatrix}0\\2\\1\end{pmatrix}=0$ $p_3\ :\ 2y+z=0$ *Geometrical Relationship:* <u>3 planes</u>/intersect at the common line l.	
	(iii) $p_2:\mathbf{r}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=4$ ---- (1) Let N be the foot of perpendicular of $S\left(0,2,0\right)$ on p_2. Let l_{NS} be the line passing through points N and S. $l_{NS}:\mathbf{r}=\begin{pmatrix}0\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ ---- (2) Sub (2) into (1): $\left[\begin{pmatrix}0\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\right].\begin{pmatrix}1\\-1\\1\end{pmatrix}=4$ ---- (*) $-2+3\lambda=4$ $\lambda=2$ $\overrightarrow{ON}=\begin{pmatrix}0\\2\\0\end{pmatrix}+2\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\0\\2\end{pmatrix}$ ---- (2) Let S' be the point of reflection of S in p_2. N is the midpoint of SS'. $\overrightarrow{ON}=\frac{1}{2}\left(\overrightarrow{OS}+\overrightarrow{OS'}\right)$ $\overrightarrow{OS'}=2\overrightarrow{ON}-\overrightarrow{OS}$	

	$\overrightarrow{OS'} = 2\begin{pmatrix}2\\0\\2\end{pmatrix} - \begin{pmatrix}0\\2\\0\end{pmatrix} = \begin{pmatrix}4\\-2\\4\end{pmatrix}$ Let m' be the line of reflection of m in p_2. The point $T(4,0,0)$ lying on m also lies on p_2. Therefore $T(4,0,0)$ also lies on m'. m' passes through $T(4,0,0)$ and $S'(4,-2,4)$ $m': \mathbf{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda\left[\begin{pmatrix}4\\0\\0\end{pmatrix} - \begin{pmatrix}4\\-2\\4\end{pmatrix}\right],\ \lambda \in \mathbb{R}$ $m': \mathbf{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\-4\end{pmatrix},\ \lambda \in \mathbb{R}$ $m': \mathbf{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\-2\end{pmatrix},\ \lambda \in \mathbb{R}$	

9 Solution [13] APGP

(i)

We first observe the following pattern:

Month	Beginning ($)	End ($)
Jan' 2019	5000	5000×1.01
Feb'2019	$5000\times1.01-100$	$(5000\times1.01-100)\times1.01$
Mar'2019	$5000\times1.01^2-100\times1.01-100$	$(5000\times1.01^2-100\times1.01-100)\times1.01$

Thus, by the end of March 2019, John's account is left with $\$(5000\times1.01^3-100\times1.01^2-100\times1.01)$
$= \$4948.50$

(ii)

From (i), we can deduce that by the end of the n^{th} month, the money left in John's saving account is

$$5000\times1.01^n-\left(100\times1.01^{n-1}+100\times1.01^{n-2}+....+100\times1.01\right)$$

$$=5000\times1.01^n-100\times1.01\left(1+1.01+1.01^2+...+1.01^{n-2}\right)$$

$$=5000\times1.01^n-100\times1.01\times\frac{1\left(1.01^{n-1}-1\right)}{1.01-1}$$

$$=5000\times1.01^n-10000\times1.01\times\left(1.01^{n-1}-1\right)$$

$$=10100+5000\times1.01^n-10000\times1.01^n$$

$$=100\left(101-50\times1.01^n\right) \text{ (shown)}$$

(iii)

Consider $100\left(101-50\times1.01^n\right)\le0$

$\Rightarrow 50\times1.01^n\ge101$

$\Rightarrow 1.01^n\ge\dfrac{101}{50}$

$\Rightarrow n\ge70.66$

Thus, it will take Mr Tan 71 months to deplete his saving account and it will be by November of 2024.

Alternative solution using table:

n	Amt left in account
70	$66.18>0$
71	$-34.16<0$

Account depleted in the 71st month.

(iv)
The amount of interest earned by Mrs Tan's for each subsequent month forms an AP: 10, 10+5, 10+2×5,…..
i.e. an AP with first term 10 and common difference 5
Thus, by the end of n^{th} month, the total amount of money in Mrs Tan's saving account

$$=3000+50n+\frac{n}{2}\left[2(10)+5(n-1)\right]$$

$$= \; =3000+50n+10n+2.5n(n-1)$$

$$=3000+57.5n+2.5n^2$$

For Mrs Tan's account to be more than Mr Tan's, we let

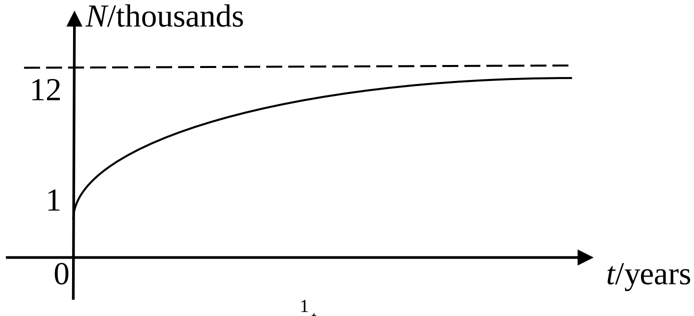
$$3000+57.5n+2.5n^2 > 100\left(101-50\times1.01^n\right) \text{ ---- (*)}$$

Then using GC: we have

n	LHS of (*)	RHS of (*)
14	4295	4352.6
15	4425	4295.2
16	4560	4327.1

It takes 15 months from Jan 2019 for Mrs' Tan's account to exceed that of Mr Tan.
Thus, it is by end of March 2020 that Mrs Tan's account will first be more than that of Mr Tan's.

10	Solution [13] DE	
	(i) Based on the given information, we have $\frac{\mathrm{d}N}{\mathrm{d}t}=2-kN\ , k \in R$ Since it is given that $\frac{\mathrm{d}N}{\mathrm{d}t}=1.5$ when N = 3, we have 1.5 = 2 −3k $\Rightarrow$ $k=\frac{1}{6}$. Thus, $\frac{\mathrm{d}N}{\mathrm{d}t}=2-\frac{1}{6}N \Rightarrow 6\frac{\mathrm{d}N}{\mathrm{d}t}=12-N$ (shown)	
	(ii) Now, $6\frac{\mathrm{d}N}{\mathrm{d}t}=12-N$ $\Rightarrow \int \frac{\mathrm{d}N}{12-N}=\int \frac{1}{6}\ \mathrm{d}t$ $\Rightarrow \frac{\ln\lvert 12-N\rvert}{-1}=\frac{1}{6}t+c$ $\Rightarrow \ln\lvert 12-N\rvert=-\frac{1}{6}t+C$ Then, we have $\lvert 12-N\rvert=e^{-\frac{1}{6}t+C}$ $\Rightarrow 12-N=\pm\ e^{-\frac{1}{6}t+C}$ $\Rightarrow 12-N=Ae^{-\frac{1}{6}t}$ where $A=\pm e^{C}$ Next, given that when $t=0$, $N=1$, we have $A=12-1=11$. Hence, the required equation connecting N and t is $N=12-11e^{-\frac{1}{6}t}$	
	(iii) Let $12-11e^{-\frac{1}{6}t}\geq 6$	

	$11e^{-\frac{1}{6}t} \le 6$ $\Rightarrow e^{-\frac{1}{6}t} \le \frac{6}{11}$ $\Rightarrow -\frac{1}{6}t \le \ln\left(\frac{6}{11}\right)$ $\Rightarrow t \ge 3.64$ Thus, a minimum of 4 years are needed for the number of flying fox to first exceed 6000.	
	(iv)  We note that as $t \to \infty$, $e^{-\frac{1}{6}t} \to 0$. Thus, $N = 12 - 11e^{-\frac{1}{6}t} \to 12$ So, in the long run, the number of flying fox approaches 12 thousands.	
	(v) One possible limitation may be that the model fails to take into account external factors that affect the population. For example (i) outburst of sudden natural disasters which will affect population of the flying fox; (ii) the increase in the hunting of flying fox over the years; (iii) adverse climate change which affect their living habitat	

RIVER VALLEY HIGH SCHOOL
2019 JC2 Preliminary Examination
Higher 2

NAME	

CLASS							INDEX NUMBER		

MATHEMATICS **9758/02**

Paper 2 **23 September 2019**

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your class, index number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
You are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
Total	

Calculator Model:

This document consists of **7** printed pages.

Section A: Pure Mathematics [40 marks]

1 A sequence u_0, u_1, u_2, ... is such that $u_{n+1} = -3u_n + An + B$, where A and B are constants and $n \geq 0$.

(i) Given that $u_0 = 4$, $u_1 = 2$ and $u_2 = 16$, find the values of A and B. [3]

It is known the nth term of the same sequence is given by $u_n = a(-3)^n + bn + c$, where a, b and c are constants.

(ii) By considering a system of linear equations, find the values of a, b and c. [3]

2 Two planes p_1 and p_2 have equations

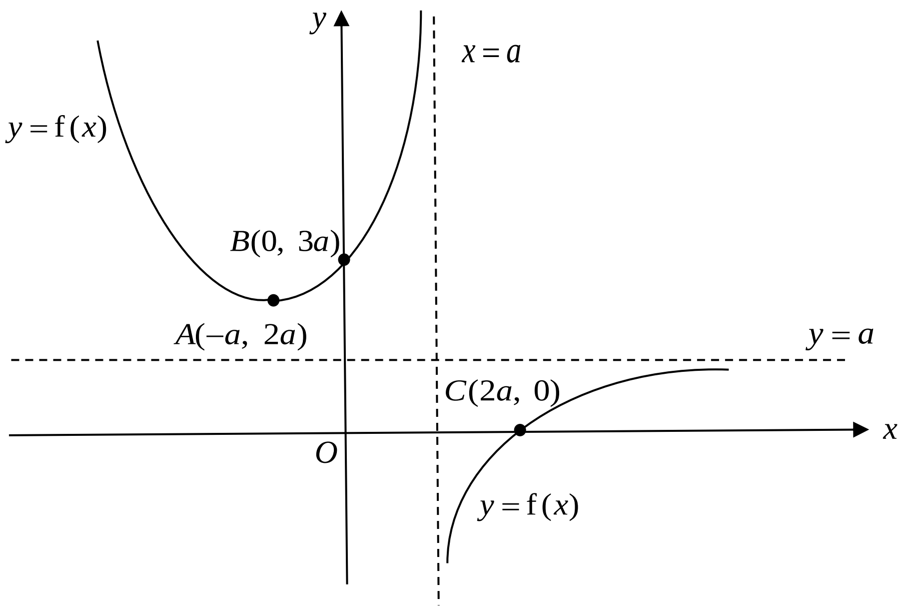
$$p_1 : \mathbf{r} = \mathbf{a} + \lambda_1\mathbf{m} + \mu_1\mathbf{n},$$
$$p_2 : \mathbf{r} = \mathbf{b} + \lambda_2\mathbf{m} + \mu_2\mathbf{n},$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

(a) The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The angle between $\mathbf{a}$ and $\mathbf{m}\times\mathbf{n}$ is 45° and that between $\mathbf{b}$ and $\mathbf{m}\times\mathbf{n}$ is 60°. Show that the distance between p_1 and p_2 is given by $\left|\dfrac{|\mathbf{b}|}{2} - \dfrac{|\mathbf{a}|}{\sqrt{2}}\right|$. [2]

(b) Q is a variable point between p_1 and p_2 such that the distance of Q from p_1 is twice that of the distance from p_2. Write down a possible position vector of Q, in terms of $\mathbf{a}$ and $\mathbf{b}$. Describe the locus of Q and write down its vector equation. [3]

3 (a) The diagram below shows the graph of $y = \mathrm{f}(x)$. The graph has a minimum point at $A(-a,\ 2a)$ where $a > 0$. It passes through the points $B(0,\ 3a)$ and $C(2a,\ 0)$. The equations of the asymptotes are $x = a$ and $y = a$. Each of the gradients of the tangents at points B and C is a.



Sketch on separate clearly labelled diagrams, the graphs of

(i) $y = \mathrm{f}(2x + a)$, [2]

(ii) $y = \mathrm{f}'(x)$, [2]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

Label in each case, the coordinates of points corresponding to the points A, B and C (if any) of $y = \mathrm{f}(x)$ and the equation of asymptote(s).

(b) Describe a sequence of transformations which transforms the graph of $2y^2 - x^2 = 1$ onto the graph of $y^2 - (x-1)^2 = 1$. [2]

4 The parametric equations of a curve are

$$x = \sin t, \quad y = \cos t, \quad \text{where } 0 < t < \frac{\pi}{2}.$$

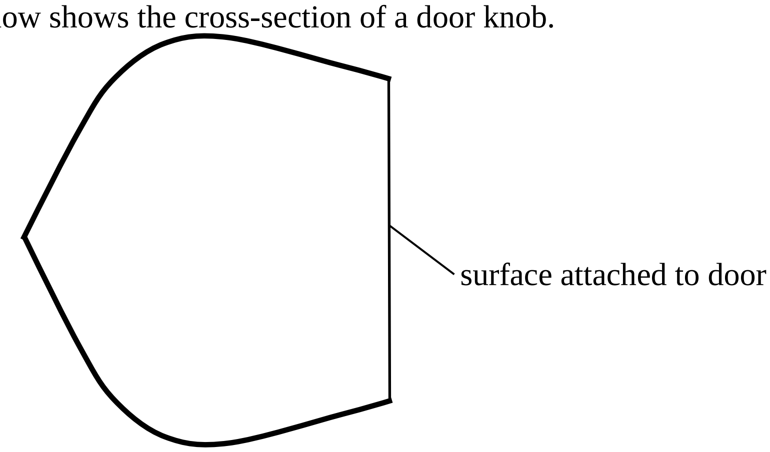
(i) Sketch the graph of the curve. [1]

(ii) Show that the equation of the tangent to the curve at the point where $t = p$ is given by the equation $y = -(\tan p)x + \sec p$. [3]

(iii) Given that the above tangent meets the x-axis at the point P and the y-axis at the point Q, find the coordinates of the mid-point M of PQ in terms of p.
Hence, determine the cartesian equation of the locus of M. [3]

(iv) Find the exact area of the region enclosed by the curve, the lines $x = \dfrac{1}{2}$, $x = \dfrac{1}{\sqrt{2}}$ and the x-axis. [3]

5 The diagram below shows the cross-section of a door knob.

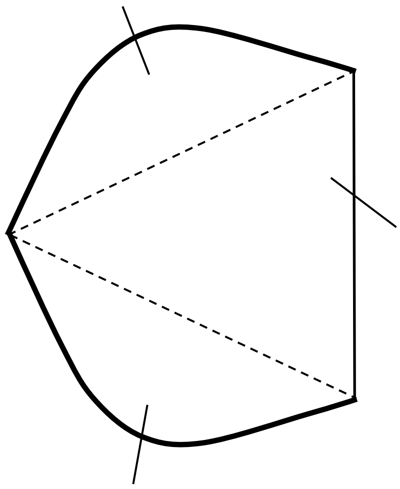


The curve part of the door knob can be modelled by the curve C with equation $y = \dfrac{2x}{1+x^2}$ between $x = 0$ and $x = 2$. The solid door knob is obtained by rotating C through four right angles about the x-axis.

(i) Sketch the graph of C for $0 \le x \le 2$. [1]

(ii) Using the substitution $x = \tan\theta$, find the exact volume of the door knob. [6]

The exterior and the interior of the door knob is made of wood and plastic respectively. The interior of the door knob consists of a solid cone.



The cost of wood and plastic used in manufacturing of the door knob is \$3 and \$1.20 per unit volume respectively. Find the cost of manufacturing one such door knob, giving your answer to the nearest dollar. [3]

Section B: Probability and Statistics [60 marks]

6 At a funfair game store, 3 red, 2 blue, 1 white, 1 yellow and 1 green beads are being arranged. The beads are identical apart from their colours.

(i) Find the number of ways to arrange all the beads in a row with all the red beads separated. [2]

(ii) Five beads, one of each colour, are string together to form a rigid ring. Find the number of distinct rings that can be formed. [2]

At another game store, the letters of the word INITIATE are being arranged in a row instead. Find the probability that all the letters are used and the letters A, E and N are arranged in alphabetical order. [2]

7 In a game, Alfred is given 6 keys of which n of the keys can unlock a box. He tries the keys randomly without replacement to unlock the box.
The random variable X denotes the number of tries Alfred takes to unlock the box.

(i) Write down an expression for $\mathrm{P}(X = 3)$ in terms of n. [1]

Use $n = 3$ for the rest of this question.
For the game, Alfred wins \$2.00 if he takes less than 4 tries to open the box else he wins \$1.00. The random variable W denotes Alfred's winnings in a game.

(ii) Find the probability distribution table for W and hence find the value of $\mathrm{E}(W)$. [2]

(iii) Find the probability that Alfred wins at least \$9.00 after playing the game 5 times. [3]

8 Two boxes, one white and one black, are used in a game. The boxes contains balls labelled '2', '5' and '10'. The balls are identical except for their numbers. The table below shows the number of balls in each box.

	Number of balls labelled		
	'2'	'5'	'10'
White box	5	2	1
Black box	5	1	2

In the first round of the game, a player draws 3 balls randomly from the white box. If the sum of the numbers on the 3 balls add up to 9 or more, the player enters the second round of the game to draw another 3 balls randomly from the black box.

(i) Find the probability that the player draws 3 different numbers in the first round. [1]

(ii) Show that the probability that sum of the numbers add up to 12 and 6, in the first and second round respectively, is $\frac{25}{1568}$.
Hence, find the probability that the sum of the numbers drawn from the two rounds adds up to 18. [4]

In the third round of the game, only the original black box with its 8 balls is used. The player draws 2 balls from it randomly. Find the expected sum of the numbers drawn. [2]

9 A multiple-choice test consists of 12 questions and each question has 5 options for a candidate to choose from. The candidate has to choose one option as the answer for each question. For every question that a candidature answered correctly, the candidate is awarded 1 mark. There are no marks awarded or deducted for wrong answers. Let X denotes the number of marks a candidate scores for the test.

(i) State an assumption on how a candidate decides which option to choose in order for X to be modelled by a binomial distribution. Explain why this assumption is necessary. [2]

Suppose the assumption stated in part (i) holds.

(ii) Find the most probable number of marks that a candidate will score in the test. [2]

(iii) A candidate found that he will score at most 5 marks. Find the probability that he will score more than 4 marks. [2]

(iv) A candidate sat for n such tests (where n is large), with each test consisting of 12 questions and each question has 5 options. Determine the least value of n so that the probability of obtaining a mean test score of at most 2.7 marks is 0.95. [2]

10 Engineers claimed that a newly developed engine is efficient. The number of hours x the engine ran on one litre of fuel was measured on 70 occasions. The results are summarized by

$$\sum(x-46)=-27, \quad \sum(x-46)^2=30939.$$

(i) Calculate unbiased estimates for the population mean and variance of the number of hours the engine runs on one litre of fuel. [2]

(ii) Determine the greatest mean number of hours the engineers should claim so that there is sufficient evidence at the 5% level of significance that the engine is efficient. [3]

(iii) The engineers collected five more data points:

56, 34, 63, 50, 54.

The engineers found that when all 75 data points are considered, they can no longer claim that the engine is efficient at the α % level of significance.
Show that the unbiased estimate for the population variance, for the set of 75 data points is 426.378, correct to 3 decimal places. Find the greatest integer value of α. [4]

11 (a) The equation of the estimated least squares regression line of y on x for a set of bivariate data is $y=a+bx$. Explain what do you understand by the least squares regression line of y on x. [2]

(b) A student wishes to determine the relationship between the length of a metal wire l in millimetres, and the duration of time t in minutes for it to completely dissolve in a particular chemical solution. After conducting the experiment, he obtained the following set of data.

l	15	30	45	60	75	90	105	120	135	150
t	0.779	1.10	1.35	1.56	1.74	1.91	2.07	2.22	2.31	2.45

(i) Draw a scatter diagram to illustrate the data. [1]

(ii) State with a reason, which of the following model is appropriate for the data collected.

(A) $t=a+bl^2$, (B) $t=a+b\ln l$, (C) $t=ae^{bl}$,

where a and b are some constants. [2]

(iii) For the above chosen model, calculate the values of a and b and the product moment correlation coefficient. [2]

(iv) Using the model in part (iii), estimate the length of metal wire that took 1.00 minute to dissolve completely in the chemical solution.
Comment on the reliability of your answer. [3]

(v) Given that 1 millimetre = 0.03937 inch, find an equation that can be used to estimate the duration of time taken to completely dissolve a metal wire of length L, where L is measured in inches. [2]

12 An orchard produces apples and pears.

(i) The masses of apples produced by the orchard have mean 70 grams and standard deviation 40 grams.

Explain why the masses of apples in the orchard are unlikely to be normally distributed. [1]

(ii) The masses of pears produced by the orchard are normally distributed. It is known that one-third of the pears produced weigh less than 148 grams, and one-third weigh more than 230 grams. Find the mean and standard deviation of the masses of pears produced in the orchard. [4]

During each production period, the orchard produces 300 apples and 400 pears and the owner is able to sell all the apples at \$0.005 per gram and all the pears at \$0.008 per gram. It is further known that the owner needs to spend a fixed amount of \$$c$ as the operating cost for each production period and that the orchard makes a profit 90% of the time.

(iii) Explain why the total mass of the 300 apples is approximately normally distributed. [1]

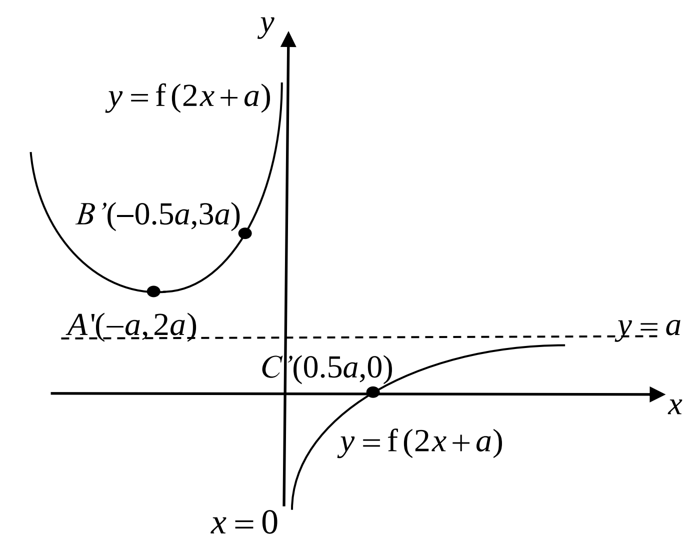
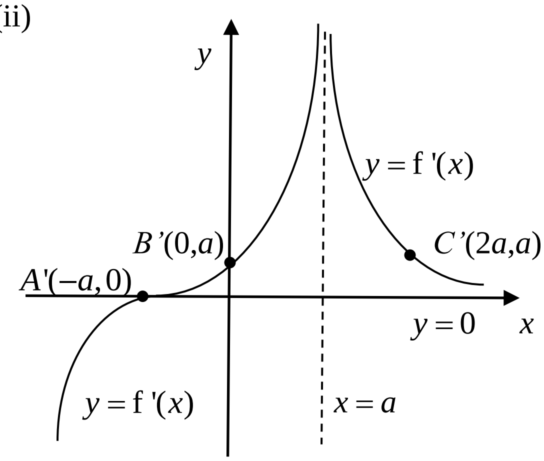
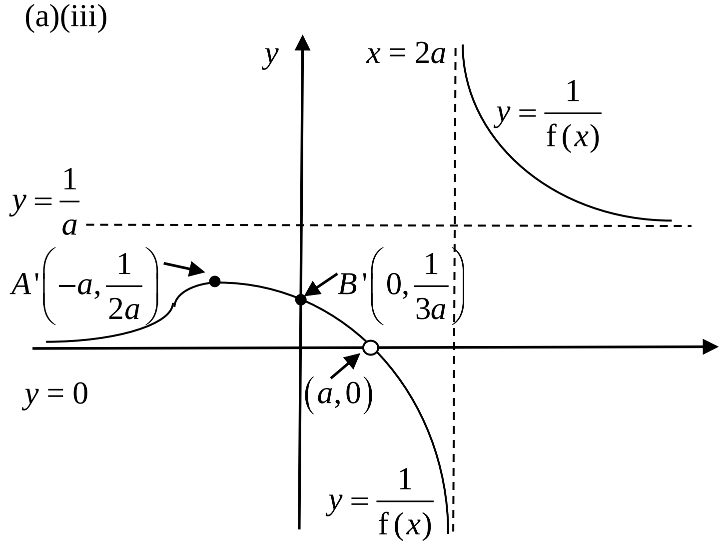
(iv) By letting R be the amount of money collected from selling the 300 apples and 400 pears, find the distribution of R and hence, determine the value of c. [6]

END OF PAPER

2019 RVHS H2 Maths Prelim P2 Solutions

1	Solution [6] System of linear Eqns	
	(i) $u_{n+1} = -3u_n + An + B$ When $n = 0$, $u_1 = -3u_0 + B$ ---- (*) $2 = -3(4) + B$ $B = 14$ When $n = 1$, $u_2 = -3u_1 + A + B$ ---- (**) $16 = -3(2) + A + 14$ $A = 8$	
	(ii) $u_0 = 4 \Rightarrow a \qquad + c = 4$ ---- (1) $u_1 = 2 \Rightarrow -3a + b + c = 2$ ---- (2) $u_2 = 16 \Rightarrow 9a + 2b + c = 16$ ---- (3) Using GC, $a = 1$, $b = 2$, $c = 3$	

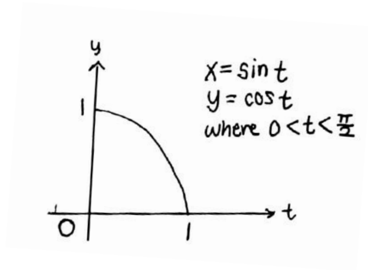
Question 2 [5] vectors		
(a)	Distance between p_1 and p_2 $= \dfrac{\left\|\overrightarrow{AB}\cdot(\mathbf{m}\times\mathbf{n})\right\|}{\|\mathbf{m}\times\mathbf{n}\|}$ $= \dfrac{\left\|\mathbf{b}\cdot(\mathbf{m}\times\mathbf{n})-\mathbf{a}\cdot(\mathbf{m}\times\mathbf{n})\right\|}{\|\mathbf{m}\times\mathbf{n}\|}$ $= \dfrac{\left\|\|\mathbf{b}\|\|(\mathbf{m}\times\mathbf{n})\|\cos 60^\circ-\|\mathbf{a}\|\|(\mathbf{m}\times\mathbf{n})\|\cos 45^\circ\right\|}{\|\mathbf{m}\times\mathbf{n}\|}$ $= \left\|\dfrac{\|\mathbf{b}\|}{2}-\dfrac{\|\mathbf{a}\|}{\sqrt{2}}\right\|$	
(b)	By ratio theorem, $\overrightarrow{OQ}=\dfrac{\mathbf{a}+2\mathbf{b}}{3}$. The locus of Q is a plane containing fixed point with position vector $\dfrac{\mathbf{a}+2\mathbf{b}}{3}$ and parallel to **m** and **n**. Equation of locus of Q: $\mathbf{r}=\dfrac{\mathbf{a}+2\mathbf{b}}{3}+\lambda\mathbf{m}+\mu\mathbf{n},\quad \lambda,\mu\in\mathbb{R}$	

3	Solution [9] Transformation	
	(a)(i) $\mathrm{f}(x) \to \mathrm{f}(x+a) \to \mathrm{f}(2x+a)$  	
	(a)(ii)  	
	(a)(iii)  	

	(b) We first note the following transformation steps: $2y^2 - x^2 = 1$ $\downarrow$ Step 1: Replace 'x' by '$x-1$' $2y^2 - (x-1)^2 = 1$ $\downarrow$ Step 2: Replace 'y' by '$\frac{y}{\sqrt{2}}$' $2\left(\frac{y}{\sqrt{2}}\right)^2 - (x-1)^2 = 1$ $y^2 - (x-1)^2 = 1$ Thus, the sequence of transformations needed are as follow 1. A translation of 1 unit in the positive *x* axis direction; 2. A scaling parallel to the *y* axis of factor $\sqrt{2}$.	

4	Solution [10] Parametric Eqn + Integration Application (Area)

(i)



(ii)

Given $x = \sin t, \quad y = \cos t$ where $0 < t < \dfrac{\pi}{2}$,

we have $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-\sin t}{\cos t} = -\tan t$

Thus, at the point where $t = p$, ie $(\sin p, \cos p)$, the equation of the tangent is

$$y - \cos p = -\tan p\left(x - \sin p\right) \text{ ---- } (*)$$

So,

$$y = -\left(\tan p\right)x + \frac{\sin^2 p}{\cos p} + \cos p$$

$$= -\left(\tan p\right)x + \frac{\sin^2 p + \cos^2 p}{\cos p}$$

$$= -\left(\tan p\right)x + \frac{1}{\cos p}$$

$$= -\left(\tan p\right)x + \sec p \text{ (shown)}$$

(iii)

The equation of the tangent is $y = -\left(\tan p\right)x + \sec p$,

Let $x = 0$, $y = \sec p$. So, $Q = \left(0, \dfrac{1}{\cos p}\right)$.

Let $y = 0$, $x = \dfrac{\sec p}{\tan p} = \dfrac{1}{\cos p} \times \dfrac{\cos p}{\sin p} = \dfrac{1}{\sin p}$.

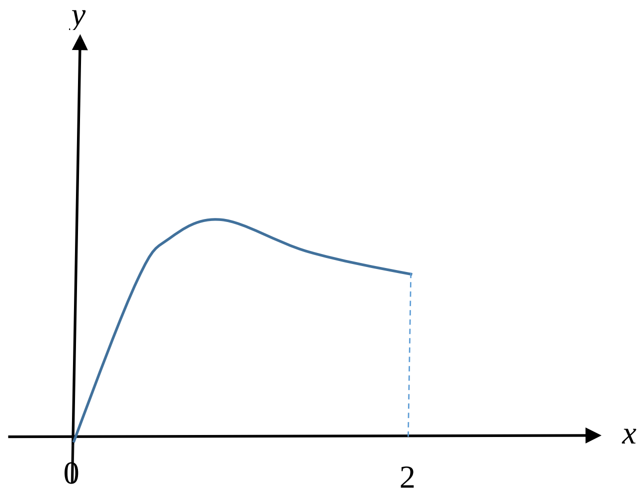
So, $P = \left(\dfrac{1}{\sin p}, 0\right)$

Hence mid point of PQ =

$$M = \left(\frac{1}{2}\left(\frac{1}{\sin p} + 0\right), \frac{1}{2}\left(\frac{1}{\cos p} + 0\right)\right) = \left(\frac{1}{2\sin p}, \frac{1}{2\cos p}\right)$$

To find the Cartesian equation of the locus of the point M,

	We let $x=\dfrac{1}{2\sin p}$ and $y=\dfrac{1}{2\cos p}$ Then $\sin p=\dfrac{1}{2x}$ and $\cos p=\dfrac{1}{2y}$ And thus, $\left(\dfrac{1}{2x}\right)^2+\left(\dfrac{1}{2y}\right)^2=1$ Hence the Cartesian equation of the locus of M is $\dfrac{1}{x^2}+\dfrac{1}{y^2}=4$ or $x^2+y^2=4x^2y^2$	
	(iv) Exact area = $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} y\ \mathrm{d}x$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} y\ \dfrac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cos t)(\cos t)\ \mathrm{d}t$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 t\ \mathrm{d}t$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \dfrac{1+\cos 2t}{2}\ \mathrm{d}t$ $=\left[\dfrac{1}{2}t+\dfrac{\sin 2t}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$ $=\left(\dfrac{\pi}{8}+\dfrac{1}{4}\right)-\left(\dfrac{\pi}{12}+\dfrac{\sqrt{3}}{8}\right)$ $=\dfrac{\pi}{24}+\dfrac{1}{4}-\dfrac{\sqrt{3}}{8}$ units^2	

<table>
<tr><td colspan="3">Question 5 [10]</td></tr>
<tr><td>(i)</td><td></td><td></td></tr>
<tr><td>(ii)</td><td>

$x=\tan\theta \Rightarrow \dfrac{dx}{d\theta}=\sec^2\theta$

Volume

$=\pi\int_0^2 \left(\dfrac{2x}{1+x^2}\right)^2 dx$

$=4\pi\int_0^2 \dfrac{x^2}{\left(1+x^2\right)^2} dx$

$=4\pi\int_0^{\tan^{-1}2} \dfrac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\, d\theta$

$=4\pi\int_0^{\tan^{-1}2} \dfrac{\tan^2\theta}{\sec^2\theta}\, d\theta$

$=4\pi\int_0^{\tan^{-1}2} \sin^2\theta\, d\theta$

$=4\pi\int_0^{\tan^{-1}2} \dfrac{1-\cos 2\theta}{2}\, d\theta$

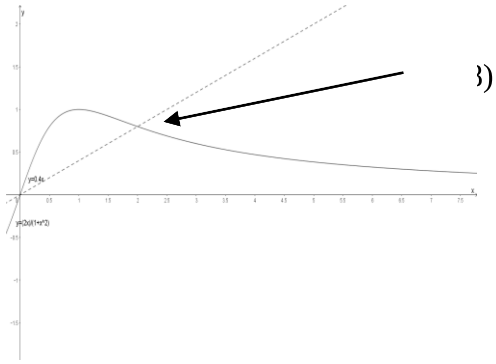
$=2\pi\left[\theta-\dfrac{\sin 2\theta}{2}\right]_0^{\tan^{-1}2}$

$=2\pi\left[\theta-\sin\theta\cos\theta\right]_0^{\tan^{-1}2}$

$=2\pi\left[\tan^{-1}2-\tfrac{2}{\sqrt{5}}\tfrac{1}{\sqrt{5}}\right]$

$=2\pi\left[\tan^{-1}2-\tfrac{2}{5}\right]$ units^3

</td><td></td></tr>
</table>

	 Volume of plastic $= \frac{1}{3}\pi\left(\frac{4}{5}\right)^2(2) = \frac{32}{75}\pi \text{ units}^3$ Cost of one paperweight $= \left(2\pi\left[\tan^{-1}2 - \frac{2}{5}\right] - \frac{32}{75}\pi\right)(3) + \left(\frac{32}{75}\pi\right)(1.2)$ $= \$11$ (to the nearest dollar)	

Question 6 [6]		
(i)	RRR BB W Y G Step 1 : Arrange BB W Y G in $\frac{5!}{2!}$ ways Step 2: Slot the RRR into the 6 appropriate slots _ to the left/ right of the arranged letters in Step:1 generally denoted by X _X_X_X_X_X_ No. of ways $=\frac{5!}{2!}\times\binom{6}{3}=1200$ Note: Number of ways in which RRR are separated $\neq$ Total number of ways $-$ All R separated ----(*) Why is this so?	
(ii)	No. of ways for the letters to form a circle = $\frac{5!}{5}$ Since clockwise and anti-clockwise are indistinguishable in a ring, No. of ways = $\frac{5!}{5}\div 2=12$ Note: For a physical 3-dimensional ring made up of beads, what happen when you flip it the other side?	
(iii)	<u>Method 1</u> III N TT A E 1. There is only 1 way to arrange A, E and N in alphabetical order. 2. Without restriction there are $(3!)$ ways to arrange A, E and N. Without restriction, total number of ways to arrange all the letters $=\frac{8!}{3!2!}$ ---- (**) To get number of ways to arrange all letters and A, E and N in alphabetical order, REMOVE all the UNWANTED arrangements in (**), using the ideas in (1) and (2).	

	No. of ways to arrange all letters and A, E and N in alphabetical order $=\left[\frac{8!}{3!2!}\right]\div(3!)=\frac{8!}{3!3!2!}$ (Note: A, E and N need not be together.) Required prob = $\frac{8!}{3!3!2!}\Big/\frac{8!}{3!2!}=\frac{1}{3!}=\frac{1}{6}$ Method 2 Step 1: Other than A, E, N, the other letters are III TT, number of ways to arrange III TT $=\frac{5!}{3!2!}$ Step 2: After arranging III TT, we create slots _ to the left / right of each letter. _X_X_X_X_X_ Step3: We now slot in A, E, N into the slots so that A, E, N are in alphabetical order. We could group A, E, N as - 1 item: [AEN] - 2 items: [AE], [N] - 2 items: [A], [NE] - 3 items: [A], [E], [N] For example to slot 2 items: [AE], [N] into _X_X_X_X_X_, Number of ways $=\frac{5!}{3!2!}\times\binom{6}{2}$ Number of ways to arrange all letters and A, E and N in alphabetical order $=\frac{5!}{3!2!}\times\left[\binom{6}{1}+\binom{6}{2}+\binom{6}{2}+\binom{6}{3}\right]$	

7	Solution [6] DRV	
	(i) $\text{P}(X=3)$ $=\dfrac{6-n}{6}\cdot\dfrac{5-n}{5}\cdot\dfrac{n}{4}$ $=\dfrac{(6-n)(5-n)n}{120}$	
	(ii) When $n=3$, $\text{P}(W=1)$ $=\text{P}(X\geq 4)$ = P(keys from first 3 trials can't unlock box) $=\left(\dfrac{3}{6}\right)\left(\dfrac{2}{5}\right)\left(\dfrac{1}{4}\right)=\dfrac{1}{20}=0.05$ *w* | 2 | 1 P(W = *w*) | 0.95 | 0.05 $\text{E}(W)=2(0.95)+(0.05)=\$1.95$ Alternatively $\text{P}(W=2)$ $=\text{P}(X=1)+\text{P}(X=2)+\text{P}(X=3)$ $=\left(\dfrac{3}{6}\right)+\left(\dfrac{3}{6}\right)\left(\dfrac{3}{5}\right)+\left(\dfrac{3}{6}\right)\left(\dfrac{2}{5}\right)\left(\dfrac{3}{4}\right)$ $=\dfrac{19}{20}=0.95$	
	(iii) Let T denote the r.v the number of times that Alfred wins \$2 from a game, out of 5 games played. $T\sim\text{B}(5,0.95)$ P(Alfred wins more at least \$9 out of 5 games) $=\text{P}(T\geq 4)$ $=1-\text{P}(T\leq 3)$ = 0.977	

<table>
<tr><td colspan="3">Question 8 [7] Probability</td></tr>
<tr><td>(i)</td><td>P(Player draws 3 different numbers in the first round)
$=\dfrac{\binom{5}{1}\binom{2}{1}}{\binom{8}{3}}=\dfrac{5}{28}$</td><td></td></tr>
<tr><td>(ii)</td><td>Probability of sum of numbers from first round is 12 and sum of numbers from second round is 6)
$=\dfrac{\binom{5}{1}\binom{2}{2}}{\binom{8}{3}}\dfrac{\binom{5}{3}}{\binom{8}{3}}$
$=\dfrac{25}{1568}$

P(Sum of the numbers drawn adds up to 18)
= P(Sum of each of first and second round is 9)
+ P(Sum of first round is 12 and second round is 6)
$=\dfrac{\binom{5}{2}\binom{2}{1}}{\binom{8}{3}}\dfrac{\binom{5}{2}}{\binom{8}{3}}+\dfrac{25}{1568}$
$=\dfrac{125}{1568}$</td><td></td></tr>
<tr><td></td><td>Let X be the sum of the 2 numbers drawn.

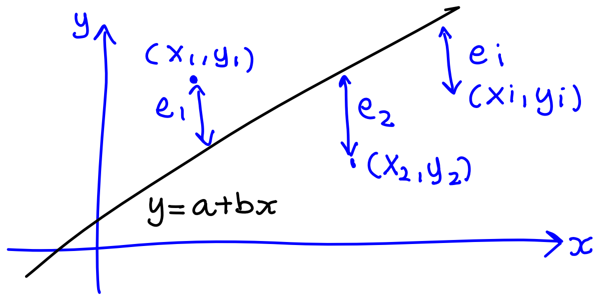
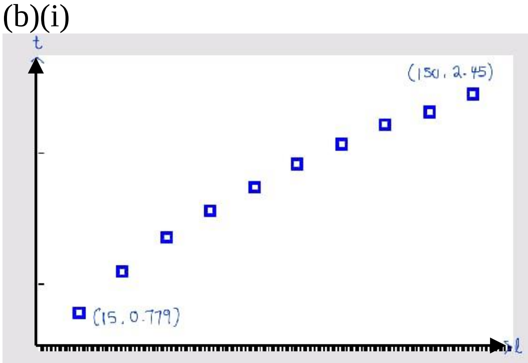
x	4	7	12	15	20
$P(X = x)$	$\dfrac{\binom{5}{2}}{\binom{8}{2}}=\dfrac{10}{28}$	$\dfrac{\binom{5}{1}}{\binom{8}{2}}=\dfrac{5}{28}$	$\dfrac{\binom{5}{1}\binom{2}{1}}{\binom{8}{2}}=\dfrac{10}{28}$	$\dfrac{\binom{2}{1}}{\binom{8}{2}}=\dfrac{2}{28}$	$\dfrac{\binom{2}{2}}{\binom{8}{2}}=\dfrac{1}{28}$

E(X)
$=4\left(\dfrac{10}{28}\right)+7\left(\dfrac{5}{28}\right)+12\left(\dfrac{10}{28}\right)+15\left(\dfrac{2}{28}\right)+20\left(\dfrac{1}{28}\right)$
$=8.75$</td><td></td></tr>
</table>

Question 9 [8] Binomial Dist		
(i)	Assumption: The candidature has no knowledge of which option is the correct answer and chooses one of the 5 options **<u>randomly/by guessing</u>**. This assumption is important to ensure that the probability of choosing a particular option is constant at 0.2.	
(ii)	Let *X* denotes the number of marks a candidate scores for the test, out of 12. X ~ B(12, 0.2) P(X=1) = 0.206 P(X=2) = 0.283 P(X=3) = 0.236 Mode of X is 2. Majority of the candidates will score 2 marks.	
(iii)	$P\left(X > 4 \mid X \leq 5\right)$ $= \dfrac{P(X = 5)}{P(X \leq 5)}$ $= 0.0542 \ \ (3\,\text{s.f})$	
(iv)	Assuming n is large, by CLT, $\overline{X} \sim N\left(2.4, \dfrac{1.92}{n}\right)$ approx. $P\left(\overline{X} \leq 2.7\right) \geq 0.95$ $n = 57, P\left(\overline{X} \leq 2.7\right) = 0.94893 < 0.95$ $n = 58, P\left(\overline{X} \leq 2.7\right) = 0.95041 > 0.95$ Thus, least value of n is 58.	

Question 10 [9] Hypo Test		
(i)	$\bar{x} = -\frac{27}{70} + 46 = 45.61428 = 45.6$ (to 3 s.f.) $s^2 = \frac{1}{69}\left(30939 - \frac{(-27)^2}{70}\right) = 448.2403727 = 448$ (to 3 s.f.)	
(ii)	Let μ_0 be the mean number of hours the engineer claimed. Test $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ at 5% significance level Test statistic: Under H_0, $Z = \frac{\bar{X} - \mu_0}{s / \sqrt{70}} \sim N(0,1)$ To reject H_0, $\frac{\bar{x} - \mu_0}{s / \sqrt{70}} > 1.644853632$ $\mu_0 < \bar{x} - 1.644853632 \frac{s}{\sqrt{70}}$ $\mu_0 < 41.45197671$ $\mu_0 < 41.5$ (to 3 s.f.) Thus, the greatest mean number of hours the engineers should claim is 41.5.	
(iii)	Based on n = 70, $\sum x = \left(-\frac{27}{70} + 46\right) \times 70 = 3193$ Based on n = 75, $\bar{x} = \frac{3193 + 56 + 34 + 63 + 50 + 54}{75} = 46$ $\sum (x - 46)^2$ $= \sum (x - \bar{x})^2$ $= 30939 + (56-46)^2 + (34-46)^2 + (63-46)^2$ $+ (50-46)^2 + (54-46)^2$ $= 31552$ $\therefore s^2 = \frac{\sum (x - \bar{x})^2}{74} = 426.3783783784 = 426.378$ (3.s.f)	

	Test $H_0 : \mu = 41.5$ against $H_1 : \mu > 41.5$ at α% significance level Test statistic: Under H_0, $Z = \dfrac{\overline{X} - 41.5}{s\ / \sqrt{75}} \sim N(0,1)$ By GC, p-value = 0.029559 Do not reject H_0, $p - value \geq \dfrac{\alpha}{100}$ $\Rightarrow \dfrac{\alpha}{100} \leq 0.029559$ $\Rightarrow \alpha \leq 2.9559$ The greatest integer value of α is 2. [If $\mu_0 = 41.4$, p-value = 0.026649, then the greatest integer value of α is 2. If $\mu_0 = 41.45197671$, p-value = 0.028230374, then the greatest integer value of α is 2.]	

11	Solution [12] Correlation & Regression	
	(a) Let the sample of bivariate data be (x_i, y_i) where $i = 1, 2, \ldots n$. Let $e_i = y_i - (a + bx_i)$ be the vertical deviation between the point (x_i, y_i) and the line $y = a + bx$.  The line $y = a + bx$ is the least square regression line for the sample of bivariate data if the sum of the squares of the vertical deviation $\sum_{i=1}^{n}(e_i)^2$ is the minimum.	
	(b)(i)  	
	(b)(ii) Model (B). As l increases, t increases at a decreasing rate, therefore model (B) is most appropriate. Note that for Model (A), the graph of $t = a + bl^2$ has a turning point located on the vertical axis, therefore Model A is not suitable.	
	(b)(iii) $t = a + b\ln l$ Using GC, $a = -1.370621901 \approx -1.37$ (3.s.f) $b = 0.7394875471 \approx 0.740$ (3 s.f) $r = 0.9871042012 \approx 0.987$ (3 s.f)	

	(b)(iv) $t=-1.370621901++0.7394875471\ln l$ ---- (1) To estimate *l* when t = 1.00, use the regression line of t on ln *l*, since *t* is the dependent variable. Sub $t=1.00$ into (1), $l=24.6743\approx 24.7$ mm. The appropriate regression line is used since *t* is the dependent variable. Since $\|r\|\approx 0.987$ is close to 1, the model based on $t=a+b\ln l$, is a good fit for the data. $t=1.00$ is within the input data range $0.779\le t\le 2.45$, the estimate is an interpolation. Therefore, the estimate is reliable.	
	(b)(v) 1 millimeter = 0.03937 inch 1 inch = $\dfrac{1}{0.03937}$ millimeters *L* inches = $\dfrac{L}{0.03937}$ millimeters $\left(L=0.03937l\right)$ $t=a+b\ln\dfrac{L}{0.03937}$ $t=a+b\left(\ln L-\ln 0.03937\right)$ $t=\left(a-b\ln 0.03937\right)+b\ln L$ $t=1.02143631+0.7394875471\ln L$ $t=1.02+0.739\ln L$ (3 s.f)	

12	Solution [12 marks] Normal Dist	
(i)	Let A be the mass of a randomly chosen apple. Suppose $A \sim N\left(70, 40^2\right)$ Then $\mathrm{P}\left(A<0\right)=0.0401\,(3\text{ s.f.})$, but it's impossible for apples to have negative mass, so the probability should be much closer to 0. Hence, the normal distribution is not a suitable distribution for the masses of apples.	
(ii)	Let Y be the mass of a pear. Let μ be the mean mass of the pears, and σ^2 be the variance in the mass of the pears. $Y \sim N\left(\mu, \sigma^2\right)$ Given, $\mathrm{P}\left(Y<148\right)=\mathrm{P}\left(Y>230\right)=\frac{1}{3}$ By symmetry, $\mu=\frac{148+230}{2}=189$ Let $Z=\frac{Y-189}{\sigma} \sim N\left(0,1\right)$ $\mathrm{P}\left(Y<148\right)=\frac{1}{3}$ $\mathrm{P}\left(Z<\frac{-41}{\sigma}\right)=\frac{1}{3}$ From GC, $\frac{-41}{\sigma}=-0.430727$ $\sigma=95.18783606$ $\sigma=95.2\text{ g}\quad(3\text{ s.f.})$	
(iii)	Let T_A be the total mass of the 300 apples. Then $T_A=A_1+A_2+...+A_{300}$ and $T_A \sim \mathrm{N}\left(300\times70, 300\times40^2\right)$ approx. by the Central Limit Theorem (CLT) as n = 300 is large.	
(iv)	Similarly, by letting T_P be the total mass of the 400 pears, we have $T_P \sim N\left(400\times189, 400\times9060.724134\right)$	

	So, we have $T_A \sim \mathrm{N}\left(21000, 480000\right)$ and $T_P \sim N\left(75600, 3624289.654\right)$ Thus, we have $R = 0.005T_A + 0.008T_P$ and $R \sim \mathrm{N}(0.005\times 21000+0.008\times 75600,$ $0.005^2 \times 480000 + 0.008^2 \times 3624289.654)$ or $R \sim \mathrm{N}(709.8, 243.9545379)$ Since c is the running cost of the orchard, $\mathrm{P}(R > c) = 0.9$ From GC, c = 689.7833896 Therefore, the cost of running the orchard is \$689.78 (2 d.p) Note: In accounting terms, the amount of money collected from sales in a business, is referred to as the revenue.	

ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS **HIGHER 2** **9758/01**

Wednesday **28 August 2019** **3 hrs**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:____(_____) **C.G.:** __________

TUTOR'S NAME: ____

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: ______________________

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

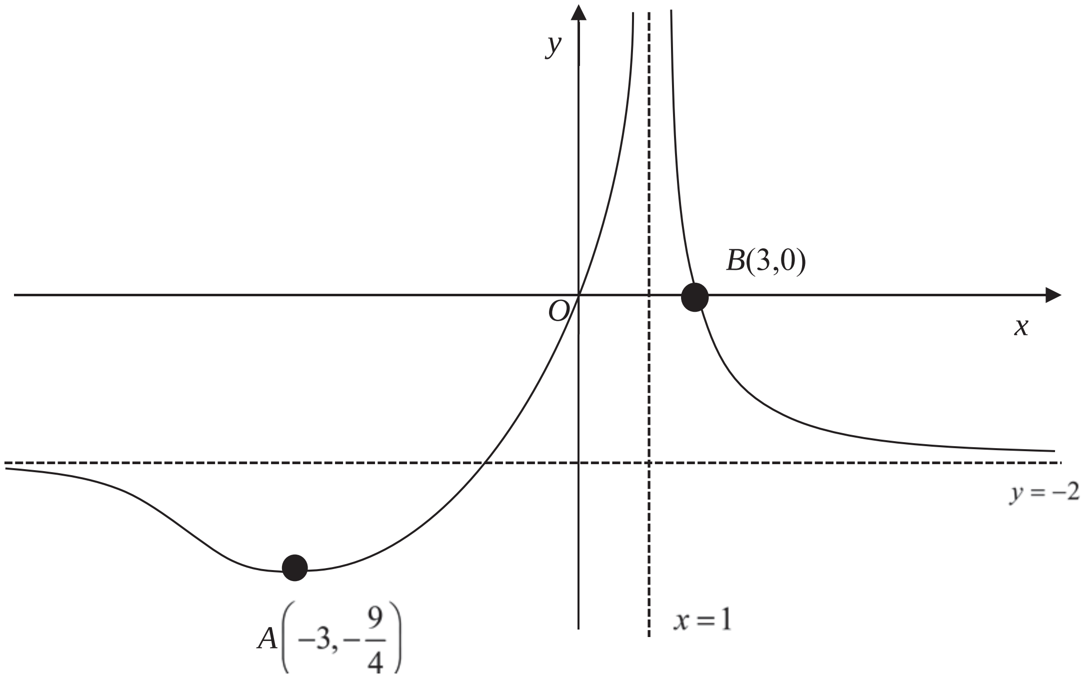
The number of marks is given in brackets [ ] at the end of each question or part question.

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**TOTAL**
Marks											
	6	**7**	**10**	**9**	**10**	**10**	**12**	**12**	**12**	**12**	**100**

This document consists of **27** printed pages and **1** blank page including this page.

[Turn Over]

1 The diagram below shows the graph of $y = 2\mathrm{f}(3-x)$. The graph passes through the origin O, and two other points $A\left(-3,-\frac{9}{4}\right)$ and $B(3,0)$. The equations of the vertical and horizontal asymptotes are $x=1$ and $y=-2$ respectively.



(a) State the range of values of k such that the equation $\mathrm{f}(3-x)=k$ has exactly two negative roots. **[1]**

(b) By stating a sequence of two transformations which transforms the graph of $y=2\mathrm{f}(3-x)$ onto $y=\mathrm{f}(3+x)$, find the coordinates of the minimum point on the graph of $y=\mathrm{f}(3+x)$. Also, write down the equations of the vertical asymptote(s) and horizontal asymptote(s) of $y=\mathrm{f}(3+x)$. **[5]**

2 **(i)** On the same axes, sketch the curves with equations $y=\left|2x^2+6x+4\right|$ and $y=3-4x$, indicating any intercepts with the axes and points of intersection. Hence solve the inequality $3-4x<\left|2x^2+6x+4\right|$. **[4]**

(ii) Find the exact area bounded by the graphs of $y=3-4x$, $y=\left|2x^2+6x+4\right|$, $x=-3$ and $x=-1$. **[3]**

[Turn Over

3 The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \frac{x-4}{x-1}\ ,\ x \in \mathbb{R}, x \neq 1$$

$$\mathrm{g}: x \mapsto x^2+2x+2\ ,\ x \in \mathbb{R}, x > -1$$

(i) Show that f has an inverse. **[1]**

(ii) Show that $\mathrm{f} = \mathrm{f}^{-1}$ and hence evaluate $\mathrm{f}^{101}(101)$. **[5]**

(iii) Prove that the composite function fg exists and find its range. **[4]**

4 It is given that $y = \sqrt{\mathrm{e}^{\cos x}}$.

(i) Show that $2\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0$. Hence find the Maclaurin's expansion of y up to and including the term in x^2. **[4]**

Deduce the series expansion for $\mathrm{e}^{\sin^2\left(\frac{x}{2}\right)}$ up to and including the term in x^2. **[3]**

(ii) Using the series expansion from **(ii)**, estimate the value of $\int_0^{\sqrt{2}} \mathrm{e}^{\sin^2\left(\frac{x}{2}\right)}\mathrm{d}x$ correct to 3 decimal places. **[2]**

5 A curve C is determined by the parametric equations

$$x = at^2, \quad y = 2at, \quad \text{where } a > 0.$$

(i) Sketch C. **[1]**

(ii) Find the equation of the normal at a point P, with non-zero parameter p. **[2]**

Show that the normal at the point P meets C again at another point Q, with parameter q, where $q = -p - \frac{2}{p}$. Hence show that $|PQ|^2 = \frac{16a^2}{p^4}(p^2+1)^3$. **[4]**

(iii) Another point R on C with parameter r, is the point of intersection of C and the circle with diameter PQ. By considering the gradients of PR and QR, show that

$$p^2 - r^2 + 2\left(\frac{r}{p}\right) = 2.$$ **[3]**

[Turn Over

6 **(a)** **(i)** Express $p=-1-\sqrt{3}\mathrm{i}$ in exponential form. **[1]**

(ii) Without the use of a calculator, find the two smallest positive whole number values of n for which $\dfrac{(p^*)^n}{\mathrm{i}p}$ is a purely imaginary number. **[4]**

(b) Without the use of a calculator, solve the simultaneous equations

$$z-w+6+7\mathrm{i}=0 \text{ and } 2w-\mathrm{i}z^*-19-3\mathrm{i}=0,$$

giving z and w in the form $x+y\mathrm{i}$ where x and y are real. **[5]**

7 The position vectors, relative to an origin O, at time t in seconds, of the particles P and Q are $(\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + 0\mathbf{k}$ and $\left(\dfrac{3}{2}\cos\left(t+\dfrac{\pi}{4}\right)\right)\mathbf{i} + \left(3\sin\left(t+\dfrac{\pi}{4}\right)\right)\mathbf{j} +$ $\left(\dfrac{3\sqrt{3}}{2}\cos\left(t+\dfrac{\pi}{4}\right)\right)\mathbf{k}$ respectively, where $0 \le t \le 2\pi$.

(i) Find $|\overrightarrow{OP}|$ and $|\overrightarrow{OQ}|$. **[2]**

(ii) Find the cartesian equation of the path traced by the point P relative to the origin O and hence give a geometrical description of the motion of P. **[2]**

(iii) Let θ be the angle POQ at time t. By using scalar product, show that

$$\cos\theta = \frac{3\sqrt{2}}{8} - \frac{1}{4}\cos\left(2t+\frac{\pi}{4}\right).$$ **[3]**

(iv) Given that the length of projection of $\overrightarrow{OQ}$ onto $\overrightarrow{OP}$ is $\sqrt{5}$ units, find the acute angle θ and the corresponding values of time t. **[5]**

[Turn Over

8 **(a)** Meredith owns a set of screwdrivers numbered 1 to 17 in decreasing lengths. The lengths of the screwdrivers form a geometric progression. It is given that the total length of the longest 3 screwdrivers is equal to three times the total length of the 5 shortest screwdrivers. It is also given that the total length of all the odd-numbered screwdrivers is 120 cm. Find the total length of all the screwdrivers, giving your answer correct to 2 decimal places. **[4]**

(b) Meredith is building a DIY workbench, and she needs to secure several screws by twisting them with a screwdriver drill. Each time Meredith presses the button on the drill, the screw is rotated clockwise by u_n radians, where n is the number of times the button is pressed. Each press rotates the screw more than the previous twist, and on the first press, the screw is rotated by $\frac{2\pi}{3}$ radians. It is given that $\cos u_{n+1} = \frac{1}{2}\cos u_n - \frac{\sqrt{3}}{2}\sin u_n$ and $\sin u_{n+1} = \frac{1}{2}\sin u_n + \frac{\sqrt{3}}{2}\cos u_n$ for all $n \geq 1$.

(i) By considering $\cos\left(u_{n+1} - u_n\right)$ or otherwise, and assuming that the increase in rotation in successive twists is less than π radians, prove that $\{u_n\}$ is an arithmetic progression with common difference $\frac{\pi}{3}$ radians. **[3]**

(ii) Each screw requires at least 25 complete revolutions to ensure that it does not fall out. Find the minimum number of times Meredith has to press the drill button to ensure the screw is fixed in place. **[3]**

(iii) The distance the screw is driven into the workbench on the nth press of the drill, d_n, is proportional to the angle of rotation u_n. If the total distance the screw is driven into the workbench after 21 presses is 144mm, find the distance the screw is driven into the workbench on the first press. **[2]**

9 **(i)** By using the substitution $x = 15\sin\theta + 15$, find the $\int_0^{15} \sqrt{15^2 - (x-15)^2}\,dx$ leaving your answer in terms of π . **[5]**

(ii) A sculptor decides to make a stool by carving from a cylindrical block of base radius

30 cm and height 35 cm using a 3D carving machine. The design of the stool based on the piecewise function $g(x)$ where

$$g(x) = \begin{cases} 30 - \frac{2}{3}\sqrt{15^2 - (x-15)^2} & \text{for } 0 \le x \le 15 \\ 30 & \text{for } 15 < x < 35. \end{cases}$$

The figure below shows the 3D image of the stool after the design ran through a 3D machine simulator.

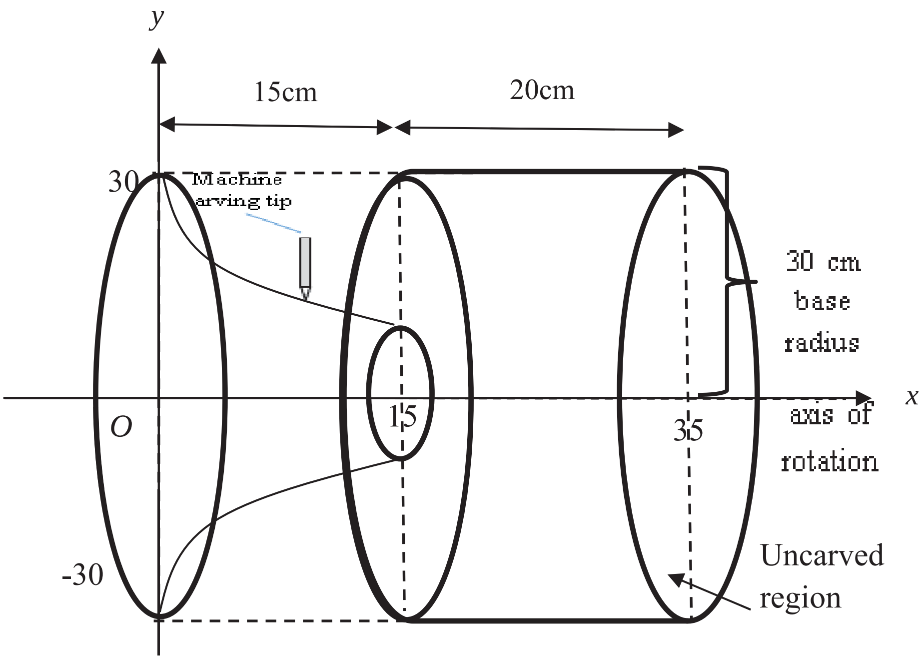


Figure 1: 3D Image of the stool

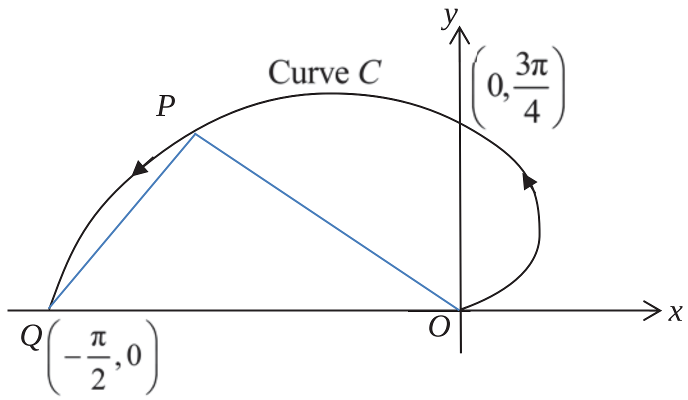
(a) Find the exact area bounded by the curve $y = g(x)$, $x = 15$ and the x-axis and y-axis. **[3]**

(b) The curve defined by the function $y = g(x)$ when rotated 2π radians about the x–axis gives the shape of the stool that the sculptor desires, as shown in Figure 1.
Find the exact volume of the stool. **[4]**

[Turn Over

10 The diagram below shows a curve C with parametric equations given by

$$x = \theta\cos 2\theta, \quad y = 3\theta\sin 2\theta, \qquad \text{for } 0 \le \theta \le \frac{\pi}{2}.$$



The area bounded by curve C and the x-axis is a plot of land which is owned by a farmer Mr Green where he used to grow vegetables. Over the past weeks, vegetables were mysteriously missing and Mr Green decided to install an automated moving surveillance camera which moves along the boundary of the farmland in an anticlockwise direction along the curve C starting from point O and ending at point Q before moving in a clockwise direction along the curve C back to O.

At a particular instant t seconds, the camera is located at a point P with parameter θ on the curve C. You may assume that the camera is at O initially. The camera should be orientated so that the field of view should span from O to Q exactly as shown.

(i) Assuming that the camera is moving at a speed given by $\frac{d\theta}{dt} = 0.01$ radians/sec, find the rate of change of the area of the triangle OPQ, A when $\theta = \frac{\pi}{6}$. **[4]**

(ii) Using differentiation, find the value of θ that would maximize A and explain why A is a maximum for that value of θ. Hence find this value of A and the coordinates of the point P corresponding to the location of the camera at that instant. **[5]**

(iii) For the image to be 'balanced', triangle OPQ is isosceles. Find the coordinates of the location where the camera should be. **[3]**

End of Paper

ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAM

MATHEMATICS **HIGHER 2** **9758/02**

Monday **16 September 2019** **3 hr**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:____(_____) **C.G.:** __________

TUTOR'S NAME: ____

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: _______________________

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Q	1	2	3	4	5	6	7	8	9	10	11	TOTAL
M												
	9	11	12	8	6	6	8	8	8	12	12	100

This document consists of **26** printed pages including this page.

[Turn Over]

Section A: Pure Mathematics [40 marks]

1 The curve C has equation $y = \dfrac{x^2 + ax + b}{x + c}$ where a, b and c are constants. The line $x = -1$ is an asymptote to C and the range of values that y can take is given by $y \le 0$ or $y \ge 4$.

(i) State the value of c and show that $a = 4$ and $b = 4$. **[4]**

(ii) Sketch C indicating clearly the equations of the asymptotes and coordinates of the turning points and axial intercepts. **[3]**

(iii) State the coordinates of the point of intersection of the asymptotes. Hence state the range of values of k such that the line $y = k(x+1)+2$ cuts C at two distinct points. **[2]**

2 A water tank contains 1 cubic meter of water initially. The volume of water in the tank at time t seconds is V cubic metres. Water flows out of the tank at a rate proportional to the volume of water in the tank and at the same time, water is added to the tank at a constant rate of k cubic metres per second.

(i) Show that $\dfrac{dV}{dt} = k\left(1 - \dfrac{a}{k}V\right)$, where a is a positive constant. **[2]**

Hence find V in terms of t. **[5]**

(ii) Sketch the solution curve for V against t, such that

(a) $a < k$.

(b) $a > k$.

For cases **(a)** & **(b)**, describe and explain what would happen to the volume of water, V in the tank eventually. **[4]**

3 The plane π_1 has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 10$, and the coordinates of A and B are (2, a, 2), (1, 0, 3) respectively, where a is a constant.

(i) Verify that B lies on π_1. **[1]**

(ii) Given that A does not lie on π_1, state the possible range of values for a. **[1]**

(iii) Given that $a = 9$, find the coordinates of the foot of the perpendicular from A to π_1. Hence, or otherwise, find the vector equation of the line of reflection of the line AB in π_1. **[5]**

The plane π_2 has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 4$.

(iv) Find the acute angle between π_1 and π_2. **[2]**

(v) Find the cartesian equations of the planes such that the perpendicular distance from each plane to π_2 is $\frac{5\sqrt{2}}{2}$. **[3]**

[Turn Over]

4 **(a)** Given that $f(r) = \dfrac{r}{2^r}$, by considering f($r+1$) – f(r), find $\displaystyle\sum_{r=1}^{n} \frac{1-r}{2^{r+1}}$. **[3]**

(b) **(i)** Cauchy's root test states that a series of the form $\displaystyle\sum_{r=0}^{\infty} a_r$ (where $a_r > 0$ for all r) converges when $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} < 1$, and diverges when $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} > 1$. When $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, the test is inconclusive. Using the test and given that $\displaystyle\lim_{n\to\infty} \sqrt[n]{n^p} = 1$ for all positive p, explain why the series $\displaystyle\sum_{r=0}^{\infty} \frac{2^r r^x}{3^r}$ converges for all positive values of x. **[3]**

(ii) By considering $(1-y)^{-2} = 1 + 2y + 3y^2 + 4y^3 + \ldots$, evaluate $\displaystyle\sum_{r=0}^{\infty} \frac{2^r r^x}{3^r}$ for the case when $x = 1$. **[2]**

Section B: Probability and Statistics [60 marks]

5 Seng Ann Joo Cooperative sells granulated sugar in packets. These packets come in two sizes: standard and large. The masses, in grams, of these packets are normally distributed with mean and standard deviation as shown in the table below.

	Mean	Standard Deviation
Standard	520	8
Large	1030	11

(i) Find the probability that two standard packets weigh more than a large packet. **[3]**

(ii) Find the probability that the mean mass of two standard packets and one large packet of sugar is between 680g and 700g. **[3]**

6 A university drama club contains 3 Biology students, 4 History students, and 6 Literature students. 5 students are to be selected as the cast of an upcoming production.

(i) In how many ways can the 5 cast members be selected so that there are at most 2 Biology students? **[2]**

(ii) Find the probability that, amongst the cast members, the number of History students exceeds the number of Literature students, given that there are at most 2 Biology students. **[4]**

[Turn Over]

7 **(i)** The discrete random variable X takes values 1, 2, 3,… , n, where n is a positive integer greater than 1, with equal probabilities.

Find, in terms of n, the mean μ, and the variance, σ^2, of X. **[4]**

[You may use the result $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.]

Let $n = 6$. An observation of X is defined as an *outlier* if $|X - \mu| > \sigma$.

(ii) 20 observations of X are made. Find the probability that there are at least 8 observations that are outliers. **[4]**

8 In a game of chance, a player has to draw a counter from a bag containing n red counters and $(40-n)$ blue counters before throwing a fair die. If a red counter is drawn, she throws a six-sided die, with faces labelled 1 to 6. If a blue counter is drawn, she throws a ten-sided die, with faces labelled 1 to 10. She wins the game if the uppermost face of the die thrown shows a number that is a perfect square.

(i) Given that $n = 15$, find the exact probability that a player wins the game. Hence, find the probability that, when 3 people play this game, exactly 2 won. **[3]**

(ii) For a general value of n, the probability that a winning player drew a blue counter is denoted by $\mathrm{f}(n)$. Show that $\mathrm{f}(n) = a + \frac{b}{360+n}$, where a and b are constants to be determined. Without further working, explain why f is a decreasing function for $0 \le n \le 40$, and interpret what this statement means in the context of the question. **[5]**

9 Many different interest groups, such as the lumber industry, ecologists, and foresters, benefit from being able to predict the volume of a tree from its diameter. The following table of 10 shortleaf pines is part of the data set concerning the diameter of a tree, x, in inches and volume of a tree y, in cubic feet.

Diameter (x inches)	5.0	5.6	7.5	9.1	9.9	10.3	11.5	12.5	16.0	18.3
Volume (y cubic feet)	3.0	7.2	10.3	17.0	23.1	27.4	26.0	41.3	65.9	97.9

(Bruce and Schumacher, 1935)

(i) Draw a scatter diagram to illustrate the data, labelling the axes clearly. **[2]**

It is thought that the volume of trees with different diameters can be modelled by one of the formulae

[Turn Over]

$$y = a + bx \quad \text{or} \quad \ln y = c + d \ln x$$

where a, b, c and d are constants.

(ii) Find the value of the product moment correlation coefficient between

(a) y and x,

(b) $\ln y$ and $\ln x$.

Leave your answers correct to 5 decimal places **[2]**

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of the models is the better model. **[1]**

(iv) It is required to estimate the value of y for which $x = 20$. Find the equation of a suitable regression line and use it to find the required estimate, correct to 1 decimal place. Explain whether your estimate is reliable. **[3]**

10 A factory manufactures a large number of erasers in a variety of colours. Each box of erasers contains 36 randomly chosen erasers. On average, 20% of erasers in the box are blue.

(i) State, in context, two assumptions needed for the number of blue erasers in a box to be well modelled by a binomial distribution. **[2]**

(ii) Find the probability that a randomly chosen box of erasers contain at most six blue erasers. **[1]**

200 randomly chosen boxes are packed into a carton. A carton is considered acceptable if at least 40% of the boxes contain at most six blue erasers each.

(iii) Find the probability that a randomly chosen carton is acceptable. **[3]**

The cartons are exported by sea. Over a one-year period, there are 30 shipments of 150 cartons each.

(iv) Using a suitable approximation, find the probability that the mean number of acceptable cartons per shipment for the year is less than 80. **[3]**

The owner decided to change the proportion of blue erasers to p. A box of erasers is chosen.

(v) Write down in terms of p, the probability that the box contains exactly one blue eraser. **[1]**

(vi) The probability that a box contains exactly one blue eraser is twice the probability that the box contains exactly two blue erasers. Write an equation in terms of p, and hence find the value of p. **[2]**

[Turn Over]

11 The time T seconds required for a computer to boot up, from the moment it is switched on, is a normally distributed random variable. The specifications for the computer state that the population mean time should not be more than 30 seconds. A Quality Control inspector checks the boot up time using a sample of 25 randomly chosen computers.

A particular sample yielded $\sum t = 802.5$ and $\sum t^2 = 26360.25$.

(i) Calculate the unbiased estimates of the population mean and variance. **[2]**

(ii) What do you understand by the term "unbiased estimate"? **[1]**

(iii) Test, at the 5% level of significance level, whether the specification is being met. Explain in the context of the question, the meaning of "5% level of significance". **[5]**

(iv) Find the range of values of $\overline{t}$ such that the specification will be met in the test carried out in part **(iii)**. **[1]**

(v) A new Quality Control policy is that when the specification is not met, all the computers will be sent back to the manufacturer for upgrading. The inspector tested a second random sample of 25 computers, and the boot up time, y seconds, of each computer is measured, with $\overline{y} = 32.4$. Using a hypothesis test at the 5% level of significance, find the range of values of the population standard deviation such that the computers will not be sent back for upgrading. **[3]**

End of Paper

ANNEX

H2 MA 2019 JC2 Prelim (Paper 1 and Paper 2)

Paper 1

Select topic from the dropdown list. If the question consists of multiple topics, choose 1 topic.

QN	TOPIC (H2) Paper 1	ANSWERS (Exclude graphs and text answers)
1	Graphs & Transformations	(a) $-\frac{9}{8}<k<-1$ (b) $\left(3,-\frac{9}{8}\right)$, $x=-1$, $y=-1$
2	Integration & Applications	(ii) 20
3	Functions	(ii) $\mathrm{f}^{-1}(x)=1-\frac{3}{x-1}=\frac{x-4}{x-1}$, $\frac{97}{100}$ (iii) $(-\infty,1)$
4	Maclaurin & Binomial Series	(i) $y=\sqrt{\mathrm{e}}-\frac{\sqrt{\mathrm{e}}}{4}x^2+\ldots$ (ii) $1+\frac{1}{4}x^2+\ldots$ (iii) 1.650
5	Integration & Applications	(ii) $y=2ap-p(x-ap^2)$ (iii) $\frac{16a^2}{p^4}(p^2+1)^3$
6	Complex Numbers	(a)(i) $2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ (ii) $n=2$ and $n=5$ (b) $w=7+2\mathrm{i}$ and $z=1-5\mathrm{i}$
7	Vectors	(i) 3 (iv) 0.730, $t=0.910, 1.45, 4.05, 4.59$
8	APGP	(a) r = 0.882854, 222.38 (b)(i) $\frac{\pi}{3}$ (ii) 16 (iii) $\frac{8}{7}$ mm
9	Integration & Applications	(a)(i) $\frac{225}{4}\pi$ (a)(ii) $\left(450-\frac{75}{2}\pi\right)$ cm^2 (b) $\left(32500\pi-2250\pi^2\right)$ cm^3

10	Differentiation & Applications	(i) 0.0327 (ii) $2.14 \text{ units}^2, (-0.449, 2.73)$ (iii) $P(-0.785, 2.55)$
11	H2 Prelim P1 Q11 Topic	Nil
12	H2 Prelim P1 Q12 Topic	Nil
13	H2 Prelim P1 Q13 Topic	Nil
14	H2 Prelim P1 Q14 Topic	Nil

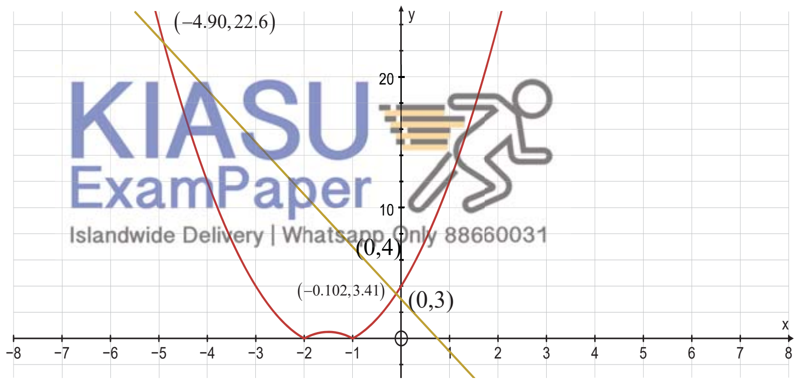
Paper 2

Select topic from the dropdown list. If the question consists of multiple topics, choose 1 topic.

QN	TOPIC (H2) Paper 2	ANSWERS (Exclude graphs and text answers)
1	Graphs & Transformations	(i) $a=4, \quad b=4$
2	Differential Equations	(i) $V=\frac{1}{a}(k-(k-a)e^{-at})$
3	Vectors	(ii) $a\in\mathbb{R}, a\neq -2.$ (iii) $l_{BA'}: \mathbf{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+\mu\begin{pmatrix}3\\7\\5\end{pmatrix}, \ \mu\in\mathbb{R}.$ (iv) 31.5° (v) $x+z=9$ or $x+z=-1$
4	Sigma Notation & MOD	(a) $\frac{n+1}{2^{n+1}}-\frac{1}{2}$ (b)(ii) 6
5	Normal Distribution	0.737, 0.943
6	PnC & Probability	(i) 1242 (ii) 0.210
7	DRV	(i) $\frac{n+1}{2}, \frac{n^2-1}{12}$ (ii) 0.339
8	PnC & Probability	(i) $\frac{5}{16}, \frac{825}{4096}$ (ii) $-9+\frac{3600}{360+n}$
9	Correlation & Regression	(ii) (a) 0.96346 (ii) (b) 0.98710 (iii) second model (iv) 121
10	Binomial Distribution	(ii) 0.401 (iii) 0.535 (iv) 0.423 (v) $36p(1-p)^{35}$

		(vi) $p=\frac{1}{36}$
11	Hypothesis Testing	(i) 32.1, 25 (iii) p-value = 0.0179 , reject H_0. (iv) $0<\overline{t}<31.6$. (v) $\sigma>7.30$
12	H2 Prelim P2 Q12 Topic	Nil
13	H2 Prelim P2 Q13 Topic	Nil
14	H2 Prelim P2 Q14 Topic	Nil

2019 H2 Math Prelim Paper 1 Solutions

Qn	Solutions
1(a)	$\mathrm{f}(3-x)=k$ Range of values of k is $-\frac{9}{8}<k<-1$.
1(b)	The graph of $y=2\mathrm{f}(3-x)$ undergoes 1) A reflection in the y-axis followed by 2) A scaling of ½ unit parallel to the y-axis $A\left(-3,-\frac{9}{4}\right)\xrightarrow{\text{(1) reflection in the } y\text{-axis}}A_1\left(3,-\frac{9}{4}\right)$ $\xrightarrow{\text{(2) scaling of a factor } \frac{1}{2} \text{ parallel to } y\text{-axis}}A_2\left(3,-\frac{9}{8}\right)$ Coordinates of the minimum point on the graph of $y=\mathrm{f}(3+x)$ are $\left(3,-\frac{9}{8}\right)$. Equation of vertical asymptote: $x=-1$ Equation of horizontal asymptote : $y=-1$
2(i)	

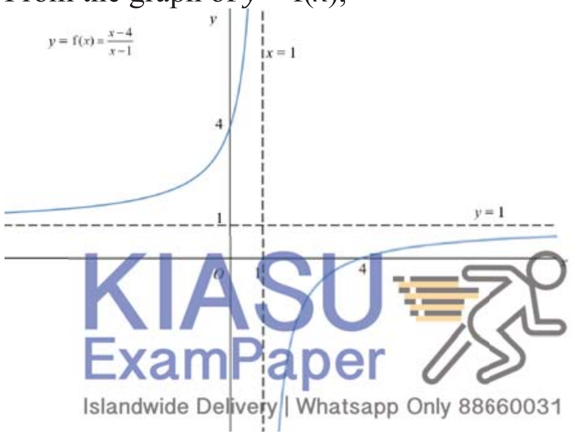
<table>
<tr><td>(ii)</td><td>

$$\int_{-3}^{-1}\left(3-4x-|2x^2+6x+4|\right)\mathrm{d}x$$
$$=\int_{-3}^{-2}\left(3-4x-2x^2-6x-4\right)\mathrm{d}x+\int_{-2}^{-1}\left(3-4x+2x^2+6x+4\right)\mathrm{d}x$$
$$=\int_{-3}^{-2}\left(-2x^2-10x-1\right)\mathrm{d}x+\int_{-2}^{-1}\left(2x^2+2x+7\right)\mathrm{d}x$$
$$=\left[-\frac{2}{3}x^3-5x^2-x\right]_{-3}^{-2}+\left[\frac{2}{3}x^3+x^2+7x\right]_{-2}^{-1}$$
$$=\left(\frac{16}{3}-20+2\right)-(18-45+3)+\left(-\frac{2}{3}+1-7\right)-\left(-\frac{16}{3}+4-14\right)$$
$$=20.$$

</td></tr>
<tr><td>3(i)</td><td>

$$\mathrm{f}(x)=\frac{x-4}{x-1}=1-\frac{3}{x-1}$$

From the graph of $y = \mathrm{f}(x)$,



Since every horizontal line $y=h, h\in\mathbb{R}, h\neq 1$ cuts the graph of f at exactly one point, f is an one-one function.

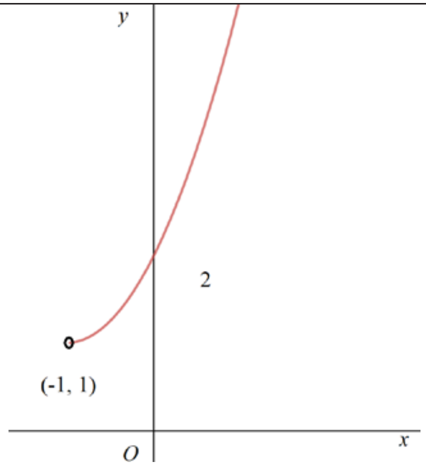
Therefore f^{-1} exists.

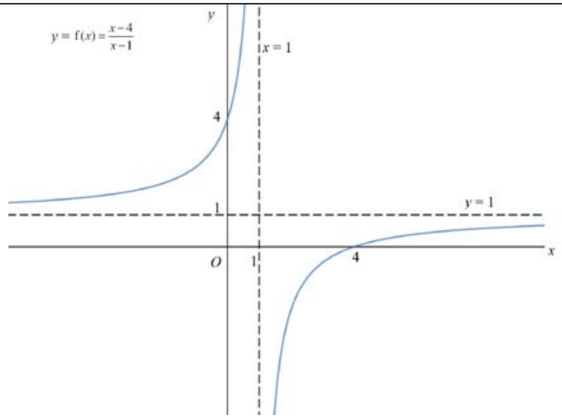
</td></tr>
<tr><td>(ii)</td><td>

Let $y=1-\frac{3}{x-1}$

</td></tr>
</table>

	$\frac{3}{x-1}=1-y$ $x-1=\frac{3}{1-y}$ $x=1+\frac{3}{1-y}$ $x=1-\frac{3}{y-1}$ Since $y=\mathrm{f}(x),\ x=\mathrm{f}^{-1}(y)$. $\therefore\ \mathrm{f}^{-1}(y)=1-\frac{3}{y-1}$ Hence $\mathrm{f}^{-1}(x)=1-\frac{3}{x-1}=\frac{x-4}{x-1},\ x\in\mathbb{R},x\neq 1$ Since the $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=(-\infty,1)\cup(1,\infty)=D_{\mathrm{f}}$ Since $\mathrm{f}^{-1}=\mathrm{f}$, $\mathrm{f}^2(x)=\mathrm{ff}^{-1}(x)=x,\ldots,\mathrm{f}^{100}(x)=x$ $\mathrm{f}^{101}(101)=\mathrm{f}\left(\mathrm{f}^{100}(101)\right)$ $=\mathrm{f}(101)$ $=\frac{101-4}{101-1}=\frac{97}{100}$
(iii)	$\mathrm{g}:x\mapsto x^2+2x+2,\ x\in\mathbb{R},x>-1$ $\mathrm{g}(x)=(x+1)^2+1$ From the graph of g,

	 Range of g = $(1, \infty)$ Domain of f = $\mathbb{R} \setminus \{1\}$ Since $(1, \infty) \subseteq \mathbb{R} \setminus \{1\}$ i.e. Range of g $\subseteq$ Domain of f Therefore the composite function fg exists. To find range of composite function fg:



$(-1,\infty) \xrightarrow{g} (1,\infty) \xrightarrow{f} (-\infty,1)$

Range of fg is $(-\infty,1)$.

4

$y=\sqrt{e^{\cos x}}$ --- (1)

$y^2 = e^{\cos x}$

Differentiate with respect to x,

$2y\frac{dy}{dx} = (-\sin x)e^{\cos x}$

$y\left(2\frac{dy}{dx} + y\sin x\right) = 0$

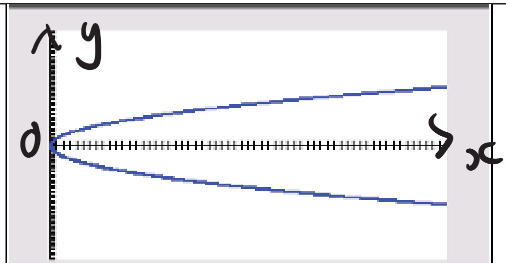
Since $y=\sqrt{e^{\cos x}} > 0$,

$2\frac{dy}{dx} + y\sin x = 0$ (Shown) --- (1)

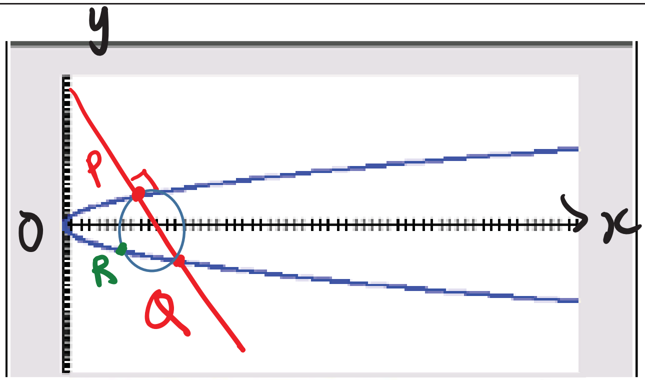
Differentiate with respect to x,

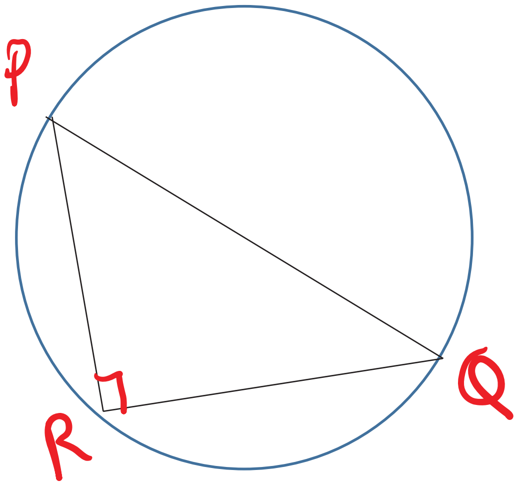
$2\frac{d^2y}{dx^2} + (\sin x)\frac{dy}{dx} + y\cos x = 0$

	When $x = 0$, $y = \sqrt{e}$ from (1) $\frac{dy}{dx} = 0$ from (2) $\frac{d^2y}{dx^2} = -\frac{\sqrt{e}}{2}$ $\therefore y = \sqrt{e} - \frac{\sqrt{e}}{4}x^2 + \ldots$
(ii)	$e^{\sin^2\left(\frac{x}{2}\right)} = e^{\frac{1-\cos x}{2}}$ $= \frac{e^{\frac{1}{2}}}{e^{\frac{1}{2}\cos x}}$ $\approx \frac{e^{\frac{1}{2}}}{e^{\frac{1}{2}} - \frac{e^{\frac{1}{2}}}{4}x^2}$ $= \frac{1}{1-\frac{1}{4}x^2}$ $= \left(1-\frac{1}{4}x^2\right)^{-1}$ $= 1+\frac{1}{4}x^2+\ldots$
(iii)	$\int_0^{\sqrt{2}} e^{\sin^2\left(\frac{x}{2}\right)} dx \approx \int_0^{\sqrt{2}} \left(1+\frac{1}{4}x^2\right) dx$ $=1.649915$ $=1.650$

5(i)	
(ii)	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{1}{\left(\frac{dx}{dt}\right)}=\frac{2a}{2at}=\frac{1}{t}$ Gradient of normal = -*t*. Equation of normal at a point *P* is given by $\frac{y-2ap}{x-ap^2}=-p$ $\Rightarrow y=2ap-p(x-ap^2)$
(iii)	If the normal at point *P* meets *C* again at point *Q*,

	$2aq = 2ap - p(aq^2 - ap^2)$ $pq^2 + 2q - (2p + p^3) = 0$ $q = \dfrac{-2 \pm \sqrt{4 + 4p(2p + p^3)}}{2p}$ $= \dfrac{-2 \pm \sqrt{4 + 8p^2 + 4p^4}}{2p}$ $= \dfrac{-2 \pm \sqrt{(2p^2 + 2)^2}}{2p}$ $= \dfrac{-2 + (2p^2 + 2)}{2p}$ or $\dfrac{-2 - (2p^2 + 2)}{2p}$ $= p$ (rejected as it is the point P) or $-p - \dfrac{2}{p}$ Therefore Q will meet C again with $q = -p - \dfrac{2}{p}$. Coordinates of $P = (ap^2, 2ap)$ Coordinates of $Q = \left(a(p + \dfrac{2}{p})^2, -2a\left(\dfrac{p^2 + 2}{p}\right)\right)$ $\|PQ\|^2$ $= (ap^2 - a((p + \dfrac{2}{p})^2)^2 + (2ap + 2a(\dfrac{p^2 + 2}{p}))^2$ $= (ap^2 - a(p^2 + 4 + \dfrac{4}{p^2}))^2 + (4ap + \dfrac{4a}{p})^2$

	$=16a^2(1+\frac{1}{p^2})^2+16a^2(\frac{p^2+1}{p})^2$ $=16a^2(\frac{(p^2+1)^2}{p^4}+(\frac{p^2+1}{p})^2)$ $=\frac{16a^2}{p^4}((p^2+1)^2+p^2(p^2+1)^2)$ $=\frac{16a^2}{p^4}(p^2+1)^3$
	



Note that PR and QR are perpendicular to each other (angle PRQ is 90^0 – angle in a semi-circle).

Gradient of PR = $\dfrac{2ap-2ar}{ap^2-ar^2}=\dfrac{2}{p+r}$

Gradient of QR = $\dfrac{-2a\left(\dfrac{p^2+2}{p}\right)-2ar}{a\left(\dfrac{p^2+2}{p}\right)^2-ar^2}=-\dfrac{2}{\left(\dfrac{p^2+2}{p}\right)-r}$

$$-\frac{2}{\left(\dfrac{p^2+2}{p}\right)-r}\bullet\frac{2}{p+r}=-1$$

	$(p+r)(p+\frac{2}{p}-r)=4$ $p^2-r^2+\frac{2r}{p}=2$. (Shown).
6(a) (i)	$p=-1-\sqrt{3}\mathrm{i}$ $=2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$
(ii)	$\frac{(p^*)^n}{\mathrm{i}p}=\frac{2^n\mathrm{e}^{\mathrm{i}\left(\frac{2n\pi}{3}\right)}}{2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}+\frac{\pi}{2}\right)}}=2^{n-1}\mathrm{e}^{\mathrm{i}\left(\frac{2n\pi}{3}+\frac{\pi}{6}\right)}$ For $\frac{(p^*)^n}{ip}$ to be purely imaginary, $\frac{2n\pi}{3}+\frac{\pi}{6}=(2k+1)\frac{\pi}{2}$ where $k\in\mathbb{Z}$ $\frac{2n\pi}{3}+\frac{\pi}{6}=k\pi+\frac{\pi}{2}$ $\frac{2n\pi}{3}=k\pi+\frac{\pi}{3}$ $2n=3k+1$ $n=\frac{3k+1}{2}$ For the two smallest $n\in\mathbb{Z}^+$, $k=1$ and $k=3$. $n=2$ and $n=5$

6 (b)	$z = w - 6 - 7\text{i}$ --- (1) $2w - \text{i}z^* - 19 - 3\text{i} = 0$--- (2) Substitute (1) into (2): $2w - \text{i}(w - 6 - 7\text{i})^* - 19 - 3\text{i} = 0$ $2w - \text{i}(w^* - 6 + 7\text{i}) - 19 - 3\text{i} = 0$ Let $w = x + \text{i}y$, $2(x + \text{i}y) - \text{i}((x - \text{i}y) - 6 + 7\text{i}) - 19 - 3\text{i} = 0$ $2(x + \text{i}y) - \text{i}((x - \text{i}y) - 6 + 7\text{i}) - 19 - 3\text{i} = 0$ $2x + 2y\text{i} - \text{i}x - y + 6\text{i} + 7 - 19 - 3\text{i} = 0$ $2x - y - 12 + \text{i}(2y - x + 3) = 0$ Compare real and imaginary components, $2x - y = 12 ---------- (1)$ $2y - x = -3 ---------- (2)$ From (2), $x = 2y + 3 -------- (3)$ Substituting (3) into (1) $2(2y + 3) - y = 12$ $3y = 6$ $y = 2$ When $y = 2$, $x = 2(2) + 3 = 7$ $w = 7 + 2\text{i}$ and $z = 7 + 2\text{i} - 6 - 7\text{i} = 1 - 5\text{i}$
7(i)	$\overrightarrow{OP} = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$ $\lvert\overrightarrow{OP}\rvert = \sqrt{\cos^2 t + \sin^2 t} = 1$

	$\vert\overrightarrow{OQ}\vert^2=\frac{9}{4}\cos^2(t+\frac{\pi}{4})+9\sin^2(t+\frac{\pi}{4})+\frac{27}{4}\cos^2(t+\frac{\pi}{4})$ $=9\cos^2(t+\frac{\pi}{4})+9\sin^2(t+\frac{\pi}{4})$ $=9$ $\Rightarrow\vert\overrightarrow{OQ}\vert=3.$
(ii)	$x=\cos t \quad y=\sin t \quad 0\le t\le 2\pi$ The cartesian equation is given by $x^2+y^2=1$. Since $z=0$, P lies on a circle centre at O and radius 1 unit in the x-y plane.
(iii)	Using scalar product, $\overrightarrow{OP}\bullet\overrightarrow{OQ}=\vert\overrightarrow{OP}\vert\vert\overrightarrow{OQ}\vert\cos\theta$ $\Rightarrow 3\cos\theta=\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix}\bullet\begin{pmatrix}\frac{3}{2}\cos\left(t+\frac{\pi}{4}\right)\\ 3\sin\left(t+\frac{\pi}{4}\right)\\ \frac{3\sqrt{3}}{2}\cos\left(t+\frac{\pi}{4}\right)\end{pmatrix}$ $=\frac{3}{2}\cos t\cos\left(t+\frac{\pi}{4}\right)+3\sin t\sin\left(t+\frac{\pi}{4}\right)$

	$\Rightarrow 3\cos\theta = \frac{3}{4}\left[\cos\left(2t+\frac{\pi}{4}\right)+\cos\frac{\pi}{4}\right]-\frac{3}{2}\left[\cos\left(2t+\frac{\pi}{4}\right)-\cos\frac{\pi}{4}\right]$ $\Rightarrow 3\cos\theta = -\frac{3}{4}\cos\left(2t+\frac{\pi}{4}\right)+\frac{3\sqrt{2}}{8}+\frac{3\sqrt{2}}{4}$ $\Rightarrow \cos\theta = \frac{3\sqrt{2}}{8}-\frac{1}{4}\cos\left(2t+\frac{\pi}{4}\right)$ (Shown).
(iv)	The length of projection of $\overrightarrow{OQ}$ onto $\overrightarrow{OP}$ is $\lvert\overrightarrow{OQ}\bullet\overrightarrow{OP}\rvert$ since $\overrightarrow{OP}$ is a unit vector. Given that the length of projection of $\overrightarrow{OQ}$ onto $\overrightarrow{OP}$ is $\sqrt{5}$ units, $\lvert\overrightarrow{OQ}\bullet\overrightarrow{OP}\rvert=\sqrt{5}$ $\Rightarrow \lvert\overrightarrow{OQ}\rvert\cos\theta=\sqrt{5}$ $\Rightarrow \cos\theta=\frac{\sqrt{5}}{3}$ $\Rightarrow \theta = 0.730$ (since θ is acute). Solving $\frac{3\sqrt{2}}{8}-\frac{1}{4}\cos\left(2t+\frac{\pi}{4}\right)=\frac{\sqrt{5}}{3}$ $\cos(2t+\frac{\pi}{4})=-0.86010$ $2t+\frac{\pi}{4}=0.53532$ (basic angle) $2t+\frac{\pi}{4}=\pi-0.53532, \pi+0.53532, \pi-0.53532+2\pi, \pi+0.53532+2\pi$ $t=0.910, 1.45, 4.05, 4.59$

8(a)	Let the length of the first screwdriver be b, and the common ratio be r. $b+br+br^2=3\left(br^{12}+br^{13}+br^{14}+br^{15}+br^{16}\right)$ $1+r+r^2-3r^{12}-3r^{13}-3r^{14}-3r^{15}-3r^{16}=0$ Using the GC, r = 0.882854 (6 s.f.) since $0<r<1$. There are 9 odd numbered screwdrivers, with common ratio of lengths being r^2. Total length = $\dfrac{b\left(1-\left(r^2\right)^9\right)}{1-r^2}=120$ $\dfrac{b\left(1-r^{18}\right)}{1-r^2}=120$ Using the GC, b = 29.6122 (6 s.f.) The total length is $\dfrac{b\left(1-r^{17}\right)}{1-r}=222.38\text{ cm}\quad(\text{to 2 d.p.})$
8 (b)(i)	$\cos\left(u_{n+1}-u_n\right)=\cos u_{n+1}\cos u_n+\sin u_{n+1}\sin u_n$ $=\dfrac{1}{2}\cos^2 u_n-\dfrac{\sqrt{3}}{2}\sin u_n\cos u_n$ $+\dfrac{1}{2}\sin^2 u_n+\dfrac{\sqrt{3}}{2}\sin u_n\cos u_n$ $=\dfrac{1}{2}$ $\therefore u_{n+1}-u_n=\cos^{-1}\dfrac{1}{2}=\dfrac{\pi}{3}$ (since the difference is less than π) is a constant independent of n. Hence $\{u_n\}$ is an arithmetic progression.

	The common difference is $\frac{\pi}{3}$. **Alternatively** $\cos u_{n+1} = \frac{1}{2}\cos u_n - \frac{\sqrt{3}}{2}\sin u_n$ $= \cos u_n \cos\frac{\pi}{3} - \sin u_n \sin\frac{\pi}{3}$ $= \cos\left(u_n + \frac{\pi}{3}\right)$ $\therefore u_{n+1} = u_n + \frac{\pi}{3}$ since the difference is less than π. $u_{n+1} - u_n$ is a constant, therefore $\{u_n\}$ is an arithmetic progression. The common difference is $\frac{\pi}{3}$.
(ii)	The total angle rotated over n twists is $\frac{n}{2}\left(2\left(\frac{2\pi}{3}\right)+(n-1)\frac{\pi}{3}\right)$. $\frac{n}{2}\left(\frac{4\pi}{3}+(n-1)\frac{\pi}{3}\right) \ge 25\times(2\pi)$ $\frac{2n\pi}{3}+\frac{\pi}{6}n(n-1)-50\pi \ge 0$ $4n+n(n-1)-300 \ge 0$ $n^2+3n-300 \ge 0$ Using the GC,

n	$n^2+3n-300$
15	-30

16	4
17	40

The minimum number of buttons presses required is 16.

(iii) Total angle rotated after 21 button presses is

$$\frac{21}{2}\left(2\left(\frac{2\pi}{3}\right)+(20)\frac{\pi}{3}\right)$$

$$=14\pi+70\pi$$

$$=84\pi$$

$84\pi k=144,\quad$ where k is a positive real constant.

The first button press rotates the screw by $\frac{2\pi}{3}$.

$$d_1=ku_1=\frac{144}{84\pi}\left(\frac{2\pi}{3}\right)=\frac{8}{7}$$

The first button press drills the screw in by $\frac{8}{7}$ mm (1.14mm).

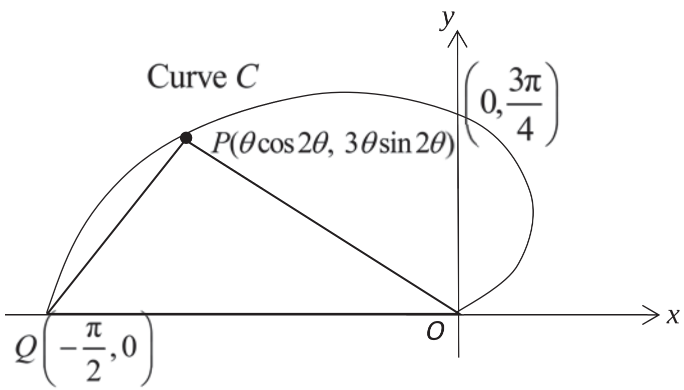
9(i) $x=15\sin\theta+15$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=15\cos\theta$$

$$\int_0^{15}\sqrt{15^2-(x-15)^2}\,\mathrm{d}x$$

	$=\int_{-\frac{\pi}{2}}^{0}\left(\sqrt{15^2-(15\sin\theta)^2}\right)(15\cos\theta)\mathrm{d}\theta$ $=225\left[\int_{-\frac{\pi}{2}}^{0}(\cos\theta)(\cos\theta)\mathrm{d}\theta\right]$ $=\frac{225}{2}\left[\int_{-\frac{\pi}{2}}^{0}\cos^2\theta\ \mathrm{d}\theta\right]$ $=\frac{225}{2}\left[\int_{-\frac{\pi}{2}}^{0}(\cos 2\theta+1)\ \mathrm{d}\theta\right]$ $=\frac{225}{2}\left[\frac{\sin 2\theta}{2}+\theta\right]_{-\frac{\pi}{2}}^{0}$ $=\frac{225}{2}\left(\frac{\pi}{2}\right)$ $=\frac{225}{4}\pi$

9(ii) (a)	Required area $=\int_0^{15} \mathrm{g}(x)\mathrm{d}x$ $=\int_0^{15}\left(30-\frac{2}{3}\sqrt{15^2-(x-15)^2}\right)\mathrm{d}x$ $=[30x]_0^{15}-\frac{2}{3}\int_0^{15}\sqrt{15^2-(x-15)^2}\,\mathrm{d}x$ $=450-\frac{2}{3}\left(\frac{225}{4}\pi\right)$ from (i) $=\left(450-\frac{75}{2}\pi\right)$ cm^2
(b)	Required Volume $=\pi(30)^2(20)+\pi\int_0^{15} y^2\mathrm{d}x$ $=\pi(30)^2(20)+\pi\int_0^{15}\left(30-\frac{2}{3}\sqrt{15^2-(x-15)^2}\right)^2 \mathrm{d}x$ $=\pi(30)^2(20)+$ $\pi\int_0^{15}\left(900-30(2)\left(\frac{2}{3}\right)\sqrt{15^2-(x-15)^2}+\left(\frac{4}{9}\right)\left(225-(x-15)^2\right)\right)\mathrm{d}x$ $=18000\pi+\pi\int_0^{15}\left(1000-40\sqrt{15^2-(x-15)^2}-\left(\frac{4}{9}\right)(x-15)^2\right)\mathrm{d}x$ $=18000\pi+\pi\left[1000x-\left(\frac{4}{9}\right)\frac{(x-15)^3}{3}\right]_0^{15}-40\pi\left(\frac{225}{4}\pi\right)$ from (i) $=\left(32500\pi-2250\pi^2\right)$ cm^3.

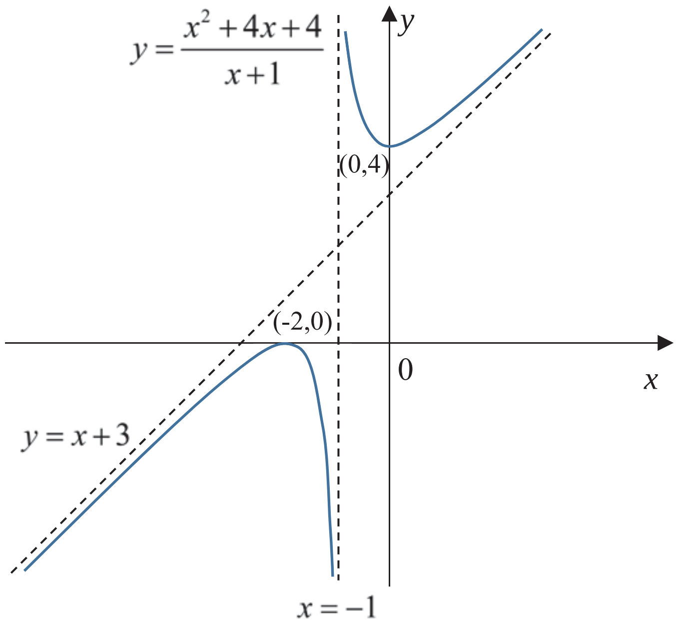
10(i)	 Point P has coordinates $(\theta\cos 2\theta,\ 3\theta\sin 2\theta)$. Let area of triangle OPQ be A. $A = \frac{1}{2}\left(\frac{\pi}{2}\right)(3\theta\sin 2\theta) = \frac{3\pi}{4}(\theta\sin 2\theta)$ $\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta)$ $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$ $= \frac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta)(0.01)$ When $\theta = \frac{\pi}{6}$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 0.0327 \text{ units}^2/\text{s}$ (3 s.f.)

(ii)	When $\frac{dA}{d\theta}=0$, $\frac{3\pi}{4}\left(2\theta\cos 2\theta+\sin 2\theta\right)=0$ Since $\frac{3\pi}{4}\neq 0$, $2\theta\cos 2\theta+\sin 2\theta=0$ Using GC, $\theta=1.0144$ (5 s.f.) $=1.01$ (3 s.f.) $\frac{d^2A}{d\theta^2}=\frac{3\pi}{4}\left(-4\theta\sin 2\theta+2\cos 2\theta+2\cos 2\theta\right)$ When $\theta=1.0144$, $\frac{d^2A}{d\theta^2}=-12.7<0$ $\therefore\ \theta=1.0144$ will result in maximum A. When $\theta=1.0144$, $A=\frac{3\pi}{4}(1.0144)\sin(2\times 1.0144)$ $=2.14$ units^2 (3 s.f.)

	When $\theta = 1.0144$, $x = 1.0144\cos\left[2(1.0144)\right] = -0.449$ $y = 3(1.0144)\sin\left[2(1.0144)\right] = 2.73$ $\therefore$ Location of the camera is at a point with coordinates $(-0.449, 2.73)$
(iii)	For triangle OPQ to be an isosceles triangle, $x = -\frac{\pi}{2} \div 2 = -\frac{\pi}{4}$ $-\frac{\pi}{4} = \theta\cos 2\theta$ Using GC, $\theta = 1.1581$ $y = 3(1.1581)\sin(2\times 1.1581) = 2.55$ $\therefore$ coordinates of $P(-0.785, 2.55)$

2019 SAJC H2 Math Paper 2 Solutions

Qn	Solution
1(i)	Since $x=-1$ is a vertical asymptote, $c=1$ $y=\dfrac{x^2+ax+b}{x+1}$ $y(x+1)=x^2+ax+b$ $x^2+(a-y)x+(b-y)=0$ The values that y can take satisfy the inequality: $(a-y)^2-4(b-y)\geq 0$ $y^2+(4-2a)y+(a^2-4b)\geq 0$ Since $y\leq 0$ or $y\geq 4$: $y=0$ and $y=4$ are roots to the equation $y^2+(4-2a)y+(a^2-4b)=0$ ....(1) Substituting $y=0$ and $y=4$ into (1) and solving: $a^2-4b=0$ $16+(4-2a)(4)=0$ $a=4,\quad b=4$
(ii)	$y=\dfrac{x^2+4x+4}{x+1}=x+3+\dfrac{1}{x+1}$

Qn	Solution
	
(iii)	Point of intersection: $(-1,2)$ For all $k\in\mathbb{R}$, the line $y=k(x+1)+2$ passes through the point (-1,2). Hence the line will cut C for $k>1$.
2(i)	Let the volume of water in the tank be V cubic metres at t seconds. $\frac{dV}{dt}=\frac{dV_{IN}}{dt}-\frac{dV_{OUT}}{dt}$ $=k-aV$ $=k\left(1-\frac{a}{k}V\right)$. (Shown) where $k>0$, and $a>0$.

Qn	Solution

$$\int \frac{1}{1-\frac{a}{k}V}\,\mathrm{d}V = \int k\,\mathrm{d}t$$

$$-\frac{k}{a}\ln\left|1-\frac{a}{k}V\right| = kt + C$$

$$\ln\left|1-\frac{a}{k}V\right| = -at - \frac{aC}{k}$$

$$1-\frac{a}{k}V = \pm e^{-\frac{ac}{k}}e^{-at} = Ae^{-at}$$

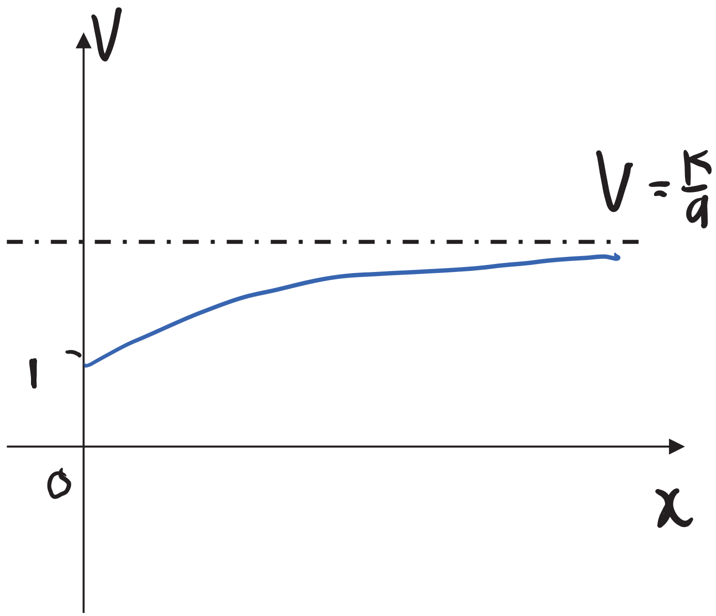
$$V = \frac{k}{a}\left(1 - Ae^{-at}\right)$$

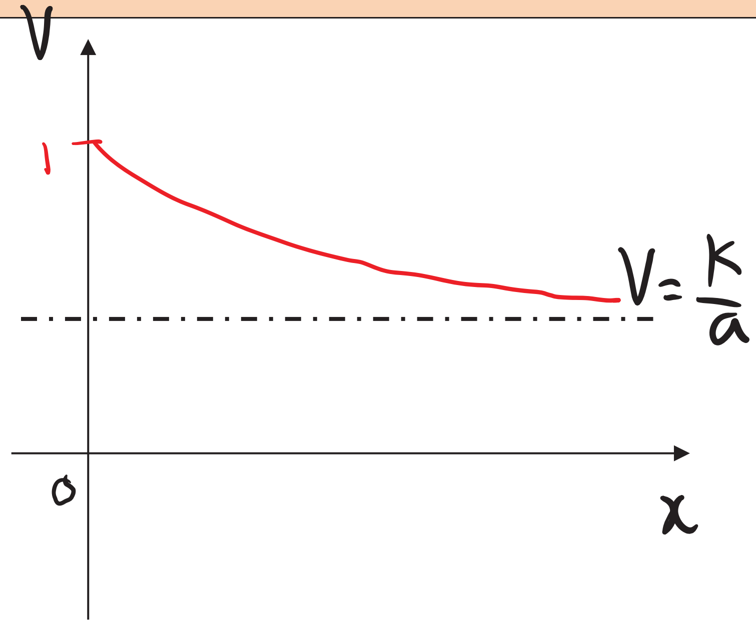
C is an arbitrary constant and $A = \pm e^{-\frac{aC}{k}}$

At $t = 0$, $V = 1$, $A = 1-\frac{a}{k}$

$$V = \frac{k}{a}\left(1-(1-\frac{a}{k})e^{-at}\right) = \frac{1}{a}(k-(k-a)e^{-at})$$

Case (i) if $a < k$

Qn	Solution
	 The volume of water in the water tank, V, increases from one cubic meter and approach $\frac{k}{a}$ cubic meters eventually. **Case (ii) if $a > k$**

Qn	Solution
	 The volume of water in the water tank, V, decreases from one cubic meter and approach $\frac{k}{a}$ cubic meters eventually.

3(i)	$\overrightarrow{OB}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=1+0+9=10$ B lies on π_1.
(ii)	$\begin{pmatrix}2\\a\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}\neq 10$ $2-a+6\neq 10$ $a\neq -2$ The range of values is $a\in\mathbb{R}, a\neq -2$.
(iii)	Let the foot of perpendicular be F. The line through AF has vector equation $l_{AF}:\mathbf{r}=\begin{pmatrix}2\\9\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix},\ \lambda\in\mathbb{R}.$ Since F lies on l_{AF}, $\overrightarrow{OF}=\begin{pmatrix}2\\9\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ for some fixed $\lambda\in\mathbb{R}$ Since F lies on π_1, $\overrightarrow{OF}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=10$ $\therefore \left[\begin{pmatrix}2\\9\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=10$ $2-9+6+11\lambda=10$ $\lambda=1$

$$\overrightarrow{OF} = \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$$

The coordinates of *F* are $(3,8,5)$.

Let the point of reflection of *A* about π_1 be *A*'.

$$\frac{\overrightarrow{OA}+\overrightarrow{OA'}}{2} = \overrightarrow{OF}$$

$$\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 6 \\ 16 \\ 10 \end{pmatrix} - \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$$

$$\overrightarrow{BA'} = \overrightarrow{OA'} - \overrightarrow{OB}$$

$$= \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$$

Line of reflection, $l_{BA'}$, has vector equation

$$l_{BA'} : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}, \ \mu \in \mathbb{R}.$$

(iv)	Acute angle between π_1 and π_2 is $\cos^{-1}\left\lvert\dfrac{\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}}{\left\lVert\begin{pmatrix}1\\-1\\3\end{pmatrix}\right\rVert\left\lVert\begin{pmatrix}1\\0\\1\end{pmatrix}\right\rVert}\right\rvert=\cos^{-1}\left\lvert\dfrac{4}{\sqrt{11}\sqrt{2}}\right\rvert=31.5^\circ\ (1\text{ d.p.})$
(v)	The desired planes have equation $\mathbf{r}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=D$, where D is the constant to be determined. Distance between the planes is given by $\left\lvert\dfrac{D}{\sqrt{2}}-\dfrac{4}{\sqrt{2}}\right\rvert=\dfrac{5}{\sqrt{2}}$ $\dfrac{D}{\sqrt{2}}-\dfrac{4}{\sqrt{2}}=\pm\dfrac{5}{\sqrt{2}}$ $D=4\pm5$ $D=9$ or $D=-1$ The possible equations are $x+z=9$ or $x+z=-1$. **<u>Alternative Solution</u>** Let a point D on the desired plane have coordinates (x,y,z). Then

	$\dfrac{\left\lvert \overrightarrow{AD}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right\rvert}{\sqrt{2}}=\dfrac{5}{\sqrt{2}}$ $\left\lvert\begin{pmatrix}x-1\\y\\z-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right\rvert=5$ $\lvert x-1+z-3\rvert=5$ $x+z-4=\pm5$ $x+z=4\pm5$ The possible equations are $x+z=9$ or $x+z=-1$.
4(a)	$\mathrm{f}(r+1)-\mathrm{f}(r)=\dfrac{r+1}{2^{r+1}}-\dfrac{r}{2^{r}}$ $=\dfrac{r+1-2r}{2^{r+1}}$ $=\dfrac{1-r}{2^{r+1}}$

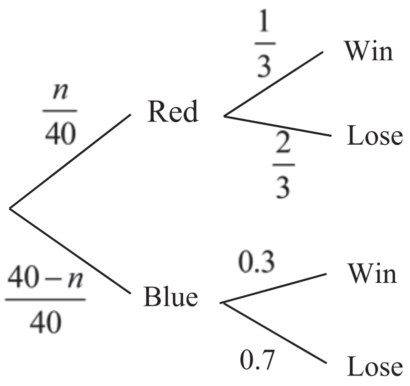
	$\displaystyle\sum_{r=1}^{n}\frac{1-r}{2^{r+1}}=\sum_{r=1}^{n}\left[\mathrm{f}(r+1)-\mathrm{f}(r)\right]$ $=\begin{bmatrix} \cancel{\mathrm{f}(2)}-\mathrm{f}(1) \\ +\cancel{\mathrm{f}(3)}-\cancel{\mathrm{f}(2)} \\ +\cancel{\mathrm{f}(4)}-\cancel{\mathrm{f}(3)} \\ \vdots \\ +\cancel{\mathrm{f}(n-1)}-\cancel{\mathrm{f}(n-2)} \\ +\cancel{\mathrm{f}(n)}-\cancel{\mathrm{f}(n-1)} \\ +\mathrm{f}(n+1)-\cancel{\mathrm{f}(n)} \end{bmatrix}$ $=\mathrm{f}(n+1)-\mathrm{f}(1)$ $=\dfrac{n+1}{2^{n+1}}-\dfrac{1}{2}$
(b)(i)	$a_n=\dfrac{2^n n^x}{3^n}$ $\displaystyle\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{2^n n^x}{3^n}}$ $\displaystyle=\lim_{n\to\infty}\left(\frac{2\sqrt[n]{n^x}}{3}\right)$ $\displaystyle=\frac{2}{3}\lim_{n\to\infty}\left(\sqrt[n]{n^x}\right)$ $\displaystyle=\frac{2}{3}(1)\quad \text{since } \lim_{n\to\infty}\left(\sqrt[n]{n^x}\right)=1$ $=\dfrac{2}{3}<1$

	Since $\lim_{n\to\infty}\sqrt[n]{a_n}=\frac{2}{3}<1,$ by the Cauchy Test, the series converges for all real values of x
(b)(ii)	$\sum_{r=0}^{\infty}\frac{2^r r}{3^r}=0+1\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)^2+3\left(\frac{2}{3}\right)^3+4\left(\frac{2}{3}\right)^4+\ldots$ $=\frac{2}{3}\left[1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)^3+\ldots\right]$ $=\frac{2}{3}\left(1-\frac{2}{3}\right)^{-2}$ since $1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)^3+\ldots=\left(1-\frac{2}{3}\right)^{-2}$ with $y=\frac{2}{3}$ $=6$
5	Let X be the mass of a standard packet of sugar in grams. Let Y be the mass of a large packet of sugar in grams. $X\sim\mathrm{N}\left(520,8^2\right)$ $Y\sim\mathrm{N}\left(1030,11^2\right)$ $X_1+X_2-Y\sim\mathrm{N}(10,249)$ $\mathrm{P}\left(X_1+X_2>Y\right)=\mathrm{P}\left(X_1+X_2-Y>0\right)$ $=0.73687$ $=0.737$ (3 s.f.)
	$\frac{X_1+X_2+Y}{3}\sim\mathrm{N}\left(690,\frac{83}{3}\right)$ $\mathrm{P}\left(680<\frac{X_1+X_2+Y}{3}<700\right)=0.94272$ $=0.943$ (3 s.f.)
6(i)	Total number of ways to select 5 members $={}^{13}C_5$ Number of ways to select 5 members with 3 Biology students = ${}^{10}C_2$ Number of ways to select 5 members with at most 2 Biology students = ${}^{13}C_5-{}^{10}C_2=1242$

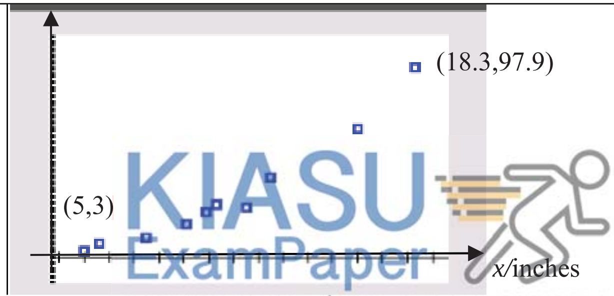
(ii)	Let the number of Biology, History, and Literature students be B, H, L respectively. $\mathrm{P}(\mathrm{H}>\mathrm{L} \mid \mathrm{B}\le 2)=\dfrac{\mathrm{P}((\mathrm{H}>\mathrm{L})\cap(\mathrm{B}\le 2))}{\mathrm{P}(\mathrm{B}\le 2)}$ $=\dfrac{\mathrm{n}((\mathrm{H}>\mathrm{L})\cap(\mathrm{H}+\mathrm{L}\ge 3))}{\mathrm{n}(\mathrm{B}\le 2)}$ Number of ways to select cast members with H > L when there are 3 humanities students = ${}^{3}C_{2}\times({}^{4}C_{2}\times{}^{6}C_{1}+{}^{4}C_{3}\times{}^{6}C_{0})=120$ Number of ways to select cast members with H > L when there are 4 humanities students = ${}^{3}C_{1}\times({}^{4}C_{3}\times{}^{6}C_{1}+{}^{4}C_{4}\times{}^{6}C_{0})=75$ Number of ways to select cast members with H > L when there are 5 humanities students = ${}^{3}C_{0}\times({}^{4}C_{3}\times{}^{6}C_{2}+{}^{4}C_{4}\times{}^{6}C_{1})=66$ Required Probability $=\dfrac{\mathrm{n}((\mathrm{H}>\mathrm{L})\cap(\mathrm{H}+\mathrm{L}\ge 3))}{\mathrm{n}(\mathrm{B}\le 2)}$ $=\dfrac{120+75+66}{1242}$ $=\dfrac{261}{1242}=0.210\ \ (3\text{ s.f.})$
7(i)	$\mathrm{P}(X=1)=\mathrm{P}(X=2)=....=\mathrm{P}(X=n)=\dfrac{1}{n}$ $\mathrm{E}(X)=\sum_{x=1}^{n}x\mathrm{P}(X=x)=\sum_{x=1}^{n}x\cdot\left(\dfrac{1}{n}\right)$ $=\dfrac{1}{n}\left(\dfrac{n(n+1)}{2}\right)$ $=\dfrac{n+1}{2}$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$

	$\mathrm{E}(X^2)=\sum_{x=1}^{n} x^2\cdot\left(\frac{1}{n}\right)$ $=\frac{1}{n}\left(\frac{n}{6}(n+1)(2n+1)\right)$ $=\frac{(n+1)(2n+1)}{6}$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2$ $=\frac{(n+1)(2n+1)}{6}-\left(\frac{n+1}{2}\right)^2$ $=\frac{(n+1)}{12}\left(2(2n+1)-3(n+1)\right)$ $=\frac{(n+1)(n-1)}{12}$ $=\frac{n^2-1}{12}$
(ii)	$\mathrm{E}(X)=\frac{7}{2}$ and $\mathrm{Var}(X)=\frac{6^2-1}{12}=\frac{35}{12}$ $\mathrm{P}\left(\lvert X-\mu\rvert>\sigma\right)=\mathrm{P}\left(\left\lvert X-\frac{7}{2}\right\rvert>\sqrt{\frac{35}{12}}\right)$ $=\mathrm{P}\left(X>\frac{7}{2}+\sqrt{\frac{35}{12}}\right)+\mathrm{P}\left(X<\frac{7}{2}-\sqrt{\frac{35}{12}}\right)$ $=\mathrm{P}(X>5.21)+\mathrm{P}(X<1.79)$ $=\mathrm{P}(X=6)+\mathrm{P}(X=1)$ $=\frac{2}{6}=\frac{1}{3}$

	Let S be the random variable "no. of observations, out of 20, such that the total score is an outlier". $S \sim \text{B}(20, \frac{1}{3})$ $\text{P}(S \geq 8) = 1 - \text{P}(S \leq 7) = 0.339$

8(i)	 P(a player wins the game) $= \frac{15}{40} \times \frac{1}{3} + \frac{25}{40} \times \frac{3}{10}$ $= \frac{5}{16}$ P(exactly 2 of 3 players win) $= {}^{3}C_{2} \left(\frac{5}{16}\right)^{2} \left(\frac{11}{16}\right)$ $= \frac{825}{4096}$ Alternatively,

	Let X be the random variable "the number of people who wins the game out of 3" $X \sim \mathrm{B}(3, \frac{5}{16})$ $\mathrm{P}(X = 2) = 0.201$ (to 3 s.f.)
(ii)	P(a player wins the game) $= \frac{n}{40} \times \frac{1}{3} + \frac{40-n}{40} \times \frac{3}{10}$ $= \frac{10n}{1200} + \frac{360-9n}{1200}$ $= \frac{360+n}{1200}$ $\mathrm{f}(n)$ = P(player draws blue | player wins) $= \frac{\mathrm{P}(\text{player draws blue and wins})}{\mathrm{P}(\text{player wins})}$ $= \frac{\frac{40-n}{40} \times \frac{3}{10}}{\frac{360+n}{1200}}$

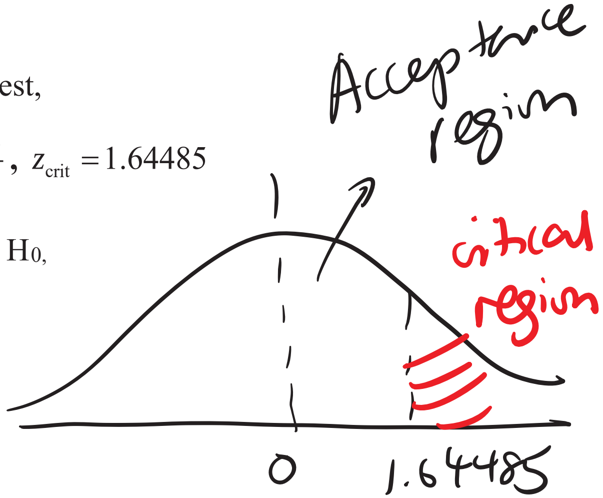
	$=\frac{120-3n}{400}\times\frac{1200}{360+n}$ $=\frac{3(120-3n)}{360+n}$ $=\frac{360-9n}{360+n}$ $=\frac{-9(360+n)+3600}{360+n}$ $=-9+\frac{3600}{360+n}$ As n increases, $\frac{3600}{360+n}$ decreases, hence f(n) decreases. Hence f is decreasing for all n, $0\le n\le 40$. This means that as the number of red counters increase, the probability that a winning player drew a blue counter decreases.
9(i)	
(ii)	(a) product moment correlation coefficient, $r=0.96346$ (b) product moment correlation coefficient, $r=0.98710$

16

(iii)	The second model $\ln y = c + d\ln x$ is the better model because its product moment correlation coefficient is closer to one as compared to the product moment correlation coefficient of the first model. From the scatter plot, it can be seen that the data seems to indicate a non-linear (curvilinear) relationship between y and x. Hence the model $y = a + bx$ is not appropriate.
(iv)	We have to use the regression line $\ln y$ against $\ln x$. From GC, the equation is $\ln y = -2.5866 + 2.4665\ln x$ When $x = 20$, $\ln y = -2.5866 + 2.4665\ln 20$ $y = e^{4.8025} = 121.82 = 121$ $x = 20$ is outside the data range and hence the relationship $\ln y = c + d\ln x$ may not hold. Hence the estimate may not be reliable.
10(i)	Assumptions 1. Every eraser is equally likely to be blue. 2. The colour of a randomly selected eraser is independent of the colour of other erasers.
(ii)	Let Y be the number of blue erasers, out of 36. $Y \sim B(36, 0.20)$ $P(Y \le 6) = 0.40069 \approx 0.401$
(iii)	Let W be the number of boxes that contain at most six blue erasers, out of 200. $W \sim B(200, 0.40069)$ $P(W \ge 40\% \text{ of } 200) = P(W \ge 80) = 1 - P(W \le 79)$ $= 0.53477 \approx 0.535$
(iv)	Let T denote the number of cartons where each carton contains at least 40% of the boxes that contains at most six blue erasers per box. $T \sim B(150, 0.53477)$ $E(T) = 150 \times 0.53477 = 80.216$ Var $(T) = 150 \times 0.53477 \times (1 - 0.53477) = 37.319$

	Since n is large $(n=30)$, by the Central Limit Theorem, $\overline{T}=\dfrac{T_1+T_2+...+T_{30}}{30} \sim N(80.216, \dfrac{37.319}{30})$ approximately. $P(\overline{T}<80)=0.423219 \approx 0.423$ (3 sig. fig.)
(v)	Let R be the number of blue erasers, out of 36. $R \sim B(36, p)$ $P(R=1)=\binom{36}{1}p^1(1-p)^{35}=36p(1-p)^{35}$
(vi)	$P(R=2)=\binom{36}{2}p^2(1-p)^{34}=630p^2(1-p)^{34}$ $P(R=1)=2P(R=2)$ $36p(1-p)^{35}=2\times 630p^2(1-p)^{34}$ $36(1-p)=1260p$ $1-p=35p$ $1=36p$ $p=\dfrac{1}{36}$

11(i)	Let T be the random variable " time taken in seconds for a computer to boot up", with population mean μ . Unbiased estimate of the population mean, $\bar{t} = \frac{802.5}{25} = 32.1$ Unbiased estimate of the population variance, $s^2 = \frac{1}{24}\left[26360.25 - \frac{802.5^2}{25}\right] = 25$
(ii)	A statistic is said to be an unbiased estimate of a given parameter when the mean of the sampling distribution of the statistic can be shown to be equal to the parameter being estimated. For example, $\mathrm{E}(\overline{X}) = \mu$.
(iii)	Test H_0: $\mu = 30$ against $H_1: \mu > 30$ at the 5% level of significance. Under H_0, $\overline{T} \sim N(30, \frac{25}{25})$. Using GC, $\bar{t} = 32.1$ gives rise to $z_{calc} = 2.1$ and p-value = 0.0179 Since p-value = 0.0179 $\le$ 0.05, we reject H_0 and conclude that there is sufficient evidence at the 5% significance level that the specification is not being met (<u>or</u> the computer requires more than 30 seconds to boot up). "5% significance level" is the probability of wrongly concluding that the mean boot up time for the computer is more than 30 seconds when in fact it is not more than 30 seconds.
(iv)	The critical value for the test is 31.645. For the specification to be met, H_0 is not rejected. $\bar{t} < 31.6$ (3 s.f.) Since $\bar{t} > 0$, Answer is $0 < \bar{t} < 31.6$.
(v)	Under H_0, $\overline{Y} \sim N\left(30, \frac{\sigma^2}{25}\right)$

	$Z=\dfrac{\overline{Y}-(30)}{\frac{\sigma}{\sqrt{25}}}\sim \mathrm{N}(0,1)$ Using a 1 – tailed z test, $z_{\text{calc}}=\dfrac{32.4-30}{\frac{\sigma}{5}}=\dfrac{12}{\sigma}$, $z_{\text{crit}}=1.64485$ In order not to reject H_0, $z_{\text{calc}}<1.64485$ $\dfrac{12}{\sigma}<1.64485$ $\sigma>7.2955$ $\sigma>7.30$ 

TEMASEK JUNIOR COLLEGE,
SINGAPORE
JC 2
Preliminary Examination 2019

CANDIDATE NAME | | CG |

MATHEMATICS

9758/01

Higher 2

30 Aug 2019

Paper 1

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiner's Use	
Question Number	Marks Obtained
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total Marks	

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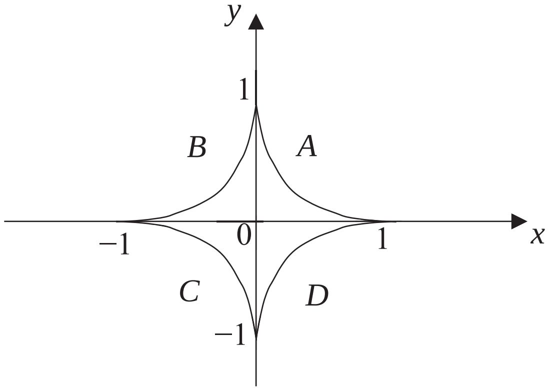
PASSION PURPOSE DRIVE

1 The diagram below shows a shape which is symmetrical about the x- and y-axes. The shape is made up of four curves, A, B, C and D.



The curve A has equation $\sqrt{x}+\sqrt{y}=1$ for $0 \le x \le 1$ and $0 \le y \le 1$.

(i) State the equations and the range of values of x and y for curves B and C. [3]

(ii) The curves A and B are scaled by a factor $\frac{1}{2}$ parallel to the x-axis and the curves C and D are scaled by a factor 2 parallel to the y-axis. Sketch the resulting shape. [2]

2 The position vectors of A, B, C and D are $\begin{pmatrix}\alpha\\1\\5\end{pmatrix}$, $\begin{pmatrix}1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}-2\\3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\\beta\\7\end{pmatrix}$ respectively, where α and β are real numbers. Given that BD is a perpendicular bisector of AC, find the values of α and β. [5]

3 The hyperbola C passes through the point (2, 0) and has oblique asymptotes $y=-2x$ and $y=2x$.

(i) Sketch C, showing the relevant features of the curve. [2]

(ii) Write down the equation of C. [1]

(iii) By adding a suitable curve to your sketch in part **(i)**, solve the inequality

$$\sqrt{\frac{x^2}{4}-1}<\sqrt{x-1}.$$ [3]

4 A curve C has equation $y = \dfrac{e^x}{x-k}$, $x \neq k$, where k is a positive real number.

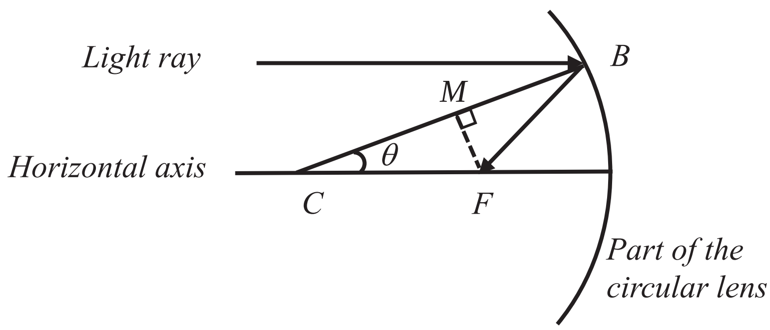
Show algebraically that C has exactly one stationary point, and show that the stationary point lies in the first quadrant. [3]

Sketch C for $x > k$, indicating clearly the equation of the asymptote and the coordinates of the stationary point. [2]

Deduce that $\displaystyle\int_{k+\frac{1}{2}}^{k+\frac{3}{2}} \frac{e^x}{x-k}\,dx < \frac{1}{3}\left(3e^{k+\frac{1}{2}} + e^{k+\frac{3}{2}}\right)$ for all positive real values of k. [2]

5 In geometric optics, the paraxial approximation is a small-angle approximation used in Gaussian optics and ray tracing of light through an optical system such as a lens.

In the diagram below, a light ray parallel to the horizontal axis is reflected at point B on the circular lens centred at point C and has radius r cm. Let $\angle BCF = \theta$ radians. FM is the perpendicular bisector of CB.



(i) Show that $CF = \dfrac{r}{k\cos\theta}$, where k is a real constant to be determined. [1]

(ii) Hence find the series expansion for CF if θ is sufficiently small for θ^3 and terms in higher powers of θ to be neglected. [2]

Suppose that the source of the light ray is now repositioned such that $\angle BCF = \left(\theta + \dfrac{\pi}{6}\right)$ radians.

(iii) Find the corresponding series expansion for CF, up to and including the term in θ^2. [4]

6 An arithmetic sequence has first term a and common difference d, where a and d are non-zero. The ninth, tenth and thirteenth terms of the arithmetic sequence are the first three terms of a geometric sequence.

(i) Show that $a = -\frac{15}{2}d$. [3]

(ii) The sum of the first n terms of the arithmetic sequence is denoted by S_n. Find the value of S_{16}. [2]

(iii) Given that the k^{th} term of the arithmetic sequence is the fourth term of the geometric sequence, find the value of k. [3]

7 A curve C has parametric equations

$$x = t^2, \qquad y = t\,\mathrm{e}^{t^2}, \qquad \text{for } t \geq 0.$$

(i) Find the equation of the tangent to C at the point P with coordinates $\left(p^2, p\,\mathrm{e}^{p^2}\right)$, where $p \neq 0$. Hence, or otherwise, find the exact equation of the tangent L to C which passes through the origin. [5]

(ii) **(a)** Find the cartesian equation of C. [1]

(b) Find the exact volume of the solid formed when the region bounded by C and L is rotated through 2π radians about the x-axis. [5]

8 **Do not use a calculator in answering this question.**

The complex numbers z and w are given by $z = \dfrac{(1+\mathrm{i})^4}{(1-\mathrm{i})^2}$ and $w = \dfrac{8}{\left(\sqrt{3}+\mathrm{i}\right)^2}$.

(i) Express z and w in polar form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give r and θ in exact form. [4]

(ii) Given that z^2, w and w^* are the roots of the equation $x^3 + bx^2 + cx + d = 0$ where b, c and d are real values, find the equation. [3]

(iii) Sketch on an Argand diagram with origin O, the points P, Q and R representing the complex numbers z, w and $z+w$ respectively. [2]

(iv) By considering the quadrilateral $OPRQ$ and the argument of $z+w$, deduce that

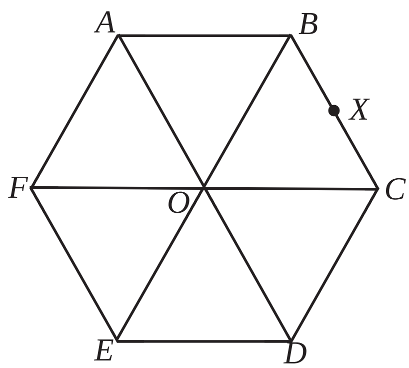
$$\tan\frac{5\pi}{12} = 2 + \sqrt{3}.$$ [3]

9 **(a)** Vectors **u** and **v** are such that $\mathbf{u} \cdot \mathbf{v} = -1$ and $(\mathbf{u} \times \mathbf{v}) + \mathbf{u}$ is perpendicular to $(\mathbf{u} \times \mathbf{v}) + \mathbf{v}$.

Show that $|\mathbf{u} \times \mathbf{v}| = 1$. [3]

Hence find the angle between **u** and **v**. [3]

(b) The figure shows a regular hexagon *ABCDEF* with *O* at the centre of the hexagon. *X* is the midpoint of *BC*.



Given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, find $\overrightarrow{OF}$ and $\overrightarrow{OX}$ in terms of **a** and **b**. [2]

Line segments *AC* and *FX* intersect at the point *Y*. Determine the ratio *AY* : *YC*. [4]

10 Mr Ng wants to hang a decoration on the vertical wall above his bookshelf. He needs a ladder to climb up.

The rectangle *ABCD* is the side-view of the bookshelf and *HK* is the side-view of the ladder where $AB = 24$ cm and $BC = 192$ cm (see **Figure 1**). The ladder touches the wall at *H*, the edge of the top of the bookshelf at *B* and the floor at *K*.

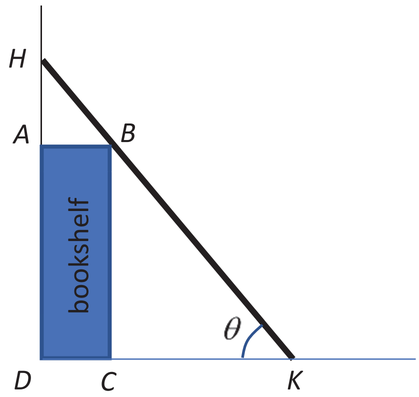


Figure 1

(i) Given that $\angle HKD = \theta$, show that the length , *L* cm of the ladder is given by

$$L = \frac{24}{\cos\theta} + \frac{192}{\sin\theta} \quad .$$ [1]

(ii) Use differentiation to find the exact value of the shortest length of the ladder as θ varies. [4]
[You do not need to verify that this length of the ladder is the shortest.]

Take *L* to be 270 for the rest of this question.

The ladder starts to slide such that *H* moves away from the wall and *K* moves towards *E* (see **Figure 2**). The ladder maintains contact with the bookshelf at *B*.

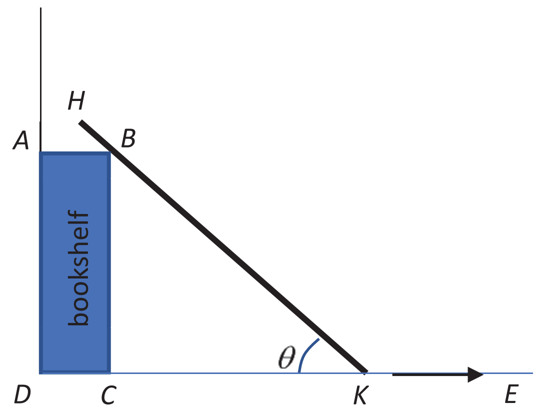


Figure 2

The horizontal distances from the wall to *H* and from the wall to *K* are *x* cm and *y* cm respectively.

(iii) By expressing $y - x$ in terms of θ, determine whether the rate of change of y is greater than the rate of change of *x*. [3]

(iv) Given that the rate of change of θ is -0.1 rad s^{-1} when $CK = 160$ cm, find the rate of change of x at this instant. [5]

11 The daily food calories, L, taken in by a human body are partly used to fulfill the daily requirements of the body. The daily requirements is proportional to the body mass, M kg, with a constant of proportionality p. The rate of change in body mass is proportional to the remaining calories.

It is given that the body mass, M kg, at time t days satisfies the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = k\left(L - pM\right),$$

where k and L are constants.

John's initial body mass is 100 kg. Find, in terms of p, the daily food calories needed to keep his body mass constant at 100 kg. [1]

To lose weight, John decides to start on a diet where his daily food calorie intake is 75% of the daily calories needed to keep his body mass constant at 100 kg.

(i) Show that $M = 75 + 25\mathrm{e}^{-pkt}$. [4]

(ii) John attained a body mass of 90 kg after 50 days on this diet. If it takes him n more days to lose at least another 10 kg, find the smallest integer value of n. [5]

(iii) John's goal with this diet plan is to achieve a body mass of 70 kg. With the aid of a graph, explain why he can never achieve his goal. [2]

(iv) By considering $\dfrac{\mathrm{d}^2M}{\mathrm{d}t^2}$, comment on his rate of body mass loss as time passes. [2]

TEMASEK JUNIOR COLLEGE,
SINGAPORE
JC 2
Preliminary Examination 2019

CANDIDATE NAME		CG	

MATHEMATICS

9758/02

Higher 2

16 Sep 2019

Paper 2

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiner's Use	
Question Number	Marks Obtained
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total Marks	

This document consists of **24** printed pages and **no** blank pages.

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PASSION PURPOSE DRIVE

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1 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{x^2}{2-x},\ x \in \mathbb{R},\ 0 \le x < 2\ .$$

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [4]

It is given that

$$\mathrm{g} : x \mapsto \frac{1}{1+\mathrm{e}^{-x}},\ x \in \mathbb{R},\ x \ge 0\ .$$

(ii) Show that fg exists. [2]

(iii) Find the range of fg. [2]

2 Express $\dfrac{6r+7}{r(r+1)}$ as partial fractions. [1]

(i) Hence use method of differences to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ in terms of N.

(There is no need to express your answer as a single algebraic fraction.) [3]

(ii) Give a reason why the series $\displaystyle\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ converges, and write down its value. [2]

(iii) Use your answer in part **(i)** to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)$. [3]

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3 A curve C with equation $y = \mathrm{f}(x)$ satisfies the equation

$$(x^2+2x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

and passes through the point $(0, \pi)$.

(i) By further differentiation, find the Maclaurin expansion of $\mathrm{f}(x)$ in ascending powers of x up to and including the term x^3. [5]

(ii) Solve the differential equation $(x^2+2x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$, given that $y = \pi$ when $x = 0$, leaving y in terms of x.

Hence show that

$$\tan^{-1}(x+1) \approx \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3$$

for small values of x. [4]

(iii) With the aid of a sketch, explain why $\int_0^2 \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 \, \mathrm{d}x$ gives a more accurate approximation of $\int_0^2 \tan^{-1}(x+1) \, \mathrm{d}x$ than $\int_0^2 \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 \, \mathrm{d}x$. [2]

4 The points A, B, C and D have coordinates $(1, 0, 3)$, $(-1, 0, 1)$, $(1, 1, 3)$ and $(1, k, 0)$ respectively, where k is a positive real number. The plane p_1 contains A, B and C while the plane p_2 contains A, B and D.

Given that p_1 makes an angle of $\frac{\pi}{3}$ with p_2, show that $k = \frac{\sqrt{6}}{2}$. [5]

The point X lies on p_2 such that the vector $\overrightarrow{XC}$ is perpendicular to p_1. Find $\overrightarrow{XC}$. [5]

Hence find the exact area of the triangle AXC. [2]

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Section B: Probability and Statistics [60 marks]

5 Anand, Beng, Charlie, Dayanah and 6 other people attend a banquet dinner, and are to sit at a round table.

(i) Dayanah will only sit next to Anand, Beng or Charlie (and no one else), and Anand, Beng and Charlie do not want to sit next to each other. Find the number of ways the 10 people can seat themselves around the table. [4]

(ii) As part of dinner entertainment, 4 people from the table are chosen to participate in a game.

Among Anand, Beng and Charlie, if any one of them is chosen, the other two will refuse to participate in the game. Furthermore, Dayanah refuses to participate unless at least one of Anand, Beng or Charlie is also chosen.

Find the number of ways the 4 people can be chosen for the game. [3]

6 A bag contains four identical counters labelled with the digits 0, 1, 2, and 3. In a game, Amira chooses one counter randomly from the bag and then tosses a fair coin. If the coin shows a Head, her score in the game is the digit labelled on the counter chosen. If the coin shows a Tail, her score in the game is the negative of the digit labelled on the counter chosen. T denotes the score in a game.

(i) Find the probability distribution of T. [2]

(ii) Amira tosses the coin and it shows a Tail. Find the probability that $T < -1$. [3]

(iii) Amira plays the game twice. Find the probability that the sum of her two scores is positive. [3]

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7 It is generally accepted that a person's diet and cardiorespiratory fitness affects his cholesterol levels. The results of a study on the relationship between the cholesterol levels, C mmol/L, and cardiorespiratory fitness, F, measured in suitable units, on 8 individuals with similar diets are given in the following table.

Cardiorespiratory Fitness (F units)	55.0	50.7	45.3	40.2	34.7	31.9	27.9	26.0
Cholesterol (C mmol/L)	4.70	4.98	5.30	5.64	6.04	6.30	6.99	6.79

(i) Draw a scatter diagram of these data. Suppose that the relationship between F and C is modelled by an equation of the form $\ln C = aF + b$, where a and b are constants. Use your diagram to explain whether a is positive or negative. [4]

(ii) Find the product moment correlation coefficient between ln C and F, and the constants a and b for the model in part **(i)**. [3]

(iii) Bronz is a fitness instructor. His cardiorespiratory fitness is 52.0 units. Estimate Bronz's cholesterol level using the model in **(i)** and the values of a and b in part **(ii)**. Comment on the reliability of the estimate. [2]

(iv) Bronz then had a medical checkup and found his actual cholesterol level to be 6.2 mmol/L. Assuming his cholesterol level is measured accurately, explain why there is a great difference between Bronz's cholesterol level and the estimated value in **(iii)**. [1]

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8 A research laboratory uses a data probe to collect data for its experiments. There is a probability of 0.04 that the probe will give an incorrect reading. In a particular experiment, the probe is used to take 80 readings, and X denotes the number of times the probe gives an incorrect reading.

(i) State, in context, two assumptions necessary for X to be well modelled by a binomial distribution. [2]

(ii) Find the probability that between 5 and 10 (inclusive) incorrect readings are obtained in the experiment. [3]

When the probe gives an incorrect reading, it will give a reading that is 5% greater than the actual value.

(iii) Suppose the 80 readings are multiplied together to obtain a Calculated Value. Find the probability that the Calculated Value is at least 50% more than the product of the 80 actual values. [5]

9 A Wheel Set refers to a set of wheel rim and tyre. The three types of wheel sets are the Clincher Bike Wheel Set, Tubular Bike Wheel Set and Mountain Bike Wheel Set.
The weight of a rim of a Clincher Bike Wheel Set follows a normal distribution with mean 1.5 kg and standard deviation 0.01 kg. The weight of its tyre follows a normal distribution with mean 110 g and standard deviation 5 g.

(i) Let C be the total weight in grams of a randomly chosen Clincher Bike Wheel Set in grams. Find $\mathrm{P}(C > 1620)$. [3]

(ii) State, in the context of the question, an assumption required in your calculation in **(i)**. [1]

Let T be the total weight in grams of a Tubular Bike Wheel Set, where $T \sim \mathrm{N}(\mu,\ 15^2)$.

(iii) The probability that the weight of a randomly chosen Clincher Bike Wheel Set exceeds a randomly chosen Tubular Bike Wheel Set by more than 150 g is smaller than 0.70351 correct to 5 decimal places. Find the range of values that μ can take. [5]

Let M be the total weight in grams of a randomly chosen Mountain Bike Wheel Set with mean 1800 g and standard deviation 20 g.

(iv) Find the probability that the mean weight of 50 randomly chosen Mountain Bike Wheel Sets is more than 1795 g. [3]

10 **(a)** Two random samples of different sample sizes of households in the town of Aimek were taken to find out the mean number of computers per household there. The first sample of 50 households gave the following results.

Number of computers	0	1	2	3	4
Number of households	5	12	18	10	5

The results of the second sample of 60 households were summarised as follows.

$$\sum y = 118 \qquad \sum y^2 = 314,$$

where y is the number of computers in a household.

(i) By combining the two samples, find unbiased estimates of the population mean and variance of the number of computers per household in the town. [4]

(ii) Describe what you understand by 'population' in the context of this question. [1]

(b) Past data has shown that the working hours of teachers in a city are normally distributed with mean 48 hours per week. In a recent study, a large random sample of n teachers in the city was surveyed and the number of working hours per week was recorded. The sample mean was 46 hours and the sample variance was 131.1 hours^2. A hypothesis test is carried out to determine whether the mean working hours per week of teachers has been reduced.

(i) State appropriate hypotheses for the test. [1]

The calculated value of the test statistic is $z = -1.78133$ correct to 5 decimal places.

(ii) Deduce the conclusion of the test at the 2.5 % level of significance. [2]

(iii) Find the value of n. [3]

(iv) In another test, using the same sample, there is significant evidence at the α% level that there is a change in the mean working hours per week of teachers in that city. Find the smallest possible integral value of α. [2]

Section A: Pure Mathematics [40 marks]

1 The function f is defined by

$$f : x \mapsto \frac{x^2}{2-x},\ x \in \mathbb{R},\ 0 \le x < 2\ .$$

(i) Find $f^{-1}(x)$ and write down the domain of f^{-1}. [4]

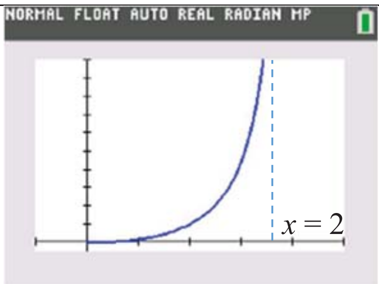
It is given that

$$g : x \mapsto \frac{1}{1+e^{-x}},\ x \in \mathbb{R},\ x \ge 0\ .$$

(ii) Show that fg exists. [2]

(iii) Find the range of fg. [2]

1(i)	Let $y = \frac{x^2}{2-x}$. $2y - xy = x^2 \quad \Rightarrow \quad x^2 + xy - 2y = 0$ Then $x = \frac{-y \pm \sqrt{y^2+8y}}{2}$ Alternatively, $2y - xy = x^2 \quad \Rightarrow \quad x^2 + xy - 2y = 0$ $\Rightarrow \left(x+\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right)^2 - 2y = 0$ $\Rightarrow \left(x+\frac{y}{2}\right)^2 = \frac{y^2}{4} + 2y$ $\Rightarrow x + \frac{y}{2} = \pm\sqrt{\frac{y^2}{4}+2y}$ $\Rightarrow x + \frac{y}{2} = \pm\sqrt{\frac{y^2}{4}+2y}$ $\Rightarrow x = \frac{-y \pm \sqrt{y^2+8y}}{2}$ Since $0 \le x < 2$, $x = \frac{-y + \sqrt{y^2+8y}}{2}$. $f^{-1}(x) = \frac{-x+\sqrt{x^2+8x}}{2}$ $D_{f^{-1}} = R_f = [0, \infty)$	Many students are uncertain how to make x the subject. Note that this can be done by either using the quadratic formula or completing square. Students must note that it is essential to explain why $x = \frac{-y-\sqrt{y^2+8y}}{2}$ is rejected. Some students did not provide the final answer. While they able to obtain $x = \frac{-y+\sqrt{y^2+8y}}{2}$, they fail to write down the expression for $f^{-1}(x)$ and $D_{f^{-1}}$ to answer the question completely.
1(ii)	$R_g = [\frac{1}{2}, 1)$ $D_f = [0, 2)$	It is necessary to write down the R_g and D_f to answer this

	Since $R_g \subset D_f$, the function fg exists.	part of the question. Simply stating $R_g \subset D_f$ is insufficient.
1(iii)	 Observe that $y = \mathrm{f}(x)$ is a strictly increasing function; so we only need to find $\mathrm{f}(\frac{1}{2}) = \frac{1}{6}$ and $\mathrm{f}(1) = 1$. $R_{fg} = [\frac{1}{6}, 1)$	Many students did not indicate that f is an increasing function. Simply writing $R_g = [\frac{1}{2}, 1)$ leading to $R_{fg} = [\frac{1}{6}, 1)$ is insufficient to obtain the full credit.

2 Express $\dfrac{6r+7}{r(r+1)}$ as partial fractions. [1]

(i) Hence use method of differences to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ in terms of N. (There is no need to express your answer as a single algebraic fraction.) [3]

(ii) Give a reason why the series $\displaystyle\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ converges, and write down its value. [2]

(iii) Use your answer in part **(i)** to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)$. [3]

2	$\dfrac{6r+7}{r(r+1)} = \dfrac{7}{r} - \dfrac{1}{r+1}$	Most students got this right.

(i)	$\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)=\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\left(\frac{7}{r}-\frac{1}{r+1}\right)\right)$ $=\sum_{r=1}^{N}\left(\left(\frac{7^{-r}}{r}-\frac{7^{-r-1}}{r+1}\right)\right)$ $=(\frac{7^{-1}}{1}-\frac{7^{-2}}{2}$ $+\frac{7^{-2}}{2}-\frac{7^{-3}}{3}$ $+\frac{7^{-3}}{3}-\frac{7^{-4}}{4}$ $+\cdots$ $+\frac{7^{-N+1}}{N-1}-\frac{7^{-N}}{N}$ $+\frac{7^{-N}}{N}-\frac{7^{-N-1}}{N+1})$ $=\frac{7^{-1}}{1}-\frac{7^{-N-1}}{N+1}=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$	A common error here is to express $\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\left(\frac{7}{r}-\frac{1}{r+1}\right)\right)$ as $\sum_{r=1}^{N}\left(\frac{1}{7}\right)^{r+1}\cdot\sum_{r=1}^{N}\left(\frac{7}{r}-\frac{1}{r+1}\right)$. Students must take note that $\sum_{r=1}^{N}\left(f(r)\cdot g(r)\right)$ is not equal to $\sum_{r=1}^{N}f(r)\cdot\sum_{r=1}^{N}g(r)$.
(ii)	As $N\to\infty,\left(\frac{1}{7}\right)^{N+1}\to 0,\frac{1}{N+1}\to 0$ Hence as $N\to\infty$, $\frac{7^{-N-1}}{N+1}=\left(\frac{1}{7}\right)^{N+1}\left(\frac{1}{N+1}\right)\to 0$ Then $\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)=\frac{1}{7}$, $\therefore\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ converges and it converges to $\frac{1}{7}$.	Students must know that the highlighted statement is the reason why the series converges, and therefore must be clearly written down in your answer.
(iii)	Method 1: $\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)$ replace r by r-1 $=\sum_{r-1=1}^{r-1=N}\left(\left(\frac{1}{7}\right)^{r-1+1}\frac{6(r-1)+13}{(r-1+1)(r-1+2)}\right)$ $=\sum_{r=2}^{N+1}\left(\left(\frac{1}{7}\right)^{r}\frac{6r+7}{(r)(r+1)}\right)=7\sum_{r=2}^{N+1}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{(r)(r+1)}\right)$ $=7\left[\sum_{r=1}^{N+1}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{(r)(r+1)}\right)-\left(\frac{1}{7}\right)^{2}\frac{13}{(1)(2)}\right]$	Many students are aware that they must replace r by r–1 in Method 1 or replace r by r+1 in Method 2. However, the plan is poorly executed with many making errors in its replacement. Some who have correctly replaced the r, are unsure how to proceed from there.

$$=7\left[\sum_{r=1}^{N+1}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{(r)(r+1)}\right)-\left(\frac{1}{7}\right)^{2}\frac{13}{(1)(2)}\right]$$

$$=7\left[\frac{1}{7}-\frac{7^{-(N+1)-1}}{(N+1)+1}-\frac{13}{49(2)}\right]=\frac{1}{14}-\frac{7^{-N-1}}{N+2}=\frac{1}{14}-\frac{1}{7^{N+1}(N+2)}$$

Method 2:
From (i),

$$\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$$

Replace r by r+1, we have

$$\sum_{r+1=1}^{r+1=N}\left(\left(\frac{1}{7}\right)^{(r+1)+1}\frac{6(r+1)+7}{(r+1)((r+1)+1)}\right)=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$$

$$\sum_{r=0}^{N-1}\left(\left(\frac{1}{7}\right)^{r+2}\frac{6r+13}{(r+1)(r+2)}\right)=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$$

Replace N by N+1, we have

$$\sum_{r=0}^{N}\left(\left(\frac{1}{7}\right)^{r+2}\frac{6r+13}{(r+1)(r+2)}\right)=\frac{1}{7}-\frac{1}{7^{N+2}(N+2)}$$

$$\left[\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+2}\frac{6r+13}{(r+1)(r+2)}\right)\right]+\underbrace{\left(\frac{1}{7}\right)^{2}\frac{13}{2}}_{\text{Term when } r=0}=\frac{1}{7}-\frac{1}{7^{N+2}(N+2)}$$

$$\frac{1}{7}\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)=\frac{1}{7}-\frac{1}{7^{N+2}(N+2)}-\frac{13}{98}$$

$$\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)=7\left(\frac{1}{98}-\frac{1}{7^{N+2}(N+2)}\right)$$

$$=\frac{1}{14}-\frac{1}{7^{N+1}(N+2)}$$

3 A curve C with equation $y=\mathrm{f}(x)$ satisfies the equation

$$(x^2+2x+2)\frac{\mathrm{d}y}{\mathrm{d}x}=2$$

and passes through the point $(0,\pi)$.

(i) By further differentiation, find the Maclaurin expansion of $\mathrm{f}(x)$ in ascending powers of x up to and including the term x^3. [5]

(ii) Solve the differential equation $(x^2+2x+2)\frac{\mathrm{d}y}{\mathrm{d}x}=2$, given that $y=\pi$ when $x=0$, leaving y in terms of x.
Hence show that

$$\tan^{-1}(x+1)\approx\frac{\pi}{4}+\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{12}x^3$$

for small values of x. [4]

(iii) With the aid of a sketch, explain why $\int_0^2 \frac{\pi}{4}+\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{12}x^3\ \mathrm{d}x$ gives a more accurate approximation of $\int_0^2 \tan^{-1}(x+1)\ \mathrm{d}x$ than $\int_0^2 \frac{\pi}{4}+\frac{1}{2}x-\frac{1}{4}x^2\ \mathrm{d}x$. [2]

3(i)	(i) $(x^2+2x+2)\frac{\mathrm{d}y}{\mathrm{d}x}=2$ Differentiating wrt x, $(x^2+2x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x+2)\frac{\mathrm{d}y}{\mathrm{d}x}=0$ Differentiating wrt x, $(x^2+2x+2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+(2x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow\ (x^2+2x+2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+(4x+4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=0$ When $x=0$, $y=\pi$ since curve passes through $(0,\pi)$. $2\frac{\mathrm{d}y}{\mathrm{d}x}=2 \quad\Rightarrow\quad \frac{\mathrm{d}y}{\mathrm{d}x}=1$ $2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2(1)=0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1$ $2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+4(-1)+2(1)=0 \Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=1$ Thus the Maclaurin expansion is $y=\mathrm{f}(x)=\pi+x-\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots$ i.e., $y=\mathrm{f}(x)=\pi+x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$	Question was very well done. Only a handful of students expressed $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{x^2+2x+2}$ and performed direct differentiation. Many students integrate the expression to find y, which is the question for (ii). This is an indicator that they do not know what the question is asking for.

<table>
<tr>
<td>(ii)</td>
<td>
(ii) $(x^2+2x+2)\dfrac{dy}{dx}=2$
$\Rightarrow \quad \int dy = 2\int \dfrac{1}{x^2+2x+2}\,dx$
$\Rightarrow \quad y = 2\int \dfrac{1}{(x+1)^2+1^2}\,dx$
$\Rightarrow \quad y = 2\tan^{-1}(x+1)+c$
Sub $x=0$, $y=\pi$: $\quad \pi = 2\tan^{-1}(1)+c$
$\Rightarrow \quad c = \pi - 2\left(\dfrac{\pi}{4}\right)=\dfrac{\pi}{2}$
Thus, $y = 2\tan^{-1}(x+1)+\dfrac{\pi}{2}$.
$\Rightarrow \quad 2\tan^{-1}(x+1)+\dfrac{\pi}{2}=\pi+x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\ldots$
$\Rightarrow \quad \tan^{-1}(x+1)=\dfrac{1}{2}\left(\pi-\dfrac{\pi}{2}+x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\ldots\right)$
$\Rightarrow \quad \tan^{-1}(x+1)\approx\dfrac{1}{4}\pi+\dfrac{1}{2}x-\dfrac{1}{4}x^2+\dfrac{1}{12}x^3$ (Shown)
for small values of x
</td>
<td>
Apart from algebraic errors, students were able to find the particular solution.

Quite a number of students forgot to "$+C$", which affected the rest of their answers.

Most students were able to see the relationship between their answers in (i) and (ii).
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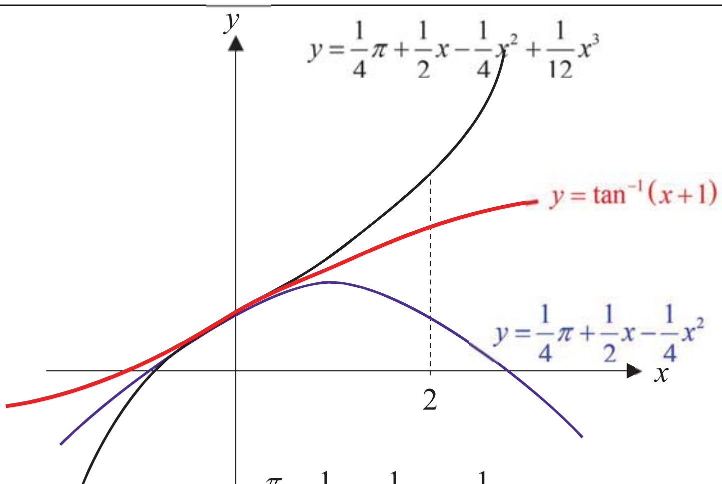
<tr>
<td>(iii)</td>
<td>


Since the curve $y=\dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2+\dfrac{1}{12}x^3$ is closer to $y=\tan^{-1}(x+1)$ than the curve $y=\dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2$ for $0\le x\le 2$, the area under the cubic curve, $\int_0^2 \dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2+\dfrac{1}{12}x^3\,dx$, will give a <u>better</u> approximation to $\int_0^2 \tan^{-1}(x+1)\,dx$, the area under the curve $y=\tan^{-1}(x+1)$, <u>than</u> the area under the quadratic curve, $\int_0^2 \dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2\,dx$.

</td>
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This part was poorly performed. Students should realise that they must sketch $y=\tan^{-1}(x+1)$ in order to make any comparison.

For the sketches, many students did not pay attention to the details which led to loss of marks. The more common ones are:

1. not realizing that they have the same y-intercept,

2. Not showing that the quadratic curve has a turning point before $x=2$, and cuts the x-axis after $x=2$.

3. Not showing that the cubic curve is concave up.
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	Students plotted / labelled the graphs as (for e.g.) $\int_0^2 \frac{\pi}{4}+\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{12}x^3\ \mathrm{d}x$ which is a definite integral (i.e. just a value). This has no meaning for comparison. A number of students drew rectangles under the graph, which clearly shows regurgitation without understanding. For the qualitative explanation, students should make it clear that the comparison of how close the curves are is limited from $x=0$ to $x=2$. Also, the link between the definite integral to the area must be clear.

4 The points A, B, C and D have coordinates $(1, 0, 3)$, $(-1, 0, 1)$, $(1, 1, 3)$ and $(1, k, 0)$ respectively, where k is a positive real number. The plane p_1 contains A, B and C while the plane p_2 contains A, B and D.

Given that p_1 makes an angle of $\frac{\pi}{3}$ with p_2, show that $k=\frac{\sqrt{6}}{2}$. [5]

The point X lies on p_2 such that the vector $\overrightarrow{XC}$ is perpendicular to p_1. Find $\overrightarrow{XC}$. [5]

Hence find the exact area of the triangle AXC. [2]

Solution:

A standard question which can provide a good source of marks, but students seemed to be unprepared for the third question on Vectors in Paper 2.

4	$\overrightarrow{AB}=\begin{pmatrix}-1\\0\\1\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=-2\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}1\\1\\3\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\overrightarrow{AD}=\begin{pmatrix}1\\k\\0\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}0\\k\\-3\end{pmatrix}$ Normal vector $\mathbf{n}_1$ of $p_1=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$ Normal vector $\mathbf{n}_2$ of $p_2=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\k\\-3\end{pmatrix}=\begin{pmatrix}-k\\3\\k\end{pmatrix}$ Since p_1 makes an angle of $\frac{\pi}{3}$ with p_2, $\frac{\lvert\mathbf{n}_1.\mathbf{n}_2\rvert}{\lvert\mathbf{n}_1\rvert\lvert\mathbf{n}_2\rvert}=\cos\frac{\pi}{3}$	Many students used the correct method but made numerous errors seen in vector product operation to obtain the normal to planes p_1 and p_2. This had a knock-on effect on the other parts of the question. This part required the direct application of formula to obtain the angle between planes in order to show that $k=\frac{\sqrt{6}}{2}$. However students made errors in obtaining normals to planes p_1 and p_2. Also some students wrongly

	$\dfrac{\left\|\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-k\\3\\k\end{pmatrix}\right\|}{\sqrt{2}\sqrt{2k^2+9}}=\dfrac{1}{2}$ $\dfrac{\|2k\|}{\sqrt{2}\sqrt{2k^2+9}}=\dfrac{1}{2}$ Squaring both sides, we have $\dfrac{4k^2}{2\left(2k^2+9\right)}=\dfrac{1}{4}$ $8k^2=2k^2+9$ $k^2=\dfrac{3}{2}$ Since k is a positive real number, we have $k=\sqrt{\dfrac{3}{2}}=\dfrac{\sqrt{6}}{2}$ (shown)	stated that the formula gave the sine of the angle. A handful students were not able to find the correct pair of vectors to "cross-product" Example, to find normal to p_1 which contains *A, B and C*: $\overrightarrow{OA}\times\overrightarrow{OB}$ [WRONG] A mark was deducted for not including $k=-\dfrac{\sqrt{6}}{2}$ and rejecting the answer accordingly.
	$\overrightarrow{XC}$ is perpendicular to p_1 $\Rightarrow$ $\overrightarrow{XC}$ is parallel to $\mathbf{n}_1$ $\Rightarrow \overrightarrow{XC}=m\begin{pmatrix}-1\\0\\1\end{pmatrix}$ $\Rightarrow \overrightarrow{OX}=\overrightarrow{OC}-m\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\3\end{pmatrix}-m\begin{pmatrix}-1\\0\\1\end{pmatrix}$	This part was badly attempted. There was a lack of clarity in presentation for this part. Many students were able to write out the equation of line *XC*: **r** = $\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}$, but instead of $\overrightarrow{OX}=\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}$ [RIGHT], many wrote $\overrightarrow{XC}=\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}$ [WRONG] This was an indication of poor understanding of the equation of line!

	Equation of p_2 is $\mathbf{r}.\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\begin{pmatrix}1\\0\\3\end{pmatrix}.\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\sqrt{6}$ **i.e.** $\mathbf{r}.\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\sqrt{6}$ Since X lies on p_2, we have $\left[\begin{pmatrix}1\\1\\3\end{pmatrix}-m\begin{pmatrix}-1\\0\\1\end{pmatrix}\right].\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\sqrt{6}$ $\Rightarrow\left(-\frac{\sqrt{6}}{2}+3+\frac{3\sqrt{6}}{2}\right)-m\left(\frac{\sqrt{6}}{2}+0+\frac{\sqrt{6}}{2}\right)=\sqrt{6}$ $\Rightarrow\sqrt{6}+3-m\left(\sqrt{6}\right)=\sqrt{6}$ $\Rightarrow m\left(\sqrt{6}\right)=3$ $\therefore m=\frac{3}{\sqrt{6}}$ $\Rightarrow\ \overrightarrow{XC}=\frac{3}{\sqrt{6}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$	
	$\left\|\overrightarrow{XC}\right\|=\left\|\sqrt{\frac{3}{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\|=\sqrt{\frac{3}{2}}\sqrt{2}=\sqrt{3}$ $\left\|\overrightarrow{AC}\right\|=\left\|\begin{pmatrix}0\\1\\0\end{pmatrix}\right\|=1$ Since XC is perpendicular to p_1 and AC lies on p_1, we have XC perpendicular to AC. Therefore, area of triangle AXC is $\frac{1}{2}\left\|\overrightarrow{XC}\right\|\left\|\overrightarrow{AC}\right\|=\frac{1}{2}\left(\sqrt{3}\right)(1)=\frac{\sqrt{3}}{2}$	Many students failed to see that triangle *AXC* is a right-angled triangle and the area can be quite easily obtained by $\frac{1}{2}(base)(\perp height)$ i.e. $\frac{1}{2}(AC)(XC)$. Instead many students found the area by finding $\frac{1}{2}\left\|\overrightarrow{AC}\times\overrightarrow{XC}\right\|$.

Section B: Probability and Statistics [60 marks]

5 Anand, Beng, Charlie, Dayanah and 6 other people attend a banquet dinner, and are to sit at a round table.

(i) Dayanah will only sit next to Anand, Beng or Charlie (and no one else), and Anand, Beng and Charlie do not want to sit next to each other. Find the number of ways the 10 people can seat themselves around the table. [4]

(ii) As part of dinner entertainment, 4 people from the table are chosen to participate in a game.

Among Anand, Beng and Charlie, if any one of them is chosen, the other two will refuse to participate in the game. Furthermore, Dayanah refuses to participate unless at least one of Anand, Beng or Charlie is also chosen.

Find the number of ways the 4 people can be chosen for the game. [3]

5(i)	No. of ways to arrange 2 of A/B/C next to D, to form a unit $= {}^{3}C_{2} \times 2!$ No. of ways to arrange the unit and 6 other people in a circle $= (7 - 1)!$ No. of ways to insert the remaining person from A/B/C $= {}^{5}C_{1}$ $\therefore$ Total no. of ways = ${}^{3}C_{2} \times 2! \times (7-1)! \times {}^{5}C_{1} = 21600$	**In general, students lost marks for this whole question because they failed to understand the question.** For this part, many students interpreted "sit next to" as applying to one side (should be both sides) and therefore were incorrect. **Students should also take the effort to describe/explain their steps** to possibly gain partial credit for their methodology.
(ii)	Case 1: A, B or C not chosen Number of ways = ${}^{6}C_{4} = 15$ Case 2: Exactly one of A, B or C chosen Number of ways = ${}^{3}C_{1} \times {}^{7}C_{3} = 105$ Total number of ways = 15 + 105 = 120 [Note: Case 2 may be split into: Case 2a: Exactly one of A, B or C chosen but not Dayanah, (Number of ways = ${}^{3}C_{1} \times {}^{6}C_{3} = 60$) And Case 2b: Dayanah and exactly one of A, B or C chosen (Number of ways = ${}^{3}C_{1} \times {}^{6}C_{2} = 45$).]	A large number of students assumed that when A, B or C were chosen, D must be chosen when this is not the case. Otherwise this part was the better done of the two.

6 A bag contains four identical counters labelled with the digits 0, 1, 2, and 3. In a game, Amira chooses one counter randomly from the bag and then tosses a fair coin. If the coin shows a Head, her score in the game is the digit labelled on the counter chosen. If the coin shows a Tail, her score in the game is the negative of the digit labelled on the counter chosen. T denotes the score in a game.

(i) Find the probability distribution of T. [2]

(ii) Amira tosses the coin and it shows a Tail. Find the probability that $T < -1$. [3]

(iii) Amira plays the game twice. Find the probability that the sum of her two scores is positive. [3]

<table>
<tr>
<td>6(i)</td>
<td>
<table>
<tr><td>t</td><td>-3</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>$P(T=t)$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{4}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td></tr>
</table>
</td>
<td>Most students could do this part correctly.</td>
</tr>
<tr>
<td>(ii)</td>
<td>$P(T<-1 \mid \text{coin shows a tail})$
$= \dfrac{P(\text{counter shows 2 or 3 and coin shows a Tail})}{P(\text{coin shows a Tail})}$
$\dfrac{\left(\frac{2}{4}\right)\times\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \dfrac{1}{2}$</td>
<td>Many students could not identify that this was a conditional probability question. Details were also insufficient among the rest of the answers.</td>
</tr>
<tr>
<td>(iii)</td>
<td>Required probability $= \dfrac{1}{2}[1 - P(\text{sum of scores} = 0)]$
$= \dfrac{1}{2}\left[1-\left(\dfrac{1}{4}\times\dfrac{1}{4}+\dfrac{1}{8}\times\dfrac{1}{8}\times 6\right)\right]$
$= \dfrac{27}{64}$

Alternative:
Required probability
$= 2[P(0,1)+P(0,2)+P(0,3)] + 2[P(-1,2)+P(-1,3)] + 2P(-2,3)$
$+ 2[P(1,2)+P(1,3)+P(2,3)] + [P(1,1)+P(2,2)+P(3,3)]$
$= 2\left[\left(\dfrac{1}{4}\times\dfrac{1}{8}\times 3\right)\right] + 2\left[\left(\dfrac{1}{8}\times\dfrac{1}{8}\times 2\right)\right] + 2\left(\dfrac{1}{8}\times\dfrac{1}{8}\right) + 2\left(\dfrac{1}{8}\times\dfrac{1}{8}\times 3\right) + 3\left(\dfrac{1}{8}\times\dfrac{1}{8}\right)$
$= \dfrac{27}{64}$
</td>
<td>Most students attempted the second method, but there were many common errors:
(1) not considering permutations for each case,
(2) missing out cases (such as 1 + 1)
(3) forgetting that $P(T=0) = \frac{1}{4}$.

A smaller number tried to use a table of outcomes, but were not very successful as they did not notice that not all outcomes had equal probabilities.</td>
</tr>
</table>

7 It is generally accepted that a person's diet and cardiorespiratory fitness affects his cholesterol levels. The results of a study on the relationship between the cholesterol levels, C mmol/L, and cardiorespiratory fitness, F, measured in suitable units, on 8 individuals with similar diets are given in the following table.

Cardiorespiratory Fitness (F units)	55.0	50.7	45.3	40.2	34.7	31.9	27.9	26.0
Cholesterol (C mmol/L)	4.70	4.98	5.30	5.64	6.04	6.30	6.99	6.79

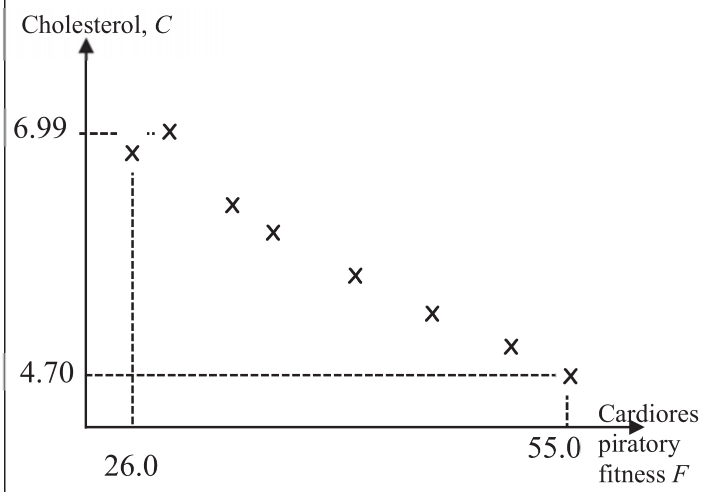
(i) Draw a scatter diagram of these data. Suppose that the relationship between F and C is modelled by an equation of the form $\ln C = aF + b$, where a and b are constants. Use your diagram to explain whether a is positive or negative. [4]

(ii) Find the product moment correlation coefficient between $\ln C$ and F, and the constants a and b for the model in part **(i)**. [3]

(iii) Bronz is a fitness instructor. His cardiorespiratory fitness is 52.0 units. Estimate Bronz's cholesterol level using the model in **(i)** and the values of a and b in part **(ii)**. Comment on the reliability of the estimate. [2]

(iv) Bronz then had a medical checkup and found his actual cholesterol level to be 6.2 mmol/L. Assuming his cholesterol level is measured accurately, explain why there is a great difference between Bronz's cholesterol level and the estimated value in **(iii)**. [1]

(i)



$\ln C = aF + b \quad \Rightarrow \quad C = e^{aF+b}$

From the scatter diagram, as F increases, C decreases at a decreasing rate, hence $a < 0$.

- Many students gave a scatter diagram of $\ln C$ vs F. However, the question asked for a scatter diagram of *these data* & the given data is C and F.
- Please ensure that the **relative positions** of the 8 data points (and not 7 or 9 points) are correct.
- Many students did not label the max/min values of C and F. Amongst those who did, a large majority did not label the max C (6.99) and/or min F (26) correctly.

This part was poorly attempted.

- Many students say that the gradient of the points in the scatter diagram (C against F) are negatively correlated so it has a negative gradient and thus a is negative. BUT "a" is the gradient of the regression line of $\ln C$ on F and not C on F.

		• Alternative solution given by some students $\ln C$ decreases as C decreases. Thus as F increases, $\ln C$ decreases. This means that the gradient of the regression line of $\ln C$ on F is negative. However, they failed to **explain clearly** that "a" is the gradient of the regression line before concluding that "a" is negative.
(ii)	Product moment correlation $r = -0.992$ (3 sf) Model is $\ln C = -0.013371F + 2.2772$ (5 sf) Thus, $a = -0.0134$ (3 sf) and $b = 2.28$ (3 sf)	Some students identified the values of a and b wrongly, i.e., wrote $a = 2.28$ in spite of the fact they had claimed that $a < 0$ in (i).
(iii)	$\ln C = -0.013371(52.0) + 2.2772$ $C = \mathrm{e}^{1.581908} = 4.86$ Estimate of Bronz's cholesterol level is 4.86 mmol/L Since $r = -0.992$ is close to -1 which suggests that the linear model is a good one and Bronz's cardiorespiratory fitness level, $F = 52.0$ lies within the data range of $[26.0, 55.0]$, the estimate is reliable.	Some students used the inaccurate values of a and b, i.e., used $\ln C = -0.0134F + 2.28$ to compute C and got an inaccurate value. Many students mentioned only one of the 2 conditions needed for a reliable estimate with the regression line.
(iv)	Bronz's **diet** could be very high in cholesterol **compared to the 8 individuals** which resulted in his actual cholesterol level of 6.20 to be higher than the value of 4.86 mmol/L estimated using the values in (iv) which are based on the 8 individuals.	This part was very poorly attempted. Students need to check if there is a reason provided by the context before giving any other reasonable explanation. In this question, a reason was given (diet) and so this is the only answer accepted. The regression line that was used to find the estimate was based on the data from the 8 people with a similar diet. So the estimate of 4.86 mmol/L would be Bronz's cholesterol level if he was on the same diet.

8 A research laboratory uses a data probe to collect data for its experiments. There is a probability of 0.04 that the probe will give an incorrect reading. In a particular experiment, the probe is used to take 80 readings, and X denotes the number of times the probe gives an incorrect reading.

(i) State, in context, two assumptions (not conditions) necessary for X to be well modelled by a binomial distribution. [2]

(ii) Find the probability that between 5 and 10 (inclusive) incorrect readings are obtained in the experiment. [3]

8(i)	1. The **probability** of the probe giving an incorrect reading is **constant** at 0.04 for each reading. 2. Trial of whether a reading by the probe is incorrect is **independent** of other readings.	Not well answered. Despite the many reminders, common errors are: 1. Probability of getting an incorrect reading is independent of … 2. Two outcomes: correct or incorrect reading (given in question, not assumption) 3. Fixed number (80) of readings (given in question, not assumption) 4. Trial is on "reading correct or incorrect", not the probe (only one probe) nor the experiment (only one experiment).
(ii)	$X \sim B(80, 0.05)$ $P(5 \leq X \leq 10) = P(X \leq 10) - P(X \leq 4)$ $= 0.216$ (3sf)	Generally well done for this part. Common error is interpreting "inclusive" to apply only to 10 and not 5, i.e. wrongly gave $P(5 < X \leq 10)$. Some students wrongly gave $P(5 \leq X \leq 10) = P(X \leq 10) \times P(X \geq 5)$ which is not true as the events $X \leq 10$ and $X \geq 5$ are not independent.

When the probe gives an incorrect reading, it will give a reading that is 5% greater than the actual value.

(iii) Suppose the 80 readings are multiplied together to obtain a Calculated Value. Find the probability that the Calculated Value is **at least** 50% more than the product of the 80 actual values. [5]

Solution

(iii)	Let the actual i^{th} value be V_i. Product of 80 actual values = $V_1 \times V_2 \times \ldots \times V_{80}$ For an incorrect reading, V_i is read as $(1.05)V_i$ (5% greater) If there are x incorrect readings, Calculated Value = $V_1 \times V_2 \times \ldots \times V_{80}\,(1.05)^x$. $V_1 \times V_2 \times \ldots \times V_{80}(1.05)^x \geq 1.5\,(V_1 \times V_2 \times \ldots \times V_{80})$ $\Rightarrow (1.05)^x \geq 1.5$ $\Rightarrow x\ln(1.05) \geq \ln 1.5$ $\Rightarrow x \geq \dfrac{\ln 1.5}{\ln 1.05} = 8.31$ i.e. at least 9 incorrect readings Required probability = $P(X \geq 9)$ $= 1 - P(X \leq 8)$ $= 0.00468$ (3sf)	Most students did not know how to interpret the question. Many students tried to apply Central Limit Theorem for sample sum – note that CLT does not apply to product of sample. Of those who could interpret the question, many did not explain how they obtained $(1.05)^x \geq 1.5$ Some students presented 'product of the 80 actual values' wrongly as V^{80} instead of $V_1 \times V_2 \times \ldots \times V_{80}$ (80 different readings)

9 A Wheel Set refers to a set of wheel rim and tyre. The three types of wheel sets are the Clincher Bike Wheel Set, Tubular Bike Wheel Set and Mountain Bike Wheel Set.
The weight of a rim of a Clincher Bike Wheel Set follows a normal distribution with mean 1.5 kg and standard deviation 0.01 kg. The weight of its tyre follows a normal distribution with mean 110 g and standard deviation 5 g.

(i) Let C be the total weight in grams of a randomly chosen Clincher Bike Wheel Set in grams. Find $\mathrm{P}(C > 1620)$. [3]

(ii) State, in the context of the question, an assumption required in your calculation in **(i)**. [1]

Let T be the total weight in grams of a Tubular Bike Wheel Set, where $T \sim \mathrm{N}(\mu,\ 15^2)$.

(iii) The probability that the weight of a randomly chosen Clincher Bike Wheel Set exceeds a randomly chosen Tubular Bike Wheel Set by more than 150 g is smaller than 0.70351 correct to 5 decimal places. Find the range of values that μ can take. [5]

Let M be the total weight in grams of a randomly chosen Mountain Bike Wheel Set with mean 1800 g and standard deviation 20 g.

(iv) Find the probability that the mean weight of 50 randomly chosen Mountain Bike Wheel Sets is more than 1795 g. [3]

9(i)	$C \sim \mathrm{N}(1500 + 110,\ 10^2 + 5^2)$ $\mathrm{P}(A > 1620) = 0.185546 \approx 0.186$ (3sf)	Well attempted. Only a few students did not convert kilograms to grams. Note that $10^2 + 5^2 \neq (10+5)^2$
(ii)	Assume that the weight of a randomly chosen Clincher Bike rim and tyre are independent of each other. (This is required for the calculation of Var(C)) (Recall Var(X+Y) = Var(X) + Var(Y) if X and Y are independent) 	Poorly done. Many gave assumptions such as: - The weight of a randomly chosen Clincher Bike Wheelset is independent of another set. (There's only one set in (i)) - Assume that a set has only one rim and one wheel (This was already stated in the question)

(iii)	$C-T \sim N\left(1610-\mu,\ 125+15^2\right) \Rightarrow C-T \sim N\left(1610-\mu,\ 350\right)$ $\mathrm{P}\left(C-T>150\right)<0.70351$ $\mathrm{P}\left(Z \geq \dfrac{150-1610+\mu}{\sqrt{350}}\right)<0.70351$ $\mathrm{P}\left(Z \geq \dfrac{-1460+\mu}{\sqrt{350}}\right)<0.70351$ $\dfrac{-1460+\mu}{\sqrt{350}} \geq -0.534523$ $\mu \geq 1449.9999$ $\mu \geq 1450$	Note that GC table is not allowed as μ is not an integer value. Many students failed to standardise correctly. Many wrote 0.534523 instead of – 0.534523, indicating that they are still not able to identify the correct area for the invnorm command in GC.
(iv)	Since $n=50$ is large, by Central Limit Theorem, $\overline{M} \sim N\left(1800,\ \dfrac{400}{50}\right)$ approximately $\overline{M} \sim N\left(1800,\ 8\right)$ approximately $P\left(\overline{M}>1795\right)=0.96146 \approx 0.961\,(3\text{sf})$	Many students wrote $M \sim N\left(1800,\ 20^2\right)$ which is incorrect as it is not given in the question that M is normally distributed. Not many students knew that CLT has to be used for this part. Some applied CLT to M.
Students are able to find the variance and expectation of the random variables but would need to work on problems which require them to standardise (especially those that involve inequality)		

10 (a) Two random samples of different sample sizes of households in the town of Aimek were taken to find out the mean number of computers per household there. The first sample of 50 households gave the following results.

Number of computers	0	1	2	3	4
Number of households	5	12	18	10	5

The results of the second sample of 60 households were summarised as follows.

$$\sum y = 118 \qquad \sum y^2 = 314\,,$$

where y is the number of computers in a household.

(i) By combining the two samples, find unbiased estimates of the population mean and variance of the number of computers per household in the town. [4]

(ii) Describe what you understand by 'population' in the context of this question. [1]

(b) Past data has shown that the working hours of teachers in a city are normally distributed with mean 48 hours per week. In a recent study, a large random sample of n teachers in the city was surveyed and the number of working hours per week was recorded. The sample mean was 46 hours and the sample variance was 131.1 hours^2. A hypothesis test is carried out to determine whether the mean working hours per week of teachers has been reduced.

(i) State appropriate hypotheses for the test. [1]

The calculated value of the test statistic is $z = -1.78133$ correct to 5 decimal places.

(ii) Deduce the conclusion of the test at the 2.5 % level of significance. [2]

(iii) Find the value of n. [3]

(iv) In another test, using the same sample, there is significant evidence at the $\alpha\%$ level that there is a change in the mean working hours per week of teachers in that city. Find the smallest possible integral value of α. [2]

This question is poorly done in general. Students display a lack of understanding of hypothesis testing concepts and procedures, coupled with a perpetual incompetency in obtaining the unbiased estimates from given data. Many of the solutions presented also reveal a lack in comprehension by students on what the questions are asking for.		
10(a)(i)	Let X be the number of computers in each household in the first sample. Using GC, $\sum fx = 98$, $\sum fx^2 = 254$ Unbiased estimate of the population mean $= \dfrac{98+118}{50+60} = \dfrac{108}{55} (\approx 1.96)$ Unbiased estimate of the population variance $= \dfrac{1}{110-1}\left[(254+314) - \dfrac{(98+118)^2}{110}\right] = \dfrac{7912}{5995} (\approx 1.32)$	The most common error for the unbiased estimate for the population mean from the combined sample is taking the averages of the unbiased estimates for the respective sample sizes, i.e. $\dfrac{1}{2}\left(\dfrac{98}{50}+\dfrac{118}{60}\right)$. The most common error for the unbiased estimate for the population variance is using the formula for pooled variance given in the MF26.
(a)(ii)	The population in this question refers to <u>all the households in the town</u>.	Majority of students are able to correctly identify what the population is. Common errors include failing to state that the households are from the town (and not anywhere else) and for referring the population to computers instead. A small handful gave the definition of a random sample here, which indicates a lack in comprehension what the question is asking for.
(b)(i)	Let μ be the population mean working hours of teachers in the city. $H_0 : \mu = 48$ $H_1 : \mu < 48$	This part is generally well done.
(ii)	Critical region $= \{z : z \le -1.960\}$ Since $z_{cal} = -1.781 > -1.960$, we do not reject H_0 **OR** p-value $= P(Z \le -1.781) = 0.0375 > 0.025$, we do not reject H_0 There is <u>insufficient</u> evidence at <u>2.5% level of significance</u> that the <u>mean working hours per week of teachers in the city has been reduced</u>.	Most students know that the finding the critical region is key to stating the correct conclusion, and most are able to obtain the correct critical value (not the region though). This leads to an overall confusion with whether to reject/accept H_0,

		or whether to reject/accept H_1. Subsequently, the confusion with whether there is sufficient/insufficient evidence persists as well. There is a sizable number of students who are of the impression that the decision for rejection of H_0 is the conclusion.
(iii)	$s^2 = \frac{n}{n-1}(131.1)$ Given z= $-1.781 \Rightarrow \frac{46-48}{\left(\sqrt{\frac{131.1}{n-1}}\right)} = -1.781$ $-2 = -1.781\sqrt{\frac{131.1}{n-1}}$ $\sqrt{n-1} = \frac{-1.781\sqrt{131.1}}{-2}$ $n = 105$	Most students use 131.1 as the population variance, therefore losing marks for accuracy.
(iv)	p-value for the two tailed test = $2 \times \mathrm{P}(Z \le -1.781) = 0.0749$ (accept 0.075) For H_0 to be rejected, $p\text{-value} \le \frac{\alpha}{100}$ $\alpha \ge 7.49$ Smallest value of α = 8	Students who rightly approach this part using a 2-tailed test are able to obtain the correct result. There is a small handful who mistakenly compared the test statistic -1.78133 to the level of significance.

TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME	
CIVICS GROUP	

H2 MATHEMATICS **9758/01**

Paper 1 **18 September 2019**

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiners' Use	
1	
2	
3	
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9	
10	
11	
Total	

[Turn Over

1 Using an algebraic method, solve the inequality $\dfrac{4x^2-x-1}{(2x-1)(x+1)} \geq 1$. [4]

Hence or otherwise, solve the inequality

$$\frac{4e^{2x}-e^x-1}{(2e^x-1)(e^x+1)} \geq 1,$$

leaving your answers in exact form. [2]

2 Referred to the origin O, A is a fixed point with position vector $\mathbf{a}$, and $\mathbf{d}$ is a non-zero vector. Given that a general point R has position vector $\mathbf{r}$ such that $\mathbf{r} \times \mathbf{d} = \mathbf{a} \times \mathbf{d}$, show that $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, where λ is a real constant. Hence give a geometrical interpretation of $\mathbf{r}$. [3]

Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$. By writing $\mathbf{r}$ as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, use $\mathbf{r} \times \mathbf{d} = \mathbf{a} \times \mathbf{d}$ to form three equations which represent cartesian equations of three planes. State the relationship between these three planes. [3]

3 **(i)** The sum of the first n terms of a sequence is denoted by S_n. It is known that $S_5 = -30$, $S_{14} = 168$ and $S_{18} = 9S_{10}$. Given that S_n is a quadratic polynomial in n, find S_n in terms of n. [4]

(ii) The nth term of the sequence is denoted by T_n. Find an expression for T_n in terms of n. Hence find the set of values of n for which $|T_n| < 12$. [4]

4 **(i)** Given that f is a strictly increasing continuous function, explain, with the aid of a sketch, why

$$\frac{1}{n}\left\{f\left(\frac{0}{n}\right)+f\left(\frac{1}{n}\right)+\ldots+f\left(\frac{n-1}{n}\right)\right\} < \int_0^1 f(x)\,dx,$$

where n is a positive integer. [3]

(ii) Hence find the least exact value of k such that $\dfrac{1}{n}\left(e^{\frac{0}{n}}+e^{\frac{2}{n}}+e^{\frac{4}{n}}+\ldots+e^{\frac{2n-2}{n}}\right) < k$, where n is a positive integer. [2]

5 It is given that $f(x) = 4x - x^3$.

(i) On separate diagrams, sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$, showing clearly the coordinates of any axial intercept(s) and turning point(s). [4]

(ii) Find the exact value of the constant k for which $\int_0^k |f(x)|\,dx = \int_{-2}^{2} f(|x|)\,dx$. [4]

6 Show that $2\cos(r\theta)\sin\theta \equiv \sin[(r+1)\theta] - \sin[(r-1)\theta]$. [1]

By considering the method of differences, find $\sum_{r=1}^{n} \cos(r\theta)$ where $0 < \theta < \frac{\pi}{2}$.

(You need not simplify your answer.) [3]

Hence evaluate the sum

$$\cos\left(\frac{19}{6}\pi\right) + \cos\left(\frac{20}{6}\pi\right) + \cos\left(\frac{21}{6}\pi\right) + \cdots + \cos\left(\frac{56}{6}\pi\right) + \cos\left(\frac{57}{6}\pi\right),$$

leaving your answer in exact form. [4]

7 The function f is defined by

$$f(x) = \begin{cases} n - x & \text{for } n \le x < n+1, \text{where } n \text{ is any positive odd integer,} \\ x - \dfrac{n}{2} & \text{for } n \le x < n+1, \text{where } n \text{ is any positive even integer.} \end{cases}$$

(i) Show that $f(1.5) = -0.5$ and find $f(2.5)$. [2]

(ii) Sketch the graph of $y = f(x)$ for $1 \le x < 5$. [2]

(iii) Does f have an inverse for $1 \le x < 5$? Justify your answer. [2]

(iv) The function g is defined by $g : x \mapsto \dfrac{2x-1}{x+1}, x \in \mathbb{R}, x \neq -1$. For $2 \le x < 3$, find an expression for $gf(x)$ and hence, or otherwise, find $(gf)^{-1}\left(\dfrac{2}{3}\right)$. [4]

[Turn Over

8 At the start of an experiment, a particular solid substance is placed in a container filled with water. The solid substance will begin to gradually dissolve in the water. Based on experimental data, a student researcher guesses that the mass, x grams, of the remaining solid substance at time t seconds after the start of the experiment satisfies the following differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{k-1}(x-1)(x-k),$$

where k is a real constant and $k>3$.

(i) Show that a general solution of this differential equation is $\ln\left|\frac{x-k}{x-1}\right|=t+C$, where C is an arbitrary real constant. [3]

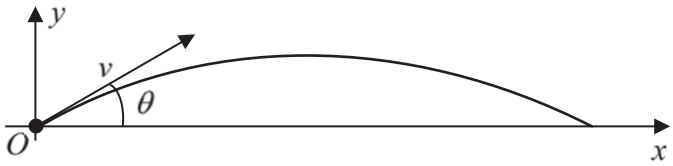
For the rest of the question, let $k=4$. It is given that the initial mass of the solid substance is 3 grams.

(ii) Express the particular solution of the differential equation in the form $x=\mathrm{f}(t)$. [4]

(iii) Find the exact time taken for the mass of the solid substance to become half of its initial value. [2]

(iv) Sketch the part of the curve with the equation found in part **(ii)** which is relevant in this context. [2]

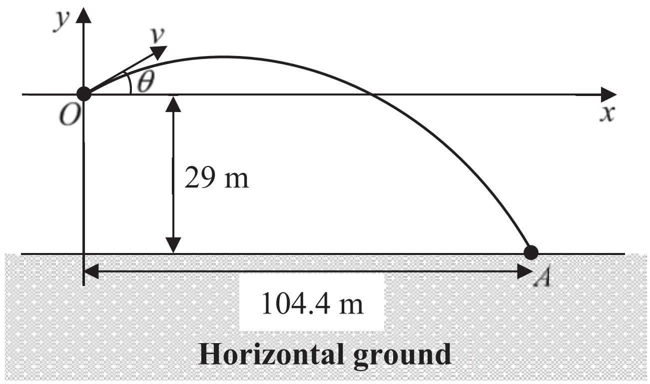
9



From a point O, a particle is projected with velocity $v\,\text{ms}^{-1}$ at a fixed angle of elevation θ from the horizontal, where v is a positive real constant and $0 < \theta < \frac{\pi}{2}$. The horizontal displacement, x metres, and the vertical displacement, y metres, of the particle at time t seconds may be modelled by the parametric equations

$$x = (v\cos\theta)t, \qquad y = (v\sin\theta)t - 5t^2.$$

(i) Using differentiation, find the maximum height achieved by the particle in terms of v and θ. (You need not show that the height is a maximum.) [3]



The particle is now projected from point O situated at a height of 29 m above the horizontal ground. The particle hits the ground at A which is at a horizontal distance of 104.4 m from O. The maximum height (measured from horizontal ground) that the particle reaches is 57.8 m. The diagram above shows the path of the particle (not drawn to scale).

(ii) Find the time taken for the particle to hit the ground at A and find the corresponding value of v. [5]

(iii) Find the exact gradient of the tangent at A. [2]

[Turn Over

10 Two houses, A and B, have timber cladding on the end of their shed roofs, consisting of rectangular planks of decreasing length.

(i) The first plank of the roof of house A has length 350 cm and the lengths of the planks form a geometric progression. The 20^{th} plank has length 65 cm. Show that the total length of all the planks must be less than 4128 cm, no matter how many planks there are. [4]

House B consists of only 20 planks which are identical to the first 20 planks of house A.

(ii) The total length of all the planks used for house B is L cm. Find the value of L, leaving your answer to the nearest cm. [2]

(iii) Unfortunately the construction company misunderstands the instructions and covers the roof of house B wrongly, so that the lengths of the planks are in arithmetic progression with common difference d cm. If the total length of the 20 planks is still L cm and the length of the 20^{th} plank is still 65 cm, find the value of d and the length of the longest plank. [4]

It is known that house C has timber cladding on the end of its shed roof, consisting of rectangular planks of increasing length. The first plank of the roof of house C has length 65 cm and the lengths of the planks are in arithmetic progression with common difference 11 cm. The total length of the first N planks of the roof of house C exceeds 20 640 cm. Find the least value of N. [3]

11 A circle with a fixed radius r cm is inscribed in an isosceles triangle ABC where $\angle ABC = \theta$ radians and $AB = BC$. The circle is in contact with all three sides of the triangle at the points D, E and F, as shown in Fig. 1.

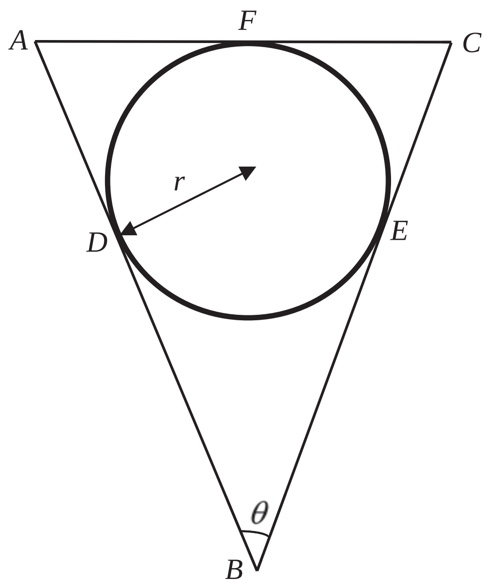


Fig. 1

(i) Show that the length BD can be expressed as $r\cot\dfrac{\theta}{2}$ cm. [1]

(ii) By finding the length AD in terms of r and θ, show that the perimeter of the triangle ABC can be expressed as $4r\cot\left(\dfrac{\pi}{4}-\dfrac{\theta}{4}\right)+2r\cot\dfrac{\theta}{2}$ cm. [2]

(iii) Using differentiation, find the exact value of θ such that the perimeter of the triangle ABC is minimum. Find the minimum perimeter of triangle ABC, leaving your answer in the form $a\sqrt{b}r$ cm, where a and b are positive integers to be determined. [6]

[Turn Over

Fig. 2 shows a decorative item in the shape of a sphere with a fixed radius inscribed in an inverted right circular cone with base radius R cm and slant height $2R$ cm. The sphere is in contact with the slopes and the base of the cone.

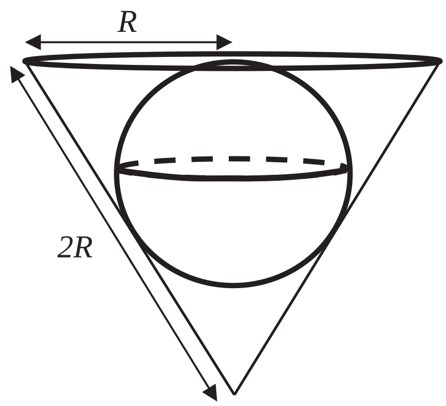


Fig. 2

To make the item glow in the dark, the sphere is filled entirely with fluorescent liquid. However, due to a manufacturing defect, the fluorescent liquid leaks into the bottom of the inverted cone at a rate of 2 cm^3 per minute.

(iv) Assuming that the leaked liquid in the inverted cone will not reach the exterior of the sphere, find the exact rate of increase of the depth of the leaked liquid in the inverted cone when the volume of the leaked liquid in the inverted cone is 24π cm^3. Express your answer in terms of π. [6]

[The volume of a cone of base radius r and height h is given by $\frac{1}{3}\pi r^2 h$.]

End of Paper

TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ____

CIVICS GROUP: ____

H2 MATHEMATICS 9758/02

Paper 2

25 SEPTEMBER 2019

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

For Examiners' Use	
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Total	

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

[Turn Over

Section A: Pure Mathematics [40 marks]

1 Given that $y = \sqrt{\left(5 - e^{2x}\right)}$, show that

$$y\frac{dy}{dx} = -e^{2x} \ . \qquad [1]$$

By further differentiation of this result, find the Maclaurin series for y up to and including the term in x^2. [4]

2 Let z_1 and z_2 be the roots of the equation

$$z^2 - ikz - 1 = 0$$

where $0 < k < 2$ and $0 < \arg(z_1) < \arg(z_2) < \pi$.

(i) Find z_1 and z_2 in cartesian form, $x + iy$, where x and y are real constants in terms of k. [3]

For the rest of the question, let $\arg(z_1) = \theta$, $0 < \theta < \frac{\pi}{2}$.

Let w be a complex number such that wz_1 is purely imaginary and $\arg\left(\frac{z_1}{w}\right) = \frac{7\pi}{6}$.

(ii) Show that $\arg(z_1) = \frac{\pi}{3}$. [3]

(iii) Find z_2, leaving your answer in the exact form. [3]

3 The plane p_1 contains the point *A* with coordinates $(1,\ 2,\ 8)$ and the line *l* with equation

$\mathbf{r} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, where λ is a real parameter.

(i) Show that a cartesian equation of plane p_1 is $3x - y + z = 9$. [2]

The foot of perpendicular from point *A* to line *l* is denoted as point *F*.

(ii) Find the coordinates of point *F*. [3]

(iii) Point *B* has coordinates $(1, -4,\ 2)$. Find the exact area of triangle *ABF*. [2]

(iv) Point *C* has coordinates $(-1,\ 6,\ 6)$. By finding the shortest distance from point *C* to p_1, find the exact volume of tetrahedron *ABFC*. [4]

[Volume of tetrahedron = $\dfrac{1}{3} \times$ base area $\times$ perpendicular height]

(v) Point *D* lies on the line segment *AC* such that $AD : DC = 1 : 3$. Another plane p_2 is parallel to p_1 and contains point *D*. Find a cartesian equation of p_2. [2]

4 A designer is tasked to design a 3-dimensional ornament for the company. He then programs two curves, C_1 and C_2, into the computer software. The curve C_1 has the equation $y=\sqrt{\left(2-x^2\right)}$ and the curve C_2 has the equation $y=x^3$. The coordinates of the point of intersection of C_1 and C_2 is $(1,1)$.

(i) Find the exact area of the finite region bounded by C_1, C_2 and the positive x-axis. [You may use the substitution $x=\sqrt{2}\sin\theta$ where $0\le\theta\le\frac{\pi}{2}$.] [6]

(ii) The designer wants to know how much material is needed to construct the 3-dimensional ornament. He finds out that the surface area generated by the segment of a curve $y=\mathrm{f}(x)$ between $x=a$ and $x=b$ rotated through 360° about the x-axis is given by

$$S=\int_a^b 2\pi y\sqrt{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)}\,\mathrm{d}x \text{ where } y=\mathrm{f}(x),\ y\ge 0,\ a\le x\le b.$$

The 3-dimensional ornament is formed when the finite region bounded by C_1, C_2 and the positive x-axis is rotated through 360° about the x-axis. Find the exact surface area of the 3-dimensional ornament. [7]

Section B: Probability and Statistics [60 marks]

5 A biased die in the form of a regular tetrahedron has its four faces labelled 2, 3, 4 and 5, with one number on each face. The die is tossed and X is the random variable denoting the number on the face which the die lands. The probability distribution of X is shown in the table below, where $0<v<u<1$.

x	2	3	4	5
$\mathrm{P}(X=x)$	u	v	u	v

(i) Find $\mathrm{E}(X)$ in terms of u. [2]

(ii) Given that $\mathrm{Var}(X)=1.16$, find u and v. [4]

6 A computer game consists of at most 3 rounds. The game will stop when a player clears 2 rounds or does not clear 2 consecutive rounds. The probability that a player clears round 1 is 0.6. The conditional probability that the player clears round 2 given that he clears round 1 is half the probability that he clears round 1. The conditional probability that the player clears round 2 given that he does not clear round 1 is the same as the probability that he clears round 1.

(i) Find the probability that a player plays 3 rounds. [1]

(ii) Find the probability that a player clears round 1 given that he does not clear round 2. [2]

(iii) The total probability that a player plays 3 rounds and clears round 3 is 0.2. Find the probability that a player clears exactly 2 rounds. [2]

In order to play the computer game, Eric needs to type a 6-digit passcode to unlock the game. The 6-digit passcode consists of digits 0-9 and the digits do not repeat.

How many possible passcodes can there be if

(iv) the 6-digit passcode is odd? [2]

(v) there are exactly 3 odd digits in the 6-digit passcode? [2]

7 MHL bakery sells mini breads that weighs an average of 45 grams each. A customer claims that the bakery is overstating the average weight of mini breads. To test this claim, a random sample of 80 mini breads are selected from MHL bakery and the weight, x grams, of each mini bread is measured. The results are summarised as follows.

$$n = 80 \qquad \sum x = 3571 \qquad \sum x^2 = 159701$$

Calculate unbiased estimates of the population mean and variance of the weight of mini breads. [2]

Test, at the 4% level of significance, whether there is sufficient evidence to support the customer's claim. [4]

From past records, it is known that the weights of the mini breads from MHL bakery are normally distributed with standard deviation 1.5 grams. To further investigate the customer's claim, the bakery records the weights of another 20 randomly selected mini breads and the average weight for the second sample is k grams.
Based on the combined sample of 100 mini breads, find the range of values of k such that the customer's claim is valid at the 4% level of significance. [4]

8 In a chemical reaction, the amount of catalyst used, x grams, and the resulting reaction times, y seconds, were recorded and the results are given in the table.

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
y	62.1	51.2	44.1	39.1	35.0	k	33.0	31.4	29.5

The equation of the regression line of y on x is $y = 68.8067 - 7.12667x$, correct to 6 significant figures.

(i) Show that $k = 37.3$, correct to 1 decimal place. [2]

(ii) Draw a scatter diagram for these values, labelling the axes clearly. [1]

It is suggested that the relationship between x and y can be modelled by one of the following formulae

(A) $y = a + bx$

(B) $y = c + dx^2$

(C) $y = e + \dfrac{f}{x}$

where a, b, c, d, e and f are constants.

(iii) Find the value of the product moment correlation coefficient for each model. Explain which is the best model and find the equation of a suitable regression line for this model. [5]

(iv) By using the equation of the regression line found in part **(iii)**, estimate the reaction time when the amount of catalyst used is 4.2 grams. Comment on the reliability of your estimate. [2]

9 The masses in grams of Envy apples have the distribution $\mathrm{N}\left(\mu,\sigma^2\right)$.

(i) For a random sample of 8 Envy apples, it is given that the probability that the sample mean mass is less than 370 grams is 0.25, and the probability that the total mass of these 8 Envy apples exceeds 3000 grams is 0.5. Find the values of μ and σ. [4]

For the rest of the question, use $\mu = 380$ and $\sigma = 20$.

(ii) Find the probability that the total mass of 8 randomly chosen Envy apples is between 2900 grams and 3100 grams. [2]

The masses in grams of Bravo apples have the distribution $\mathrm{N}\left(250,18^2\right)$.

To make a fruit platter, a machine is used to slice the apples and remove the cores. After slicing and removing the cores, the mass of an Envy apple and the mass of a Bravo apple will be reduced by 30% and 20% respectively. A fruit platter consists of 8 randomly chosen Envy apples and 12 randomly chosen Bravo apples.

(iii) Find the probability that the total mass of fruits, after slicing and removing the cores, in a fruit platter exceeds 4.5 kg. [4]

(iv) State an assumption needed for your calculations in parts **(ii)** and **(iii)**. [1]

To beautify the fruit platter, fruit carving is done on the apples after slicing and removing their cores. The carving reduced the masses of each apple (after slicing and removing its core) by a further 10%.

Let p be the probability that the total mass of fruits in a fruit platter, with carving done, exceeds 4.1 kg. Without calculating p, explain whether p is higher, lower or the same as the answer in part **(iii)**. [1]

10 **(a)** In a packet of 10 sweets, it is given that six of them are red, three of them are yellow and the remaining one is blue. 5 sweets are chosen randomly from the packet of sweets and R denotes the number of sweets that are red. Explain clearly why R cannot be modelled by a binomial distribution. [1]

(b) In a food company, a large number of sweets are produced daily and it is given that $100p\%$ of the sweets produced are red. The sweets are packed into packets of 10 each. Assume now that the number of sweets that are red in a packet follows a binomial distribution.

(i) It is given that the probability of containing exactly five red sweets in a randomly chosen packet of sweets is 0.21253. Show that p satisfies an equation of the form $p(1-p)=k$, where k is a constant to be determined. Hence find the possible values of p. [3]

For the rest of the question, use $p = 0.6$.

(ii) A packet of sweets is randomly chosen. Find the probability that there is at most 8 red sweets given that there is more than 2 red sweets. [3]

(iii) Two packets of sweets are selected at random. Find the probability that one of the packets contains at most 5 red sweets and the other packet contains at least 5 red sweets. [3]

(iv) Two packets of sweets are selected at random. Find the probability that the difference in the number of red sweets in the two packets is at least 8. [3]

End of Paper

1 Express $\dfrac{12}{x+1}-(7-x)$ as a single simplified fraction. [1]

Without using a calculator, solve $\dfrac{12}{x+1} \leq 7-x$. [3]

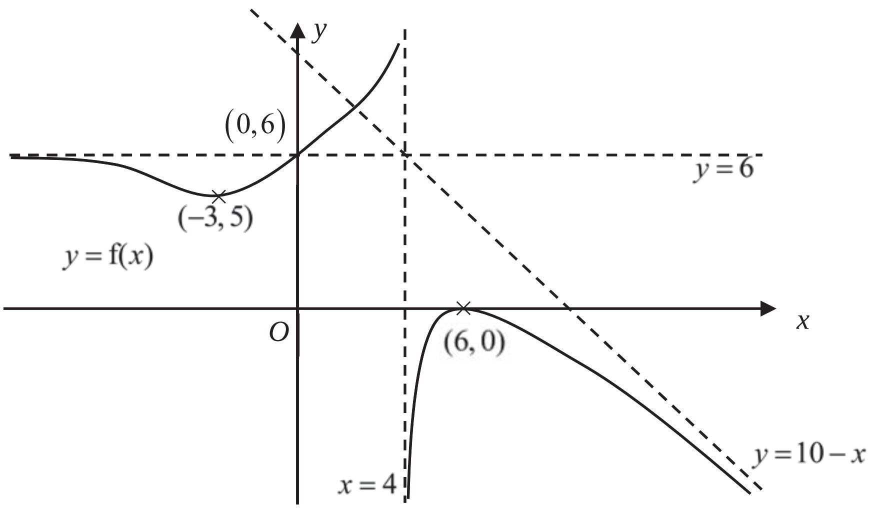
2 **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}\left(x^2\right)$. [1]

(ii) Hence, or otherwise, evaluate $\int_0^1 x\tan^{-1}\left(x^2\right)\mathrm{d}x$ exactly. [3]

3 **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(3x^2 2^x\right)$. [2]

(ii) Find the equation of the tangent to the curve $y = 3x^2 2^x$ at the point where $x = 1$, giving your answer in exact form. [3]

4 The graph for $y = \mathrm{f}(x)$ is given below, where $y = 10 - x$, $y = 6$ and $x = 4$ are asymptotes. The turning points are $(-3, 5)$ and $(6, 0)$, and the graph intersects the y-axis at $(0, 6)$.



On separate diagrams, sketch the graphs of

(i) $y = \mathrm{f}(|x|)$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

5 Referred to the origin O, points P and Q have position vectors $3\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$ respectively. Point M is a point on QP extended such that $PM{:}QM$ is 2:3.

(i) Find the position vector of point M in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(ii) Find $\overrightarrow{PQ}\times\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

(iii) State the geometrical meaning of $\dfrac{\left|\overrightarrow{PQ}\times\overrightarrow{OM}\right|}{\left|\overrightarrow{PQ}\right|}$. [1]

6 A curve C has equation $y=\mathrm{f}(x)$, where the function f is defined by

$$\mathrm{f}: x \mapsto \frac{12-3x}{x^2+4x-5}, \quad x\in\mathbb{R}, x\neq -5, x\neq 1.$$

(i) Find algebraically the range of f. [3]

(ii) Sketch C, indicating all essential features. [4]

(iii) Describe a pair of transformations which transforms the graph of C on to the graph of $y=\dfrac{9-x}{x^2-6x}$. [2]

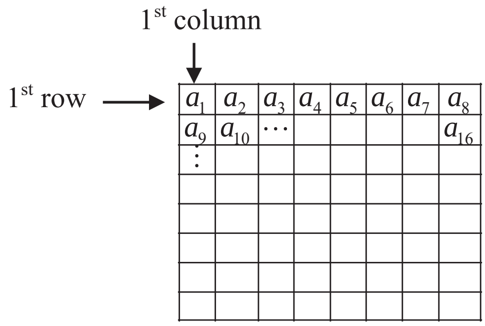
7 Given that $\sin^{-1} y=\ln(1+x)$, where $0<x<1$, show that $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{1-y^2}$. [2]

(i) By further differentiation, find the Maclaurin expansion of y in ascending powers of x, up to and including the term in x^2. [4]

(ii) Use your expansion from **(i)** and integration to find an approximate expression for $\displaystyle\int \frac{\sin(\ln(1+x))}{x}\,\mathrm{d}x$. Hence find an approximate value for $\displaystyle\int_0^{0.5} \frac{\sin(\ln(1+x))}{x}\,\mathrm{d}x$. [3]

[Turn over

8 **(a)** A sequence of numbers $a_1, a_2, a_3, \ldots, a_{64}$ is such that $a_{n+1} = a_n + d$, where $1 \le n \le 63$ and d is a constant. The 64 numbers fill the 64 squares in the 8×8 grid in such a way that a_1 to a_8 fills the first row of boxes from left to right in that order. Similarly, a_9 to a_{16} fills the second row of boxes from left to right in that order.



Given that the sums of the numbers in the **first row** and in the **third column** are 58 and 376 respectively, find the values of a_1 and d. [4]

(b) A geometric series has first term *a* and common ratio *r*, where *a* and *r* are non-zero. The sum to infinity of the series is 2. The sum of the six terms of this series from the 4th term to the 9th term is $-\dfrac{63}{256}$. Show that $512r^9 - 512r^3 - 63 = 0$.

Find the two possible values of *r*, justifying the choice of your answers. [5]

9 One of the roots of the equation $z^3 - az - 66 = 0$, where *a* is real, is *w*.

(i) Given that $w = b - \sqrt{2}\,\mathrm{i}$, where *b* is real, find the exact values of *a* and $\dfrac{w}{w^*}$. [6]

(ii) Given instead that $w = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$, $-\pi < \theta < -\dfrac{3\pi}{4}$, find $\left|aw^2 + 66w\right|$ and $\arg\left(aw^2 + 66w\right)$ in terms of *r* and θ. [4]

10 The point *M* has position vector relative to the origin *O*, given by $6\mathbf{i} - 5\mathbf{j} + 11\mathbf{k}$. The line l_1 has equation $x - 7 = \dfrac{y}{3} = \dfrac{z+2}{-2}$, and the plane π has equation $4x - 2y - z = 30$.

(i) Show that l_1 lies in π. [2]

(ii) Find a cartesian equation of the plane containing l_1 and *M*. [3]

The point *N* is the foot of perpendicular from *M* to l_1. The line l_2 is the line passing through *M* and *N*.

(iii) Find the position vector of *N* and the area of triangle *OMN*. [5]

(iv) Find the acute angle between l_2 and π, giving your answer correct to the nearest 0.1°. [3]

11 [It is given that the volume of a cylinder with base radius r and height h is $\pi r^2 h$ and the volume of a cone with the same base radius and height is a third of a cylinder.]

A manufacturer makes double-ended coloured pencils that allow users to have two different colours in one pencil. The manufacturer determines that the shape of each coloured pencil is formed by rotating a trapezium $PQRS$ completely about the x-axis, such that it is a solid made up of a cylinder and two cones. The volume, V cm^3, of the coloured pencil should be as large as possible.

It is given that the points P, Q, R and S lie on the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are positive constants. The points R and S are $(-a,0)$ and $(a,0)$ respectively, and the line PQ is parallel to the x-axis.

(i) Verify that $P(a\cos\theta,\ b\sin\theta)$, where $0<\theta<\frac{\pi}{2}$, lies on the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Write down the coordinates of the point Q. [2]

(ii) Show that V can be expressed as $V=k\pi\sin^2\theta(2\cos\theta+1)$, where k is a constant in terms of a and b. [3]

(iii) Given that $\theta=\theta_1$ is the value of θ which gives the maximum value of V, show that θ_1 satisfies the equation $3\cos^2\theta+\cos\theta-1=0$. Hence, find the value of θ_1. [4]

At $\theta=\frac{\pi}{6}$, the manufacturer wants to change one end of the coloured pencil to a rounded-end eraser. The eraser is formed by rotating the arc PS completely about the x-axis.

(iv) Find the volume of the eraser in terms of a and b. [3]

12 A ball-bearing is dropped from a point O and falls vertically through the atmosphere. Its speed at O is zero, and t seconds later, its velocity is v m s^{-1} and its displacement from O is x m. The rate of change of v with respect to t is given by $10-0.001v^2$.

(i) Show that $v=100\left(\frac{e^{\frac{t}{5}}-1}{e^{\frac{t}{5}}+1}\right)$. [4]

(ii) Find the value of v_0, where v_0 is the value approached by v for large values of t. [1]

(iii) By using chain rule, form an equation relating $\frac{dx}{dt}$, $\frac{dv}{dt}$ and $\frac{dv}{dx}$. Given that $v=\frac{dx}{dt}$, form a differential equation relating v and x. Show that

$$v=100\sqrt{1-e^{-\frac{x}{500}}}.$$ [5]

(iv) Find the distance of the ball-bearing from O after 5 seconds, giving your answer correct to 2 decimal places. [3]

[Turn over

Section A: Pure Mathematics [40 Marks]

1 Express $2\cos\theta\sin\frac{\theta}{2}$ in the form $\sin a\theta - \sin b\theta$, where a and b are constants to be found. [2]

Hence, find the exact value of α, where $0 < \alpha < \pi$, for which

$$\int_{\alpha}^{\pi} \left(3\cos\frac{3\theta}{2} - \cos\frac{\theta}{2}\right) e^{\cos\theta\sin\frac{\theta}{2}}\, d\theta = 4\left(\frac{1}{e} - 1\right).$$ [5]

2 **(i)** Show that $\frac{2}{2r+1} - \frac{3}{2r+3} + \frac{1}{2r+5} = \frac{Ar+B}{(2r+1)(2r+3)(2r+5)}$, where A and B are constants to be found. [2]

(ii) Hence find $\sum_{r=1}^{n} \frac{2r+9}{(2r+1)(2r+3)(2r+5)}$. (There is no need to express your answer as a single algebraic fraction.) [4]

(iii) It is given that $\sum_{r=1}^{n} \frac{2r+9}{(2r+1)(2r+3)(2r+5)}$ is within 0.01 of the sum to infinity.

Write down an inequality in terms of n, and hence find the smallest possible value of n. [3]

3 The function f is defined by

$$f : x \mapsto \frac{-x^2 + 5x - 11}{x - 2}, \quad x \in \mathbb{R}, x \neq 2.$$

(i) Find the equations of the asymptotes of the curve $y = f(x)$. [3]

(ii) Determine whether f has an inverse, justifying your answer. [2]

Given that the function g is defined by

$$g : x \mapsto f(x), \quad x \in \mathbb{R}, 2 < x \leqslant 4,$$

find $g^{-1}(x)$ and state the domain of g^{-1}. [4]

Sketch the graph of $y = g^{-1}g(x)$. [2]

4 A curve C has parametric equations

$$x = -\sqrt{t^2+4}\,, \quad y = \frac{\ln t}{t}, \quad \text{where } t > 0.$$

(i) Show that $\dfrac{dy}{dx} = \dfrac{(\ln t - 1)\sqrt{t^2+4}}{t^3}$. [3]

(ii) Find the exact coordinates of the turning point on C, and explain why it is a maximum. [4]

(iii) Sketch C. [3]

(iv) Show that the area bounded by C and the lines $x = -\sqrt{13}$ and $x = -\sqrt{5}$ is given by

$$\int_1^3 \frac{\ln t}{\sqrt{t^2+4}}\, dt.$$

Find the area, giving your answer to 4 decimal places. [3]

Section B: Probability and Statistics [60 Marks]

5 Mr and Mrs Lee participate in a game show, together with 3 other men and 5 other women. In the first round, the 10 participants are grouped into 5 pairs.

(i) Find the number of ways the pairings can be done if there is only 1 pair of the same gender. [3]

After the first round, Mr and Mrs Lee are both eliminated. The remaining 8 participants are seated around a round table. Find the number of ways this can be done if

(ii) there are no restrictions, [1]

(iii) the 3 men are not all seated together. [3]

[Turn over

6 As the use of email becomes more prevalent, the number of unsolicited email (also known as spam) received increases. Besides advertisements, spam can now be cleverly disguised as business emails and contain malware. Hence, there is a need to use spam filter.

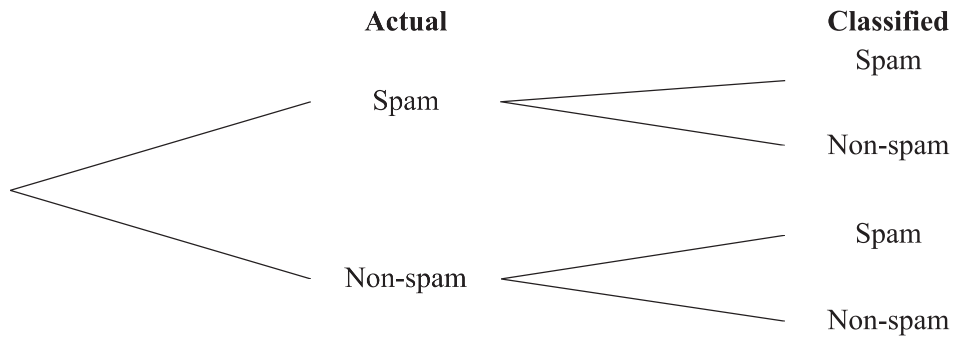
The probability that Yip receives a spam email is p. He uses a spam filter, Spam Guard Plus, to filter his emails. He has the following information:

P(an email is classified correctly) = $\frac{41}{50}$;

P(an email is classified correctly | it is classified as spam) = $\frac{38}{45}$;

P(an email is classified correctly | it is a spam email) = $\frac{19}{20}$.

(i) Show that $p = \frac{4}{5}$. Hence, complete the probability tree below. [5]



(ii) Andy and Betty notices that, on average, 30% and 70% of their emails are spam respectively. State whether Spam Guard Plus would be **(a)** more appropriate for Andy, **(b)** more appropriate for Betty, or **(c)** just as appropriate for both Andy and Betty. Justify your answer. [2]

7 Seven members of the school cross country team undergo a new training programme to improve their fitness. During a particular session, each of them has to complete a 200 metre run and to achieve as many push-ups as possible in one minute. The times taken for the 200 metre run, t seconds, together with the number of push-ups each runner achieves, n, are shown in the table.

Student	A	B	C	D	E	F	G
t	38.3	42.1	35.1	40.1	32.0	31.6	41.0
n	44	35	48	42	49	49	40

(i) Draw a scatter diagram to illustrate the data, labelling the axes. [1]

(ii) Explain using your scatter diagram why the linear model $n = at + b$ would not be appropriate. [1]

It is thought that the relationship between n and t can be modelled by one of the formulae

$$n = c(t-30)^2 + d \qquad \text{or} \qquad n = e(t-30)^3 + f$$

where c, d, e and f are constants.

(iii) The product moment correlation coefficient between n and $(t-30)^2$ is -0.980, correct to 3 decimal places. Determine, with a reason, which of the 2 models is more appropriate. [2]

(iv) It is known that student H is able to do 48 push-ups in one minute. It is required to estimate student H's timing for the 200 metre run. Find the equation of a suitable regression line, and use it to find the required estimate. Comment on the reliability of this estimate. [5]

8 A bag contains two balls numbered 3, n balls numbered 2 and three balls numbered 1. A player picks two balls at random from the bag at the same time.

If the difference between the numbers on the two balls is 2, the player receives \$6.

If the difference between the numbers on the two balls is 1, the player does not receive or lose any money.

If the numbers on the 2 balls are the same, the player loses \$1.

(i) Show that the largest value of n such that player is expected to receive money from this game is 8. [5]

For the rest of this question, take the value of n to be 8.

(ii) Show that the probability that a player loses money in a game is $\dfrac{16}{39}$. [1]

Victoria plays this game 50 times.

(iii) Find the probability that she lost money for at least 20 games. [2]

(iv) The probability that Victoria loses money in r games is more than 0.1. Find the set of values of r. [3]

[Turn over

9 Exposure to Volatile Organic Compounds (VOCs), which have been identified in indoor air, is suspected as a cause for headaches and respiratory symptoms. Indoor plants have not only a positive psychological effect on humans, but may also improve the air quality. Certain species of indoor plants were found to be effective removers of VOCs.

A commonly known VOC is Benzene. The following data gives the benzene levels, x (in ppm) in 40 test chambers containing the indoor plant *Epipremnum aureum*.

$$n = 40, \qquad \sum(x - 26.0) = -30.1, \qquad \sum(x - 26.0)^2 = 214.61.$$

The initial mean Benzene level (in ppm) without *Epipremnum aureum* was found to be 26.0.

(i) Test, at the 5% level of significance, the claim that the mean Benzene level, μ (in ppm), has decreased as a result of the indoor plant *Epipremnum aureum*. You should state your hypotheses clearly. [5]

(ii) State, giving a reason, whether there is a need to make any assumptions about the population distribution of the Benzene level in order for the test to be valid. [2]

The Benzene levels of another 50 test chambers containing the indoor plant *Epipremnum aureum* were recorded, The sample mean is $\overline{x}$ ppm and the sample variance is 8.33 ppm^2.

(iii) The acceptance region of a test of the null hypothesis $\mu = 26.0$ is $\overline{x} > 25.1$. State the alternative hypothesis and find the level of significance of the test. [4]

(iv) If the null hypothesis is $\mu = \mu_0$, where $\mu_0 > 26.0$, would the significance level of a test with the same acceptance region in part **(iii)** be larger or smaller than that found in part **(iii)**? Give a reason for your answer. [2]

10 **In this question, you should state clearly the values of the parameters of any distribution you use.**

A bus service plies from a point A in the city, through a point B, and then to its terminal station at point C.

Journey times in minutes from A to B have the distribution $\mathrm{N}\left(28, 4^2\right)$.

(i) Find the probability that a randomly selected bus journey from A to B is completed within 35 minutes. [1]

The journey times in minutes from A to C, have the distribution $\mathrm{N}\left(46.2, 4.8^2\right)$.

(ii) The journey times in minutes from B to C have the distribution $\mathrm{N}\left(\mu, \sigma^2\right)$. Given that the journey times from A to B are independent of the journey times from B to C, find the value of μ and show that $\sigma^2 = 7.04$. [3]

(iii) Find the set of values of k such that at least 90% of all journey times from A to C can be completed within k minutes. [2]

The performance of the bus operation is deemed as "unreliable" if a random sample of 70 journeys from A to C yields a mean journey time exceeding 47 minutes.

(iv) Two independent random samples of 70 journeys from A to C are taken. Find the probability that both samples will result in the performance of the bus operation to be deemed as "unreliable". [3]

To improve the reliability performance of the bus operation, more bus lanes are introduced and some bus stops along the bus route are removed. The journey times from A to B are now reduced by 10%, and the journey times from B to C now have the distribution $\mathrm{N}\left(\mu - 1, 8\right)$.

(v) Find the probability that two journeys from A to C are completed within a total of 90 minutes. [4]

YISHUN INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
Higher 2

CANDIDATE NAME	

CG		INDEX NO	

MATHEMATICS 9758/01

Paper 1 **4 September 2019**

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your CG and name on the work you hand in.
Write in dark blue or black pen.
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiners' Use

Question	1	2	3	4	5	6	7
Marks							

Question	8	9	10	11
Marks				

Total marks	

1 **(i)** Expand $\sin\left(\dfrac{\pi}{4}-2x\right)$ in ascending powers of x, up to and including the term in x^3. [3]

(ii) The first two non-zero terms found in part **(i)** are equal to the first two non-zero terms in the series expansion of $(a+bx)^{-1}$ in ascending powers of x. Find the exact values of the constants a and b. Hence find the third exact non-zero term of the series expansion of $(a+bx)^{-1}$ for these values of a and b. [3]

2 **(a)** Vectors **a** and **b** are such that $\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$ and $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$. Show that **a** and **b** are perpendicular. [2]

(b) Referred to the origin O, points C and D have position vectors $\mathbf{c}$ and $\mathbf{d}$ respectively. Point P lies on OC produced such that $OC:CP = 1:\lambda - 1$, where $\lambda > 1$. Point M lies on DP, between D and P, such that $DM:MP = 2:3$. Write down the position vector of M in terms of λ, $\mathbf{c}$ and $\mathbf{d}$. Hence, find the area of triangle OPM in the form $k\lambda|\mathbf{c}\times\mathbf{d}|$, where k is a constant to be found. [4]

3 The function f is defined by

$$\mathrm{f}(x)=\begin{cases}(x-2a)^2 & \text{for } 0\le x<2a,\\ 2ax-4a^2 & \text{for } 2a\le x<4a,\end{cases}$$

where a is a positive real constant and that $\mathrm{f}\left(x+4a\right)=\mathrm{f}\left(x\right)$ for all real values of x.

(i) Sketch the graph of $y=\mathrm{f}\left(x\right)$ for $-3a\le x\le 8a$. [3]

(ii) Hence find the value of $\int_{-2a}^{8a}\mathrm{f}\left(|\,x\,|\right)\,\mathrm{d}x$ in terms of a. [3]

4 A curve C has parametric equations

$$x = t^2 \ , \ y = \frac{1}{\sqrt{t}} \ , t > 0 \ .$$

(i) The curve $y = \frac{8}{x}$ intersects C at point A. Without using a calculator, find the coordinates of A. [2]

(ii) The tangent at the point $P\left(p^2, \frac{1}{\sqrt{p}}\right)$ on C meets the x-axis at point D and the y-axis at point E. The point F is the midpoint of DE. Find a cartesian equation of the curve traced by F as p varies. [5]

5 The equation of a curve is $2xy+(1+y)^2=x$.

(i) Find the equations of the two tangents which are parallel to the y-axis. [4]

(ii) The normal to the curve at the point $A(1,\ 0)$ meets the y-axis at the point B. Find the area of the triangle OAB. [3]

6 The sum of the first n terms of a sequence is a cubic polynomial, denoted by S_n. The first term and the second term of the sequence are 2 and 4 respectively. It is known that $S_5 = 90$ and $S_{10} = 830$.

(i) Find S_n in terms of n. [4]

(ii) Find the 54th term of the sequence. [2]

(b) **(i)** Given that $\cos(2n-1)\alpha-\cos(2n+1)\alpha=2\sin\alpha\sin 2n\alpha$ and α is not an integer multiple of π, show that

$$\sum_{n=1}^{N}\sin 2n\alpha=\frac{1}{2}\cot\alpha-\frac{1}{2}\operatorname{cosec}\alpha\cos(2N+1)\alpha\ .$$ [3]

(ii) Explain whether the series $\sum_{n=1}^{\infty} \sin\frac{2n\pi}{3}$ converges. [1]

7 **(a)(i)** Find $\int \cos(\ln x)\,\mathrm{d}x$. [3]

(ii) A curve C is defined by the equation $y = \cos(\ln x)$, for $\mathrm{e}^{-\frac{3}{2}\pi} \le x \le \mathrm{e}^{\frac{1}{2}\pi}$.

The region R is bounded by C, the lines $x = \mathrm{e}^{-\frac{\pi}{2}}$, $x = \mathrm{e}^{\frac{\pi}{2}}$ and the x-axis. Find the exact area of R. [3]

(b) A curve is defined by the equation $y=\dfrac{\sqrt{e^{\cot x}}}{\sin x}$. The region bounded by this curve, the x-axis, the lines $x=\dfrac{\pi}{6}$ and $x=\dfrac{2\pi}{3}$, is rotated 2π radians about the x-axis to form a solid. Using the substitution $u=\cot x$, find the exact volume of the solid obtained. [4]

8 **(i)** Show that $y=\dfrac{x-x^2-1}{x-2}$ can be expressed as $y=\dfrac{A}{x-2}+B(x+1)$, where A and B are constants to be found. Hence, state a sequence of transformations that will transform the curve with equation $y=\dfrac{1}{3-x}-\dfrac{x}{3}$ onto the curve with equation $y=\dfrac{x-x^2-1}{x-2}$. [3]

(ii) On the same axes, sketch the curves with equations $y=\dfrac{x-x^2-1}{x-2}$ and $y=|2x+1|$, stating the equations of any asymptotes and the coordinates of the points where the curves cross the axes. [4]

Hence, find the exact range of values of x for which $\dfrac{x-x^2-1}{x-2}<|2x+1|$. [4]

9 **(a)** The function f is defined by $\mathrm{f}: x \mapsto 2+\dfrac{3}{x},\ x \in \mathbb{R},\ x>0$.

(i) Sketch the graph of $y=\mathrm{f}(x)$. Hence, show that f has an inverse. [2]

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(iii) On the same diagram as in part **(i),** sketch the graphs of $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$. [2]

(iv) Explain why f^2 exists and find $\mathrm{f}^2(x)$. [2]

(b) The function h is defined by $\mathrm{h} : x \mapsto \dfrac{3-x}{x^2-1}$, $x \in \mathbb{R}$, $x \neq \pm 1$. Find algebraically the range of h, giving your answer in exact form. [4]

10 A tank initially contains 2000 litres of water and 20 kg of dissolved salt. Brine with C kg of salt per 1000 litres is entering the tank at 5 litres per minute and the solution drains out at the same rate of 5 litres per minute. The amount of salt in the tank at time t minutes is x kg. Assume that the solution is always uniformly mixed.

(i) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{400}(2C - x)$. Hence determine the value of C if the amount of salt in the tank remains constant at 120 kg after certain time has passed. [3]

(ii) Find x in terms of t. [5]

(iii) Sketch the graph of the particular solution, including the coordinates of the point(s) where the graph crosses the axes and the equations of any asymptotes. Find the time t when the amount of salt in the tank is 60 kg, giving your answer to the nearest minute. [3]

(iv) State one assumption for the above model to be valid. [1]

11 A factory produces power banks. The factory produces 1000 power banks in the first week. In each subsequent week, the number of power banks produced is 250 more than the previous week. The factory produces 7500 power banks in the Nth week.

(i) Find the value of N. [2]

(ii) After the Nth week, the factory produces 7500 power banks weekly. Find the total number of power banks that will be produced in the first 60 weeks. [3]

The sales manager predicts that the demand for power banks in this week is $a+bH$ if the demand for power banks in the preceding week is H, where a and b are constants and $b>1$. It is given that the demand for power banks in the first week is 50.

(iii) Show that the demand for power banks in the third week is $a+ba+50b^2$. [2]

(iv) Show that the demand for power banks in the nth week can be written as $a\left(\frac{b^{n-1}-1}{b-1}\right)+50b^{n-1}$. [1]

It is now given that $a = 300$ and $b = 1.05$.
In the first week, the number of power banks produced is still 1000. In each subsequent week, the number of power banks produced will be L more than the previous week.

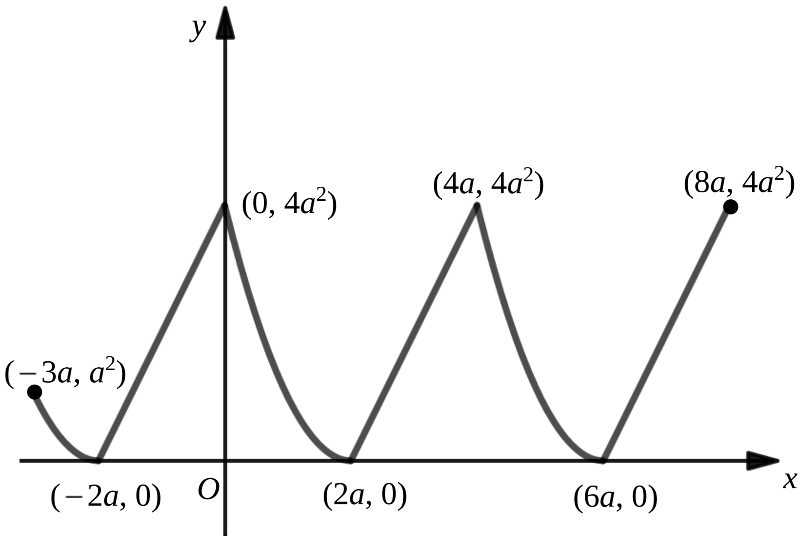
(v) The production manager decides to change the production plan so that the total production can meet the total demand in the first 60 weeks. Find the least value of L. [4]

- End of Paper -

Yishun Innova Junior College ♦ Mathematics Department
2019 JC 2 Mathematics H2 9758
Prelim Examination P1
Solutions

Qn	Solution	Remarks
1(i)	$\sin\left(\frac{\pi}{4}-2x\right)=\sin\frac{\pi}{4}\cos 2x-\cos\frac{\pi}{4}\sin 2x ------*$ $=\frac{1}{\sqrt{2}}(\cos 2x-\sin 2x)$ $=\frac{1}{\sqrt{2}}\left[\left(1-\frac{(2x)^2}{2}+...\right)-\left(2x-\frac{(2x)^3}{3!}+...\right)\right]$ $=\frac{1}{\sqrt{2}}\left(1-2x-2x^2+\frac{4x^3}{3}+...\right)$	You cannot substitute $\frac{\pi}{4}-2x$ into the standard expansion formula directly. In general, we can apply the standard expansions when x is replaced by g(x), provided g(0) = 0. For instance, $g(x)=x+x^2$.
	<u>Alternative Method</u> Let $y=\sin\left(\frac{\pi}{4}-2x\right)$ $\frac{dy}{dx}=-2\cos\left(\frac{\pi}{4}-2x\right)$ $\frac{d^2y}{dx^2}=-4\sin\left(\frac{\pi}{4}-2x\right)$ $\frac{d^3y}{dx^3}=8\cos\left(\frac{\pi}{4}-2x\right)$ When x = 0, $y=\frac{1}{\sqrt{2}}$, $\frac{dy}{dx}=-\sqrt{2}$, $\frac{d^2y}{dx^2}=-2\sqrt{2}$, $\frac{d^3y}{dx^3}=4\sqrt{2}$ $\sin\left(\frac{\pi}{4}-2x\right)=\frac{1}{\sqrt{2}}-\sqrt{2}x-\frac{2\sqrt{2}}{2}x^2+\frac{4\sqrt{2}}{3!}x^3+\ldots$ $=\frac{1}{\sqrt{2}}-\sqrt{2}x-\sqrt{2}x^2+\frac{2\sqrt{2}}{3}x^3+\ldots$ $=\frac{1}{\sqrt{2}}\left(1-2x-2x^2+\frac{4x^3}{3}+...\right)$	Be careful with the signs when you doing the higher derivatives.

(ii)	$(a+bx)^{-1} = a^{-1}\left(1+\dfrac{b}{a}x\right)^{-1}$ $= a^{-1}\left[1+(-1)\left(\dfrac{b}{a}x\right)+\dfrac{(-1)(-2)}{2}\left(\dfrac{b}{a}x\right)^2+\ldots\right]$ $=\dfrac{1}{a}\left(1-\dfrac{b}{a}x+\left(\dfrac{b}{a}\right)^2 x^2+\ldots\right)$ $\therefore a=\sqrt{2}$ and $\dfrac{b}{a}=2 \Rightarrow b=2\sqrt{2}$ Third non-zero term: $\dfrac{1}{a}\left(\dfrac{b}{a}\right)^2 x^2 = \dfrac{1}{\sqrt{2}}(2)^2 x^2 = \dfrac{4}{\sqrt{2}}x^2 = 2\sqrt{2}x^2$	Note that power is -1, not a positive integer so we need to use the series expansion of $(1+x)^n$ found in MF26 Don't forget to apply the power -1 to a after factorize a out. Third non-zero term and the coefficient of third non-zero term are different.
2(a)	$\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\|$ $\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}-\mathbf{b}\|^2$ $(\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})$ $\|\mathbf{a}\|^2+2\mathbf{a}\bullet\mathbf{b}+\|\mathbf{b}\|^2 = \|\mathbf{a}\|^2-2\mathbf{a}\bullet\mathbf{b}+\|\mathbf{b}\|^2$ $4\mathbf{a}\bullet\mathbf{b}=0$ $\mathbf{a}\bullet\mathbf{b}=0$ Hence **a** and **b** are perpendicular. <u>Alternative Method:</u> Since $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\|$, the diagonals of the parallelogram (with sides $\mathbf{a}$ and $\mathbf{b}$) are equal in length and thus the parallelogram must be a rectangle. Therefore, **a** and **b** are perpendicular.	There are two forms of vector product: dot (scalar) and cross (product). Hence expressions such as $\mathbf{a}^2$ will not make sense, as it will be ambiguous whether it means $\mathbf{a}\cdot\mathbf{a}$ or $\mathbf{a}\times\mathbf{a}$. However $\|\mathbf{a}\|^2$ is meaningful as $\|\mathbf{a}\|$ means the length of a vector, so it is a number. Next, as $\|\mathbf{a}+\mathbf{b}\|$ is the length of the vector $\mathbf{a}+\mathbf{b}$, so $\|\mathbf{a}+\mathbf{b}\| \neq \|\mathbf{a}\|+\|\mathbf{b}\|$, and therefore $\|\mathbf{a}+\mathbf{b}\|^2 \neq \|\mathbf{a}\|^2+2\|\mathbf{a}\|\|\mathbf{b}\|+\|\mathbf{b}\|^2$. It is wrong to say that $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\|$ implies $\mathbf{a}+\mathbf{b}=\mathbf{a}-\mathbf{b}$ or $\mathbf{a}+\mathbf{b}=-(\mathbf{a}-\mathbf{b})$. Take for example the vectors $\mathbf{i}$ and $\mathbf{j}$, they are obviously pointing in different directions, so neither $\mathbf{i}=\mathbf{j}$ nor $\mathbf{i}=-\mathbf{j}$, but $\|\mathbf{i}\|=\|\mathbf{j}\|=1$.

(b)	$\overrightarrow{OP}=\lambda\mathbf{c}$ $\overrightarrow{OM}=\dfrac{2\overrightarrow{OP}+3\overrightarrow{OD}}{5}=\dfrac{2\lambda\mathbf{c}+3\mathbf{d}}{5}$ **Area of triangle** $OPM=\dfrac{1}{2}\left\lvert\overrightarrow{OP}\times\overrightarrow{OM}\right\rvert$ $=\dfrac{1}{2}\left\lvert\lambda\mathbf{c}\times\left(\dfrac{2\lambda\mathbf{c}+3\mathbf{d}}{5}\right)\right\rvert$ $=\dfrac{1}{2}\left\lvert\lambda\mathbf{c}\times\dfrac{2\lambda\mathbf{c}}{5}+\lambda\mathbf{c}\times\dfrac{3\mathbf{d}}{5}\right\rvert$ $=\dfrac{1}{2}\left\lvert\lambda\mathbf{c}\times\dfrac{3\mathbf{d}}{5}\right\rvert=\dfrac{3}{10}\lambda\lvert\mathbf{c}\times\mathbf{d}\rvert$	Read carefully: P lies on OC produced. So P does not lie in between O and C. Ratio theorem is a very useful and quick way to get $\overrightarrow{OM}$. $\mathbf{c}\times\mathbf{c}$ is a vector product, so the result should be a vector. So $\mathbf{c}\times\mathbf{c}\neq 0$, it should be $\mathbf{0}$, the zero vector. Because of the definition of vector product, $\mathbf{c}\times\mathbf{c}\neq\lvert\mathbf{c}\rvert^2$, and $\mathbf{c}\times\mathbf{d}\neq\mathbf{d}\times\mathbf{c}$.
3(i)		-label all vertices and end points -Quadratic curve shape (part) for $0\le x\le 2a$ and straight line segment for $2a\le x\le 4a$
(ii)	$\int_{-2a}^{8a}\mathrm{f}\left(\lvert x\rvert\right)\,\mathrm{d}x=3\int_0^{2a}(x-2a)^2\,\mathrm{d}x+2\left(\dfrac{1}{2}(2a)(4a^2)\right)$ $=3\left[\dfrac{1}{3}(x-2a)^3\right]_0^{2a}+8a^3$ $=8a^3+8a^3$ $=16a^3$	To draw graph of f(|x|), keep RHS and reflect in the y-axis. Note that $(x-2a)^2$ is only defined for $0\le x\le 2a$ while $2ax-4a^2$ is only defined for $2a\le x\le 4a$. It is **wrong** to take for instance: $\int_{4a}^{6a}(x-2a)^2\,\mathrm{d}x$ as the area of the region from $4a\le x\le 6a$.

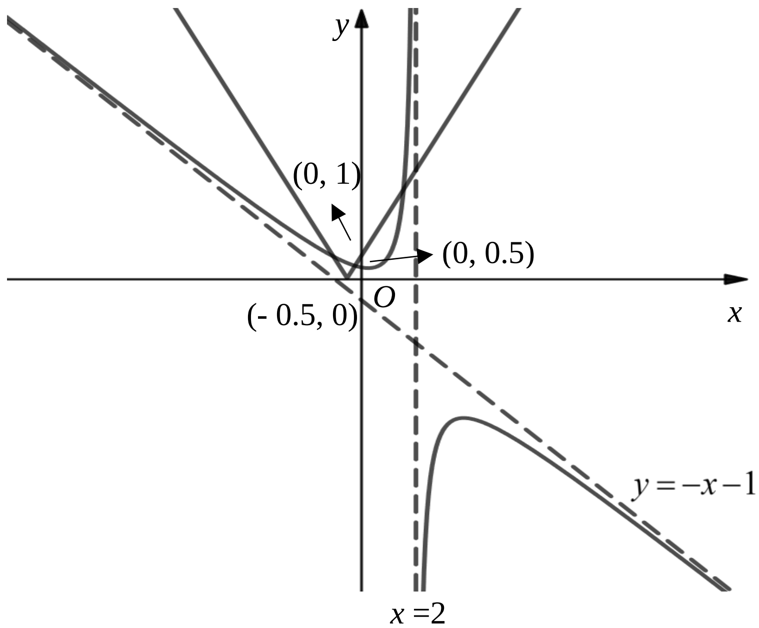
4(i)	Substitute $x=t^2$, $y=\frac{1}{\sqrt{t}}$ into $y=\frac{8}{x}$, $\frac{1}{\sqrt{t}}=\frac{8}{t^2}$ $t^{\frac{3}{2}}=8$ $t=4$ When $t=4$, $x=4^2=16$ $y=\frac{1}{\sqrt{4}}=\frac{1}{2}$ Coordinates of A is $\left(16,\frac{1}{2}\right)$.	Substitute the parametric equation of C into the Cartesian equation Avoid changing the parametric equation to Cartesian form
(ii)	$\frac{dx}{dt}=2t$, $\frac{dy}{dt}=-\frac{1}{2}t^{-\frac{3}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}t^{-\frac{3}{2}}\times\frac{1}{2t}=-\frac{1}{4}t^{-\frac{5}{2}}$ Equation of tangent at point P on curve C, $y-\frac{1}{\sqrt{p}}=-\frac{1}{4}p^{-\frac{5}{2}}\left(x-p^2\right)$ At D , $y=0$ $\therefore 0-\frac{1}{\sqrt{p}}=-\frac{1}{4}p^{-\frac{5}{2}}\left(x-p^2\right)\Rightarrow x=5p^2$ At E , $x=0$ $\therefore y-\frac{1}{\sqrt{p}}=-\frac{1}{4}p^{-\frac{5}{2}}\left(0-p^2\right)\Rightarrow y=\frac{5}{4}p^{-\frac{1}{2}}$ Coordinates of F : $\left(\frac{5p^2}{2},\frac{\frac{5}{4}p^{-\frac{1}{2}}}{2}\right)\Rightarrow\left(\frac{5p^2}{2},\frac{5}{8\sqrt{p}}\right)$ $y=\frac{5}{8\sqrt{p}}$ $\Rightarrow\sqrt{p}=\frac{5}{8y}$, substitute into $x=\frac{5p^2}{2}$ $x=\frac{5}{2}\left(\frac{5}{8y}\right)^4=\frac{3125}{8192y^4}$ Cartesian equation of the curve traced by F is $x=\frac{3125}{8192y^4}$.	Take note that we are finding the gradient of tangent $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$ Write down as header what you are trying to find, for example equation of tangent at point P. When $t=p$, Gradient of tangent at point P is $-\frac{1}{4}p^{-\frac{5}{2}}$ instead of $-\frac{1}{4}t^{-\frac{5}{2}}$ Find the midpoint F which follows the formula $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ From F, let $x=\frac{5p^2}{2}$, $y=\frac{5}{8\sqrt{p}}$ which is in the parametric form. Convert to Cartesian form so that we can trace how the path that point F moves as p varies.

5(i)	$2xy+(1+y)^2=x$ Differentiating wrt x, $2y+2x\frac{dy}{dx}+2(1+y)\frac{dy}{dx}=1$ $\frac{dy}{dx}=\frac{1-2y}{2(x+y+1)}$ When the tangent is parallel to the y-axis, $\frac{dy}{dx}$ is undefined. $\therefore 2(x+y+1)=0$ $y=-1-x$, substitute this into the equation of the curve, $2x(-1-x)+(1-1-x)^2=x$ $x^2+3x=0$ $x=0$ or $x=-3$	Do not confuse "tangent is parallel to y-axis", with x-axis: "parallel with y-axis": $\frac{dy}{dx}$ is undefined. "parallel with x-axis": $\frac{dy}{dx}=0$. Vertical lines in the Cartesian grid are of the form "x=…." instead of "y=…"
(ii)	Gradient of normal at $A=\frac{-1}{\frac{1-2(0)}{2(1+0+1)}}=-4$ Equation of normal at A: $y-0=-4(x-1)$ At B, $x=0\Rightarrow y=4$ => Coordinates of B are $(0,4)$ Area of triangle $OAB=\frac{1}{2}\times4\times1=2$ units^2	
6(a)(i)	$S_n=an^3+bn^2+cn+d$ $S_1=a(1)^3+b(1)^2+c(1)+d=2$ $\Rightarrow a+b+c+d=2$ ---- (1) $S_2=a(2)^3+b(2)^2+c(2)+d=6$ $\Rightarrow 8a+4b+2c+d=6$ ----(2) $S_5=a(5)^3+b(5)^2+c(5)+d=90$ $\Rightarrow 125a+25b+5c+d=90$ ----(3)	Don't assume that sequence is an AP/GP. It's neither. It's pointless to just write $S_5=T_1+T_2+...+T_5$ The sum is polynomial in n i.e. dependent on n and generally includes a constant term. Read question carefully. '4' is not S_2.

	$S_{10}=a(10)^3+b(10)^2+c(10)+d=830$ $\Rightarrow\ 1000a+100b+10c+d=830$ ----(4) Using GC: $a=1,\quad b=-2,\quad c=3,\quad d=0$ $S_n=n^3-2n^2+3n,\ n\geq 1,\ n\in\mathbb{Z}^+$	
(ii)	54th term of the sequence $u_{54}=S_{54}-S_{53}$ $=(54)^3-2(54)^2+3(54)-(53)^3+2(53)^2-3(53)$ $=8376$	
(b)(i)	Given that $\cos(2n-1)\alpha-\cos(2n+1)\alpha=2\sin\alpha\sin 2n\alpha$ $\sin(2n\alpha)=\dfrac{\cos(2n-1)\alpha-\cos(2n+1)\alpha}{2\sin\alpha}$ $\therefore\sum_{n=1}^{N}\sin(2n\alpha)=\sum_{n=1}^{N}\dfrac{\cos(2n-1)\alpha-\cos(2n+1)\alpha}{2\sin\alpha}$ $=\dfrac{1}{2\sin\alpha}\left[\begin{array}{l}\cos\alpha-\cos3\alpha\\+\cos3\alpha-\cos5\alpha\\+\cos5\alpha-\cos7\alpha\\+\cdots\\+\cos(2N-3)\alpha-\cos(2N-1)\alpha\\+\cos(2N-1)\alpha-\cos(2N+1)\alpha\end{array}\right]$ $=\dfrac{\cos\alpha-\cos(2N+1)\alpha}{2\sin\alpha}$ ------(*) $=\dfrac{\cos\alpha}{2\sin\alpha}-\dfrac{\cos(2N+1)\alpha}{2\sin\alpha}$ $=\dfrac{\cot\alpha}{2}-\dfrac{\operatorname{cosec}\alpha\cos(2N+1)\alpha}{2}$ (shown)	Use what was given. Don't waste time to derive it when it's already given. Note that $2\sin\alpha$ (independent of *n*) is a constant in this case Remember to cancel sufficient number of rows at the beginning and at the end.
(ii)	Let $\alpha=\frac{\pi}{3}$, $\sum_{n=1}^{N}\sin\left(\frac{2n\pi}{3}\right)$ $=\dfrac{\cot\frac{\pi}{3}}{2}-\dfrac{\operatorname{cosec}\frac{\pi}{3}\cos(2N+1)\frac{\pi}{3}}{2}$ As $N\to\infty$, $\cos(2N+1)\frac{\pi}{3}$ takes values $\frac{1}{2}$ or -1. OR cannot converge to a constant number. $\therefore\sum_{n=1}^{\infty}\sin\frac{2n\pi}{3}$ does not converge.	Note that it's the letter *N* that tends to infinity and not the letter *n*.

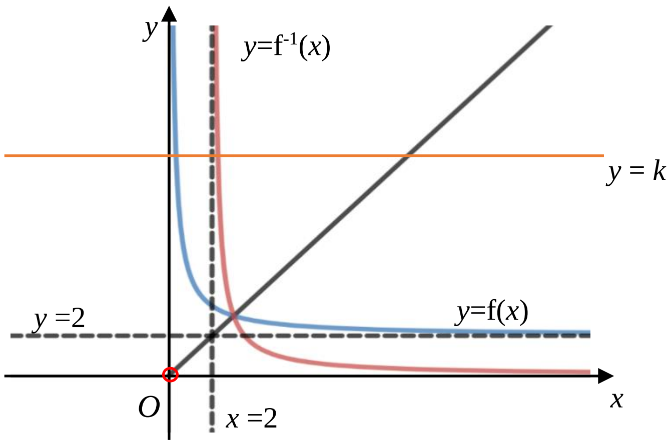
7(a)(i)	$\int \cos(\ln x)\ \mathrm{d}x = x\cos(\ln x) - \int x\left(-\frac{1}{x}\sin(\ln x)\right)\mathrm{d}x$ $= x\cos(\ln x) + \int \sin(\ln x)\ \mathrm{d}x$ $= x\cos(\ln x) + x\sin(\ln x) - \int x\left(\frac{1}{x}\cos(\ln x)\right)\ \mathrm{d}x$ $= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\ \mathrm{d}x$ $2\int \cos(\ln x))\ \mathrm{d}x = x\cos(\ln x) + x\sin(\ln x)$ $\int \cos(\ln x)\ \mathrm{d}x = \frac{1}{2}x\left(\cos(\ln x) + \sin(\ln x)\right) + C$	Use integration by parts directly. $\cos(\ln x)$ is a composite function. It is not a product of $(\cos x)(\ln x)$. Let u = $\cos(\ln x)$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$ apply integration by parts $uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ twice Bring $-\int \cos(\ln x)\ \mathrm{d}x$ to the LHS to stop the loop. Put + C in the final step
(ii)	Area = $\int_{\mathrm{e}^{-\frac{\pi}{2}}}^{\mathrm{e}^{\frac{\pi}{2}}} \cos(\ln x))\ \mathrm{d}x \quad = \frac{1}{2}\Big[\ x\cos(\ln x) + x\sin(\ln x)\ \Big]_{\mathrm{e}^{-\frac{\pi}{2}}}^{\mathrm{e}^{\frac{\pi}{2}}}$ $= \frac{1}{2}\mathrm{e}^{\frac{\pi}{2}}\left(\cos\left(\ln \mathrm{e}^{\frac{\pi}{2}}\right) + \sin\left(\ln \mathrm{e}^{\frac{\pi}{2}}\right)\right) -$ $\frac{1}{2}\mathrm{e}^{-\frac{\pi}{2}}\left(\cos\left(\ln \mathrm{e}^{-\frac{\pi}{2}}\right) + \sin\left(\ln \mathrm{e}^{-\frac{\pi}{2}}\right)\right)$ $= \frac{1}{2}\mathrm{e}^{\frac{\pi}{2}}\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \frac{1}{2}\mathrm{e}^{-\frac{\pi}{2}}\left(\cos\left(-\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right)\right)$ $= \frac{1}{2}\left(\mathrm{e}^{\frac{\pi}{2}} + \mathrm{e}^{-\frac{\pi}{2}}\right)$	Read question carefully. Integrate from $e^{-\frac{\pi}{2}}$ to $e^{\frac{\pi}{2}}$. Use (i) answer. F(Upper limit)– F(lower limit) Note that $\ln \mathrm{e}^{\frac{\pi}{2}} = \frac{\pi}{2}$, $\sin\left(-\frac{\pi}{2}\right) = -1$
(b)	Using $u = \cot x$, $\frac{\mathrm{d}u}{\mathrm{d}x} = -\mathrm{cosec}^2 x = -\frac{1}{\sin^2 x}$ $\Rightarrow \ldots\mathrm{d}u = \ldots - \frac{1}{\sin^2 x}\ \mathrm{d}x$ When $x = \frac{\pi}{6} \Rightarrow u = \frac{1}{\tan x} = \sqrt{3}$ When $x = \frac{2\pi}{3} \Rightarrow u = -\frac{1}{\sqrt{3}}$	Differentiate the given substitution. Memorise the differentiation of the 6 trigo functions. Only differentiation of sec *x* and cosec *x* formula are in MF26 Find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and hence d*x* = .. Need to change limit to u value. Write down the expression of the volume first. Then do substitution. $V = \int_{x_1}^{x_2} y^2\ \mathrm{d}x$

	Required volume is $=\pi\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}}\left(\frac{e^{\frac{1}{2}\cot x}}{\sin x}\right)^2 dx=\pi\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}}\frac{e^{\cot x}}{\sin^2 x}dx$ $=\pi\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}}e^{\cot x}\frac{1}{\sin^2 x}dx$ $=-\pi\int_{\sqrt{3}}^{-\frac{1}{\sqrt{3}}}e^u\,du$ or $\pi\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}}e^u\,du$ $=\pi\left(e^{\sqrt{3}}-e^{-\frac{1}{\sqrt{3}}}\right)$	Change to $-\pi\int_{\sqrt{3}}^{-\frac{1}{\sqrt{3}}}e^u\,du$ Integrate e^u w.r.t u is e^u
8(i)	$y=\frac{x-x^2-1}{x-2}=\frac{-3}{x-2}-(x+1)$ $y=\frac{1}{3-x}-\frac{x}{3}$ $=\frac{-1}{x-3}-\frac{1}{3}x$ 1) A scaling parallel to the y-axis by a factor of 3. 2) A translation of 1 unit in the negative x-direction.	Be careful in your algebraic manipulation when trying to express $y=\frac{x-x^2-1}{x-2}$ in the form $y=\frac{A}{x-2}+B(x+1)$. Note that you are required to **describe** the sequence of transformations that will transform the curve with equation $y=\frac{1}{3-x}-\frac{x}{3}$ to $y=\frac{x-x^2-1}{x-2}=\frac{-3}{x-2}-(x+1)$, NOT the reverse. It is **INCORRECT to use** the word **"shift"** to describe translation and the word **"flip/rotate"** to describe reflection. One of the transformations involved is scaling parallel to the y-axis with a factor of 3, it is **NOT with a factor of -3 or 3 units.**

<table>
<tr>
<td>(ii)</td>
<td>
</td>
<td>Note that the curve $y=\frac{x-x^2-1}{x-2}$ has asymptotes $y=-x-1$ and $x=2$.
You are expected to draw properly and clearly, including the behaviour of the curve $y=\frac{x-x^2-1}{x-2}$ near its asymptotes. Note that the oblique asymptote $y=-x-1$ meets the x-axis and y-axis at $(-1,0)$ and $(0,-1)$ respectively and the curve cuts the y-axis at the point $\left(0,\frac{1}{2}\right)$ which is NOT the minimum point.

You are expected to get the <u>correct shape</u> of the graph from your GC. You are required to state the coordinates of the points where the curves cross the axes. For the modulus graph $y=\left|2x+1\right|$, the coordinates are $\left(-\frac{1}{2},0\right)$ and $(0,1)$.
It has a line of symmetry $x=-\frac{1}{2}$.
You should have the sense of the relative positions for the points of intersection with the axes when sketching the two graphs, including the asymptotes on the same axes.</td>
</tr>
<tr>
<td></td>
<td>$\frac{x-x^2-1}{x-2}=-2x-1$
From GC, the first point of intersection has x-coordinate -1 .

$\frac{x-x^2-1}{x-2}=2x+1$
$x-x^2-1=(2x+1)(x-2)$
$3x^2-4x-1=0$
$x=\frac{4\pm\sqrt{16-4(3)(-1)}}{2(3)}$
$x=\frac{2\pm\sqrt{7}}{3}$</td>
<td>The question states that "Hence, find the exact range of …..", so you are required to use graphical method.</td>
</tr>
</table>

<table>
<tr><td></td><td>Thus, the range of values of x is
$x<-1,\ \dfrac{2-\sqrt{7}}{3}<x<\dfrac{2+\sqrt{7}}{3}$ or $x>2$</td><td></td></tr>
<tr><td>9(a)(i)</td><td>

Since any horizontal line $y=k,\ k\in\mathbb{R}$ intersects the graph of f at most once, f is one-one and it has an inverse.</td><td>Must sketch the graph of f for the given domain only ($x>0$) with the asymptotes.
Never use a particular line to explain one-one function and you must state the range of k. Note the 2 possible explanations.

If you use any line $y=k,\ k\in\mathbb{R}$, then the line cuts the graph of f at most once.

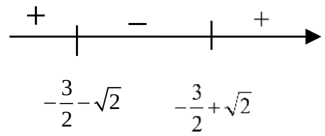
If you use any line $y=k,\ k\in R_f$, then the line cuts the graph of f exactly once.

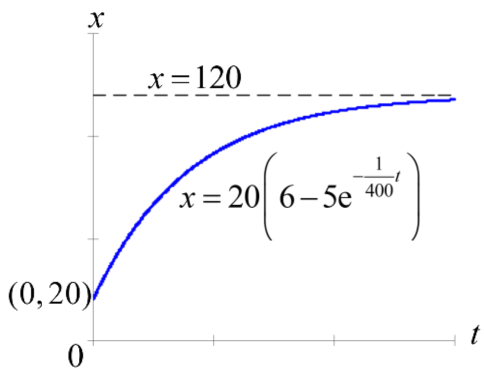
It is important to mention that f is one-one and not just f has an inverse.</td></tr>
<tr><td>(ii)</td><td>Let $y=2+\dfrac{3}{x}$
$\dfrac{3}{x}=y-2$
$x=\dfrac{3}{y-2}$
$\therefore \mathrm{f}^{-1}(x)=\dfrac{3}{x-2}$
$D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=(2,\infty)$</td><td>To find the rule of f^{-1} :

Let y = f (x) and then make x the subject</td></tr>
<tr><td>(iii)</td><td>See diagram in (i)</td><td>Must use the same scale for both axes when sketching the graphs of f and f^{-1} on the same diagram.

The graph of f^{-1} is a reflection of the graph of f in the line $y=x$</td></tr>
</table>

		The graph of $f^{-1}f$ is not simply the line $y = x$. Must sketch for the correct domain and passing through (2, 2). The graphs of f, f^{-1} and $f^{-1}f$ must intersect at the same point.
(iv)	$D_f = (0, \infty)$ and $R_f = (2, \infty)$ Since $R_f \subseteq D_f$, f^2 exists. $f^2(x) = f(f(x))$ $= 2 + \dfrac{3}{2 + \frac{3}{x}}$ $= 2 + \dfrac{3x}{2x+3}$ $= \dfrac{2(2x+3)+3x}{2x+3}$ $= \dfrac{7x+6}{2x+3}$	It is not sufficient to state $R_f \subseteq D_f$. $D_f = (0, \infty)$, $R_f = (2, \infty)$ must be stated to justify the subset.
(b)	Given $h(x) = \dfrac{3-x}{x^2-1}$ To find the range of g, the graph must intersect the horizontal line $y = k$, therefore, $D \geq 0$ Let $k = \dfrac{3-x}{x^2-1}$ $kx^2 - k = 3 - x$ $kx^2 + x - (k+3) = 0$ $(1)^2 + 4k(k+3) \geq 0$ $1 + 12k + 4k^2 \geq 0$ $4k^2 + 12k + 1 \geq 0$ Consider $4k^2 + 12k + 1 = 0$, $k = \dfrac{-12 \pm \sqrt{12^2 - 4(4)(1)}}{2(4)}$ $= \dfrac{-12 \pm \sqrt{128}}{8}$ $= \dfrac{-3 \pm 2\sqrt{2}}{2} = -\dfrac{3}{2} \pm \sqrt{2}$	This question required an algebraic approach so the GC cannot be used to obtain the graph. It is very tedious to sketch the graph without using a GC as the stationary points would have to be found by differentiation. Students would also need to show the nature of the stationary points before they can sketch the graph on their own. Moreover, it would take some time to find the exact y coordinates of the stationary points in order to obtain the range of h. Thus, students are strongly encouraged to use the discriminant instead (see solution)

	$4k^2+12k+1\geq 0$ $\therefore\ k\leq -\frac{3}{2}-\sqrt{2}\ OR\ k\geq -\frac{3}{2}+\sqrt{2}$  Hence, the range of h is $\left(-\infty,-\frac{3}{2}-\sqrt{2}\right]\cup\left[-\frac{3}{2}+\sqrt{2},\infty\right)$	
10(i)	Concentration of the brine entering the tank is $\frac{C}{1000}$ kg/L Solution leaving the tank is $\frac{x}{2000}$ kg/L. The rate at which salt enters the tank is $\frac{5C}{1000}=\frac{C}{200}$ kg/min The rate at which leaves the tank is $\frac{5x}{2000}=\frac{x}{400}$ kg/min. $\frac{dx}{dt}=\text{inflow rate }-\text{outflow rate}$ $\frac{dx}{dt}=\frac{C}{200}-\frac{x}{400}=\frac{1}{400}(2C-x)$ (Shown) When $\frac{dx}{dt}=0$, $x=120$ kg and $2C-120=0\Rightarrow C=60$ kg	
(ii)	$\int\frac{1}{120-x}\ dx=\int\frac{1}{400}\ dt$ $-\ln\lvert 120-x\rvert=\frac{1}{400}t+B$ where $B=\text{const}$ $120-x=Ae^{-\frac{1}{400}t}$ where $A=\text{const}$ When $t=0$, $x=20$ kg, $Ae^0=A=120-20=100$ $120-x=100e^{-\frac{1}{400}t}$ $x=20\left(6-5e^{-\frac{1}{400}t}\right)$	Remember to include a negative sign and modulus sign after integrating $\frac{1}{120-x}$.

(iii)	 Using GC, $t=204.33=204$ min.	
(iv)	It is assumed that there is no evaporation in the system so that the concentrations of the solution remain as stated/unaffected.	
11(i)	AP : first term = 1000, common difference = 250 $7500=1000+250(N-1)$ $N=27$	It is an AP with first term 1000 and common difference 250. $u_N=7500$. Solve for N.
(ii)	Total number of power banks produced in 60 weeks = $S_{27}+33(7500)$ $=\frac{27}{2}(1000+7500)+33(7500)$ =362250	For 1st to 27th week, it is an AP with first term 1000 and common difference 250. For 28th to 60th week, the production is at 7500 per week, so $(60-27)\times 7500$.
(iii)	Number of power banks on demand on week 1= 50 Number of power banks on demand on week 2 = a+50b Number of power banks on demand on week 3 = $a+b(a+50b)$ $=a+ba+50b^2$	For 2nd week, the demand is $a+b(50)=a+50b$ For 3rd week, demand is $a+b(\text{demand of 2nd week})$ $=a+b(a+50b)$
(iv)	Number of power banks on demand on week 4 = $a+b(a+ba+50b^2)$ = $a+ba+b^2a+50b^3$ Number of power banks on demand on week n $=a+ba+b^2a+b^3a+...+b^{n-2}a+50b^{n-1}$ $=\frac{a(b^{n-1}-1)}{b-1}+50b^{n-1}$	Continue working out for week 4 and deduce the demand of the nth week. Note that the last term of the expression is $b^{n-2}a$ for the GP. The first part of the expression is a summation of GP with first term a, common ratio b and number of terms $n-1$, which can be written as $\frac{a(b^{n-1}-1)}{b-1}$.
(v)	Total number of power banks produced in 60 weeks $=\frac{60}{2}(2(1000)+59L)$ = 30(2000+59L)	Remember to take summation of the terms from the 1st to 60th week. The total demand can be found using GC (MATH, 0: Summation).

	Total number of power banks on demand in 60 weeks $=\sum_{r=1}^{60}\left(\frac{300(1-1.05^{r-1})}{1-1.05}\right)+\sum_{r=1}^{60}\left(1.05^{r-1}(50)\right)$ = 1779181.493 For $30\left(2000+59L\right)\geq 1779181.493$ $L\geq 971.29$ Least L = 972	For total demand to be met, total production $\geq$ total demand for the first 60 weeks, solve for L.

YISHUN INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
Higher 2

CANDIDATE NAME	

CG		INDEX NO	

MATHEMATICS 9758/02

Paper 2 **19 SEPT 2019**

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your CG and name on the work you hand in.
Write in dark blue or black pen.
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiners' Use

Question	1	2	3	4	5	6
Marks						

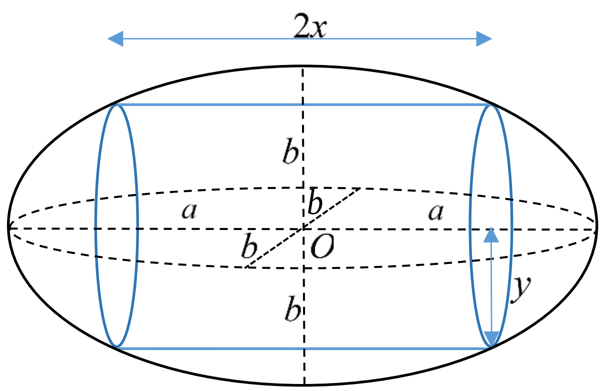
Question	7	8	9	10
Marks				

Total marks	

This document consists of 22 printed pages and 2 blank pages.

Section A: Pure Mathematics [40 marks]

1



The above diagram shows a hollow ellipsoid with centre O, enclosing a fixed volume of $\frac{4}{3}\pi ab^2$. A solid cylinder of length $2x$ and base radius y is inscribed in the ellipsoid. It is given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are positive constants with $a > b$.

Use differentiation to find, in terms of a and b, the maximum volume of the cylinder, proving that it is a maximum. Hence determine the ratio of the maximum volume of the cylinder to that of the volume enclosed by the ellipsoid. [6]

2 **(i)** Find $\int 2\sin(k+1)x \sin kx \, \mathrm{d}x$. [2]

(ii) Hence, determine in terms of k, the value of $\int_0^{\frac{\pi}{2}} \left(\sin(k+1)x - \sin kx\right)^2 \, \mathrm{d}x$, where k is an even integer. [5]

3 The plane p contains the point A with coordinates $(5, -1,\ 2)$ and the line l_1 with equation $\frac{x-3}{2} = y$, $z = 1$.

(i) The point B has coordinates $(c,\ 2,\ 2)$. Given that the shortest distance from B to l_1 is $\frac{\sqrt{205}}{5}$, find the possible values of c. [3]

(ii) Find a cartesian equation of p. [3]

The line l_2 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$, $\mu \in \mathbb{R}$.

(iii) Find the coordinates of the point at which l_2 intersects p. [3]

The line l_3 has equation $\mathbf{r} = \begin{pmatrix} a \\ 2a \\ a \end{pmatrix} + t\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$, where a is a constant.

(iv) Show that p is parallel to l_3. [1]

(v) Given that l_3 and p have no point in common, what can be said about the value of a? [1]

(vi) It is given instead that $a = 1$, find the distance between l_3 and p, leaving your answer in exact form. [2]

4 **Do not use a calculator in answering this question.**

(a) The equation $2z^3 - 3z^2 + kz + 26 = 0$, where k is a real constant, has a root $z = 1 + a\mathrm{i}$, where a is a positive real constant. Find the other roots of the equation and the values of a and k. [6]

(b) **(i)** Given that $\left(x+\mathrm{i}y\right)^2=15+8\mathrm{i}$, determine the possible values of the real numbers x and y. [3]

(ii) The roots of the equation $z^2-(2+7\mathrm{i})z=15-5\mathrm{i}$ are z_1 and z_2, with $\arg(z_1)<\arg(z_2)$. Find an exact expression for z_2, giving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [3]

(iii) Find the argument of $z_1^{\,2} z_2^{\,*}$ in exact form. [2]

Section B: Probability and Statistics [60 marks]

5 An investigation was carried out to determine the effect of rainfall on crop yield. The table below shows the average monthly rainfall, x mm, and the crop yield, y kg. The data is recorded during different months of a certain year.

x	150	163	172	175	180	187	196
y	48	70	87	92	95	89	80

(i) Draw a scatter diagram for these values. State with a reason, which of the following equations, where a and b are constants, provides the most accurate model of the relationship between x and y.

(A) $y = a\ln(x-100)+b$

(B) $y = a(x-180)^2+b$

(C) $y = \dfrac{a}{x-130}+b$ [3]

(ii) Using the model you chose in part **(i)**, write down the equation for the relationship between x and y, giving the numerical values of the coefficients. State the product moment correlation coefficient for this model. [2]

(iii) Calculate an estimate of the crop yield when the average monthly rainfall is 185 mm. Comment on the reliability of your estimate. [2]

6 A factory produces a large number of packets of cornflakes. On average, two in 7 packets contain a toy. The packets of cornflakes are sold in cartons of 12.

(a) A carton is randomly chosen.

(i) Find the probability that there are fewer than 2 toys. [1]

(ii) Find the probability that there are more than 1 but at most 6 toys. [2]

(b) Find the probability that in five randomly selected cartons, two of them contain exactly 4 toys and three of them contain exactly 2 toys. [2]

(c) A mini-mart ordered 40 cartons of cornflakes from the factory. Find the probability that none of the cartons contains fewer than 2 toys. [2]

The factory also produces a large number of packets of oats. A random sample of n packets of oats is chosen. The number of packets of oats containing a toy in the sample is denoted by A. Assume that A has the distribution $\mathrm{B}(n, p)$, where $p > 0.1$.

Given that $n = 25$ and $\mathrm{P}(A = 2 \text{ or } 3) = 0.25$, write down an equation in terms of p and find p numerically. [2]

7 A box contains five balls numbered 1, 3, 5, 6, 8. Three balls are drawn at random from the box.

(a) Find the probability that the sum of the three numbers drawn is an even number. [2]

(b) The random variable S denotes the smallest of the three numbers drawn.

(i) Determine the probability distribution of S. [2]

(ii) Find $\mathrm{E}(S)$ and $\mathrm{Var}(S)$. [2]

(iii) The mean of a random sample of 55 observations of S is denoted by $\bar{S}$. Find the probability that $\bar{S}$ is within 0.5 of $\mathrm{E}(S)$. [3]

8 **(a)** Events A and B are such that $\mathrm{P}\left(A \cup B\right) = 0.6$ and $\mathrm{P}\left(A \mid B\right) = 0.4$. Given that A and B are independent, find

(i) $\mathrm{P}\left(B\right)$, [2]

(ii) $\mathrm{P}\left(A' \mid B'\right)$. [2]

(b) 12 people are to be seated at 3 different coloured round tables.

(i) Find the number of ways that there are at least 3 people at each table. [4]

(ii) Find the probability that there are 4 people at each table given that there are at least 3 people at each table. [2]

9 A company produces car batteries. The life, in months, of a car battery of the regular type has the distribution $\mathrm{N}\left(\mu,\ \sigma^2\right)$. The mean life of 4 randomly selected car batteries of the regular type is denoted by $\overline{X}$. It is given that $\mathrm{P}\left(\overline{X} < 36.1\right) = \mathrm{P}\left(\overline{X} > 49.1\right) = 0.03355$.

(i) State the value of μ and show that $\sigma \approx 7.10$, correct to 2 decimal places. [4]

(ii) Find the smallest integer value of *k* such that more than 90% of the car batteries of the regular type have a life less than *k* months. [2]

(iii) Past experience shows that 25% of the car batteries of the regular type with lives less than 36 months are due to bad driving habits. A random sample of 100 car batteries of the regular type is selected. Find the expected number of these car batteries which will have lives each less than 36 months due to bad driving habits. [2]

(iv) After research and experimentation, the company produces a premium type of car battery using an improved manufacturing process which is able to increase the life of each car battery by 10%. Find the probability that the total life of 5 randomly chosen car batteries of the premium type is more than the total life of 6 randomly chosen car batteries of the regular type. [4]

10 A previous study revealed that the average time taken to assemble a certain type of electrical component is at least 15 minutes. The manager wants to investigate if the results of the study is valid. A random sample of 40 components is taken and the times taken to assemble the components are summarised in the following table:

Time to assemble a component (min)	9	10	12	13	15	16	17	18
Number of components	1	6	3	8	5	7	8	2

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Test at the 5% level of significance whether the results of the study is valid. You should state your hypotheses and define any symbols you use. [5]

(iii) Explain why the manager is able to conduct the test without knowing anything about the distribution of the times taken to assemble the electrical components.[1]

(iv) Explain what is meant by the phrase "5% level of significance" in this context.[1]

The manager claims that the average time taken to assemble another type of electrical component is 30 minutes. A random sample of 50 components of this type is chosen and the time taken to assemble each component is recorded. The mean and standard deviation of the sample are 29.7 minutes and k minutes respectively. Find the range of possible values of k if a test at the 8% significance level shows that there is sufficient evidence that the manager's claim is valid. [4]

~ END OF PAPER ~