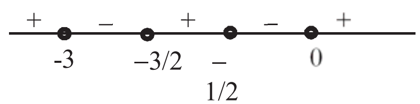
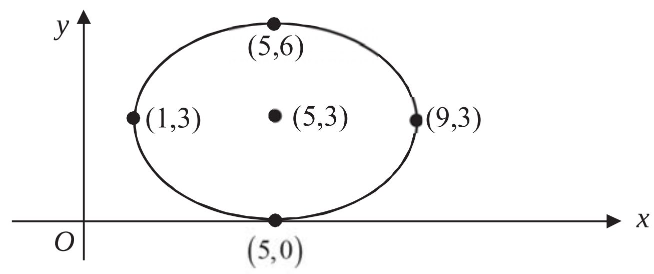
Q1. Techniques of Integration		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Integration by parts	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x^2}=-2x\mathrm{e}^{-x^2}$ $\int x^3\mathrm{e}^{-x^2}\,\mathrm{d}x$ $=\int -\frac{1}{2}x^2\cdot\left(-2x\mathrm{e}^{-x^2}\right)\mathrm{d}x$ $=-\frac{1}{2}x^2\mathrm{e}^{-x^2}-\int -x\mathrm{e}^{-x^2}\,\mathrm{d}x$ $=-\frac{1}{2}x^2\mathrm{e}^{-x^2}-\frac{1}{2}\int -2x\mathrm{e}^{-x^2}\,\mathrm{d}x$ $=-\frac{1}{2}x^2\mathrm{e}^{-x^2}-\frac{1}{2}\mathrm{e}^{-x^2}+C$ $u=-\frac{1}{2}x^2$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x}=-2x\mathrm{e}^{-x^2}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=-x$ $\quad v=\mathrm{e}^{-x^2}$	Differentiation was well done except for a few students who clearly do not know how to differentiate exponential form. Many did not observe the word 'HENCE' in the question. They went on to do by parts, splitting e^{-x^2} and x^3 and was unable to continue.

Q2. Inequalities

Assessment Objectives	Solution	Examiner's Feedback
Use of algebraic method to solve inequality.	$x^2+4x+3-\frac{x+3}{2x+1}<0$ $(x+3)(x+1)-\frac{x+3}{2x+1}<0$ $(x+3)\frac{(x+1)(2x+1)-1}{2x+1}<0$ $\frac{(x+3)\left[2x^2+3x+1-1\right]}{2x+1}<0$ $\frac{(x+3)(x)(2x+3)}{2x+1}<0$  $\therefore -3<x<-\frac{3}{2}$ or $-\frac{1}{2}<x<0$	Despite reminders, many students still went to cross-multiply for inequality. Since this questions says "without the use of a calculator", it is expected that students factorize completely and obtain the roots. Incomplete factorization will result in loss of marks. A number of students wrote $-3\le x\le -\frac{3}{2}$ or $-\frac{1}{2}<x\le 0$ as the final answer. This should not be as the question is a strict inequality to begin with. Many students also wrote 'and' instead of 'or'.
Use of replacement	$-3\le x^2\le -\frac{3}{2}$ or $-\frac{1}{2}<x^2\le 0$ $\therefore\quad x=0$	Many students could identify the correct replacement but could not go on to obtain the final answer mark for $x=0$ as the answer in the earlier part was wrong.

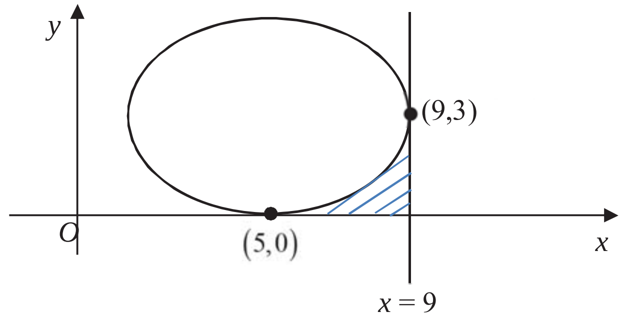
<table>
<tr><th colspan="3">Q3. Definite Integrals + Conics + Transformations</th></tr>
<tr><th>Assessment Objectives</th><th>Solution</th><th>Examiner’s Feedback</th></tr>
<tr><td>Standard graph - ellipse</td><td>

(i)



</td><td>This part was generally well attempted with some students losing marks for not indicating the coordinates of points specifically requested by the question. There is a significant number of students who sketched a hyperbola instead and given no credit at all.</td></tr>
<tr><td>Volume about x-axis</td><td>

(ii)



$$\frac{(x-5)^2}{16}+\frac{(y-3)^2}{9}=1 \Rightarrow y=3\pm\sqrt{9\left(1-\frac{(x-5)^2}{16}\right)}$$

$$\Rightarrow y=3-\sqrt{9\left(1-\frac{(x-5)^2}{16}\right)} \quad (\because y<3)$$

Volume (about x-axis)

$$=\pi\int_5^9 y^2\,dx=\pi\int_5^9\left(3-\sqrt{9\left(1-\frac{(x-5)^2}{16}\right)}\right)^2 dx$$

$$=10.84267894=10.8 \text{ (to 3 s.f.)}$$

</td><td>This part was very badly attempted or not attempted at all by students. Most students who attempted the question did not realize which area was being referred to and did not managed to get the correct expression for y, but instead assumed the negative square root to be rejected since the $y>0$. A large number of students are also confused about the limits to be used in the definite integral or did not realize in their algebraic manipulation that $\sqrt{a^2+b^2}\neq a+b$ and they proceed to obtain an overly simplified expression. A small but significant number of students set up the integral with respect to y instead when the axis of rotation was clearly defined.</td></tr>
</table>

Translation and Scaling	**(iii)**	Scaling parallel to x axis with scale factor of ½ followed by translation of 3 units in the negative y axis. **OR** Translation of 3 units in the negative y axis followed by a scaling parallel to x axis with scale factor of ½.	This part was very badly attempted. Many students used their own colloquial terms such as "Stretch", "Multiple", "Shift", "Move" which were not accepted. Some used the generic term "Transform" in place of "scale" and "translate" when they were required to describe the transformation. Many students also failed to include the key word "scale factor" in their description of the scaling step. Other common mistakes includes the confusion of ½ vs 2 for scale factor and + vs – 3 for the magnitude of the translation. Quite a number of students also omitted the axis in which the transformation was applied or stated the wrong axis.

Q4. Sigma Notation		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Simplifying expression by partial fractions	**(i)** <u>**Method 1:**</u> $\frac{3r(r+1)+1}{r^3(r+1)^3} = \frac{3r^2+3r+1}{r^3(r+1)^3}$ $= \frac{(r^3+3r^2+3r+1)-r^3}{r^3(r+1)^3}$ $= \frac{(r+1)^3-r^3}{r^3(r+1)^3}$ $= \frac{1}{r^3} - \frac{1}{(r+1)^3}$ where $A=1$ and $B=-1$. <u>**Method 2:**</u> $\frac{3r(r+1)+1}{r^3(r+1)^3} = \frac{A}{r^3} + \frac{B}{(r+1)^3}$ $3r(r+1)+1 = A(r+1)^3 + Br^3$ By comparing coefficient of r^0: $A=1$ By comparing coefficient of r^3: $A+B=0 \Rightarrow B=-1$ $\frac{3r(r+1)+1}{r^3(r+1)^3} = \frac{1}{r^3} - \frac{1}{(r+1)^3}$	This part was generally well attempted. Students should compare the coefficients on both sides to find the unknowns.
Summation of series by the method of differences	**(ii)** $\frac{7}{(1)^3(2)^3} + \frac{19}{(2)^3(3)^3} + \ldots + \frac{6n(2n+1)+1}{(2n)^3(2n+1)^3}$ $= \sum_{r=1}^{2n} \frac{3r(r+1)+1}{r^3(r+1)^3}$	This part was not well attempted. The result in (i) can be used to find the expression for the series without using sigma notation.

	$=\sum_{r=1}^{2n}\left[\frac{1}{r^3}-\frac{1}{(r+1)^3}\right]$ $=\frac{1}{1^3}-\cancel{\frac{1}{2^3}}$ $+\cancel{\frac{1}{2^3}}-\cancel{\frac{1}{3^3}}$ $+\ldots$ $+\cancel{\frac{1}{(2n-1)^3}}-\cancel{\frac{1}{(2n)^3}}$ $+\cancel{\frac{1}{(2n)^3}}-\frac{1}{(2n+1)^3}$ $=1-\frac{1}{(2n+1)^3}$	Students were generally not familiar with the sigma notation to represent the series. The sigma notation should be used with integral limits. A significant percentage of students wrote the following sigma notation: $\sum_{r=0.5}^{n}\frac{3r(r+1)+1}{r^3(r+1)^3}$ where the lower limit was 0.5 and the running index was assumed to increase by 0.5 each time. Some students were also confused by the last term in the series. Though they recognized that r has been replaced by $2n$, they did not think of $2n$ as the upper limit. Instead they represented the series wrongly by the following sigma notation $\sum_{r=1}^{n}\frac{6r(2r+1)+1}{(2r)^3(2r+1)^3}$ which only included the even terms of the given series (i.e. 2^{nd}, 4^{th}, $6^{\text{th}},\ldots,(2n)^{\text{th}}$). Some students found a positive value of B in (i) and blindly performed method of difference even though it was not possible. Credit was not given in such cases.
Limits, Sum to infinity and use of standard series expansion	**(iii)** As $2n\to\infty$, $\frac{1}{(2n+1)^3}\to 0$, $1-\frac{1}{(2n+1)^3}\to 1$	This part was not well attempted. Students were able to recognize that the question asked for the long-term average. However, they were confused by the running index r and the unknown constant n . A significant percentage of students considered the case where $r\to\infty$ instead of $2n\to\infty$.

	$\sum_{r=1}^{\infty}\left[\frac{3r(r+1)+1}{r^3(r+1)^3}+\left(\frac{1}{2}\right)^r\right]$ $=\sum_{r=1}^{\infty}\frac{3r(r+1)+1}{r^3(r+1)^3}+\sum_{r=1}^{\infty}\left(\frac{1}{2}\right)^r$ $=1+\left[\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots\right]$ $=1+\frac{1/2}{1-1/2}$ $=2$	Some students did not know how to present their analysis and solutions mathematically for the first summation. They wrongly wrote $\sum_{r=1}^{\infty}\frac{3r(r+1)+1}{r^3(r+1)^3}=1-\frac{1}{(2(\infty)+1)^3}$. As the symbol ∞ is not a number, the presentation should be $\sum_{r=1}^{\infty}\frac{3r(r+1)+1}{r^3(r+1)^3}=\lim_{n\to\infty}\left(1-\frac{1}{(2n+1)^3}\right)=1$. A significant percentage of students had the misconception that if $\left(\frac{1}{2}\right)^r\to 0$ as $r\to\infty$, $\sum_{r=1}^{\infty}\left(\frac{1}{2}\right)^r=0$. For students who recognized that they should use the formula for sum to infinity, • some were not able to identify the first term correctly where they assumed $a=1$, • some were not able to identify the value of common ratio r as they were confused with the running index r.

<table>
<tr><th colspan="3">Q5. A.P. & G.P.</th></tr>
<tr><th>Assessment Objectives</th><th>Solution</th><th>Examiner's Feedback</th></tr>
<tr><td>Finding sum of a finite arithmetic series</td><td>(i) $a = 300$, $d = 20$
Sum of seats in the first n rows ≥ 50000
$\frac{n}{2}\left(2(300)+(n-1)20\right) \geq 50000$
<table><tr><th>n</th><th>$\frac{n}{2}\left(2(300)+(n-1)20\right)$</th></tr><tr><td>57</td><td>49020</td></tr><tr><td>58</td><td>50460</td></tr><tr><td>59</td><td>51920</td></tr></table>
$n \geq 58$
Minimum no. of rows is 58.</td><td>Some students wrote
$T_n = 300+(n-1)20 \geq 50000$
instead of
$\frac{n}{2}\left(2(300)+(n-1)20\right) \geq 50000$.
Some students put "> 50000" instead of "≥ 50000".</td></tr>
<tr><td>Finding nth term of a finite geometric series</td><td>(ii) $a = 60$, $r = 0.9$
Price of seats in 45th row $= 60(0.9)^{3-1} = \$48.60$</td><td>Part (ii) is generally well done.</td></tr>
<tr><td>Finding nth term of a finite arithmetic series</td><td>(iii) Number of seats in 60th row $= 300+(60-1)20$
$= 1480$

Creating new AP starting from last row, $a = 1480$, $d = -20$
$\frac{n}{2}\left(2(1480)+(n-1)(-20)\right) = 51000$
$n = 53.28$
Number of seats in Category 3
$= S_{20} = \frac{20}{2}\left[2(1480)+(20-1)(-20)\right] = 25800$</td><td>There were confusion in the number of seats in each category as students tend to mixed up the calculations for the number of seats in the first and third category. A number of students tried to use Geometric Progression to calculate the number of seats for each category.</td></tr>
</table>

Number of seats in Category 2

$= S_{40} - S_{20} = \frac{40}{2}\left[2(1480)+(40-1)(-20)\right] - 25800 = 43600 - 25800$

$= 17800$

Number of seats in Category 1

$= 51000 - S_{40} = 51000 - 43600 = 7400$

Total revenue $= 7400(60) + 17800(60)(0.9) + 25800(60)(0.9)^2$

$= \$2659080$

<u>Alternative</u>

$a = 300$, $d = 20$

Number of seats in Category 3

$= S_{60} - S_{40}$

$= \frac{60}{2}\left[2(300)+(60-1)(20)\right] - \frac{40}{2}\left[2(300)+(40-1)(20)\right]$

$= 53400 - 27600$

$= 25800$

Number of seats in Category 2

$= S_{40} - S_{20}$

$= 27600 - \frac{20}{2}\left[2(300)+(20-1)(20)\right]$

$= 27600 - 9800$

$= 17800$

Number of seats in Category 1

$= 51000 - 25800 - 17800 = 7400$

	Total revenue $= 7400(60) + 17800(60)(0.9) + 25800(60)(0.9)^2$ $= \$2659080$	

<table>
<tr><th colspan="3">Q6. Maclaurin's Series</th></tr>
<tr><th>Assessment Objectives</th><th>Solution</th><th>Examiner's Feedback</th></tr>
<tr><td>Implicit differentiation and use of product rule</td><td>

$y=\sqrt{e^x \cos x}$

$y^2 = e^x \cos x$

Differentiating w.r.t. x ,

$2y\frac{dy}{dx} = e^x \cos x - e^x \sin x$

$2y\frac{dy}{dx} = y^2 - e^x \sin x$

Differentiating w.r.t. x ,

$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} - \left(e^x \sin x + e^x \cos x\right)$

$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} + \left(2y\frac{dy}{dx} - y^2\right) - y^2$

$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 4y\frac{dy}{dx} - 2y^2$

$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} - y^2$ (shown)

</td><td>A lot of candidates did not square the y and then implicitly differentiate, but instead tried to bulldoze their way through the repeated differentiation, to various outcomes. Much time could have been saved if the correct preparation for implicit differentiation was used.

Many students could not get to the final required form of the equation because they could not see the appropriate substitution for $e^x \sin x$ or $e^x \cos x$

Some notations which are not correct are used, eg: $\frac{dy}{dx}(y^2) = \frac{dy}{dx}(e^x \cos x)$
Or silly errors like $e^x \cos x$ becoming $ex\cos x$ and $\cos^2 x$ becoming $\cos 2x$</td></tr>
</table>

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<tr>
<td></td>
<td>Differentiating w.r.t. x ,
$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2y\frac{\mathrm{d}y}{\mathrm{d}x}$
When $x=0$, $\mathrm{f}(0)=1$, $\mathrm{f}'(0)=\frac{1}{2}$, $\mathrm{f}''(0)=-\frac{1}{4}$, $\mathrm{f}'''(0)=-\frac{5}{8}$
$y=1+\frac{x}{2}+\frac{x^2}{2}\left(-\frac{1}{4}\right)+\frac{x^3}{3!}\left(-\frac{5}{8}\right)$
$y=1+\frac{x}{2}-\frac{x^2}{8}-\frac{5x^3}{48}$</td>
<td></td>
</tr>
<tr>
<td>Verification</td>
<td>Using MF26,
$y=\sqrt{\mathrm{e}^x\cos x}$
$y=\mathrm{e}^{\frac{x}{2}}(\cos x)^{\frac{1}{2}}$
$y\approx\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}\right)\left(1-\frac{x^2}{2}\right)^{\frac{1}{2}}$
$y\approx\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}\right)\left(1+\frac{1}{2}\left(-\frac{x^2}{2}\right)\right)$
$y\approx1-\frac{x^2}{4}+\frac{x}{2}-\frac{x^3}{8}+\frac{x^2}{8}+\frac{x^3}{48}$
$y=1+\frac{x}{2}-\frac{x^2}{8}-\frac{5x^3}{48}$ (verified)</td>
<td>Some students misunderstand the question. They are supposed to verify that the expression in (ii) is correct. They interpreted it as checking IF the expression is correct and some concluded that it is not.

A worrying situation is that many students commit basic algebraic mistakes, like saying
$\left(1+x-\frac{x^3}{3}\right)^{\frac{1}{2}}=1+x^{\frac{1}{2}}-\frac{1}{3}x^{\frac{3}{2}}$</td>
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OR

Using MF26,

$y=\sqrt{e^x \cos x}$

$y \approx \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)^{\frac{1}{2}}\left(1-\frac{x^2}{2}\right)^{\frac{1}{2}}$

$y \approx \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^2}{2}-\frac{x^3}{2}\right)^{\frac{1}{2}}$

$y=\left(1+x-\frac{x^3}{3}\right)^{\frac{1}{2}}$

$y=1+\frac{1}{2}\left(x-\frac{x^3}{3}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(x-\frac{x^3}{3}\right)^2$

$+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(x-\frac{x^3}{3}\right)^3+\ldots$

$y=1+\frac{x}{2}-\frac{x^3}{6}-\frac{1}{8}\left(x^2\right)+\frac{1}{16}x^3+\ldots$

$y \approx 1+\frac{x}{2}-\frac{x^2}{8}-\frac{5x^3}{48}$ (verified)

</td>
<td>

Some students gave too many terms and had a hard time manipulating. They have to realize that they can truncate the series for terms beyond x^3. Some students truncate too much too early, ending up with the incorrect answer too.

A few students square their expression in (ii) to compare with the standard series expansion of y^2 and concluded it is correct. Credit is not given because $A^2=B^2$ does not necessarily imply $A=B$

</td>
</tr>
</table>

Q7. Applications of Differentiation		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Maxima/Minima problems	**(i)** Volume of trough $V=\frac{1}{2}(2)(10)h\sqrt{4-h^2}$ $V=10h\sqrt{4-h^2}$ $\frac{dV}{dh}=10\sqrt{4-h^2}+\frac{1}{2}(10h)\left(4-h^2\right)^{-\frac{1}{2}}(-2h)$ $\frac{dV}{dh}=10\sqrt{4-h^2}-10h^2\left(4-h^2\right)^{-\frac{1}{2}}$ To maximize volume of trough, $\frac{dV}{dh}=0$ $10h^2\left(4-h^2\right)^{-\frac{1}{2}}=10\sqrt{4-h^2}$ $10h^2=10\left(4-h^2\right)$ $4=2h^2$ $h=\sqrt{2}$ $\left(\text{reject } -\sqrt{2} \text{ since } h>0\right)$ **OR** $V^2=100h^2\left(4-h^2\right)$ $V^2=400h^2-100h^4$ $2V\frac{dV}{dh}=800h-400h^3$ To maximize volume of trough, $\frac{dV}{dh}=0$ $400h(2-h^2)=0$ $h=\sqrt{2}$ $\left(\text{reject } h=-\sqrt{2} \text{ and } h=0 \text{ since } h>0\right)$	Many students were not able to form the expression of the volume correctly. This is the most commonly seen mistake: $V=\frac{1}{2}(10)h\sqrt{4-h^2}$. When it comes to differentiation, some students made mistake in the chain rule while performing the product rule. Example: $-2h$ was often missed out.

<table>
<tr>
<td></td>
<td>1st derivative test:
$\begin{array}{|c|c|c|c|} \hline h & \sqrt{2}^- & \sqrt{2} & \sqrt{2}^+ \\ \hline \frac{\mathrm{d}V}{\mathrm{d}\theta} & + & 0 & - \\ \hline \end{array}$

OR
2nd derivative test:
$\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = 10h\left(4-h^2\right)^{-\frac{3}{2}}\left(2h^2-12\right)$
When $h=\sqrt{2}$, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2} < 0$

It is a maximum value.

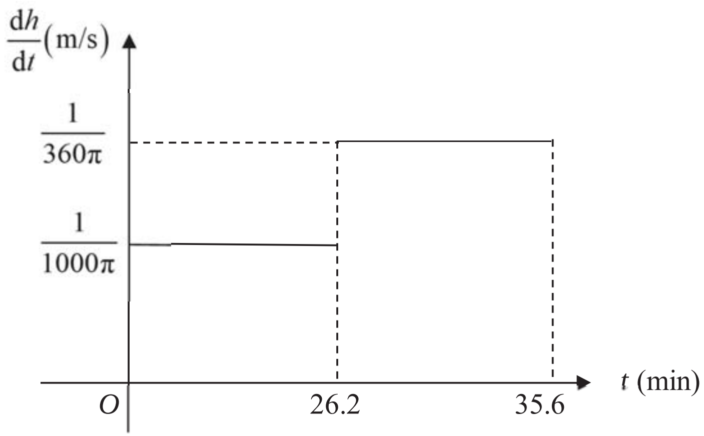
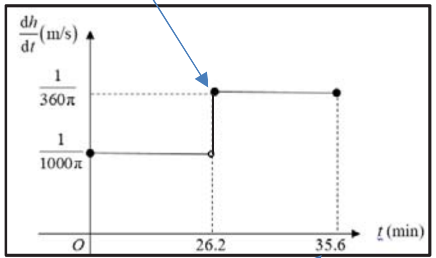
$V = 10\sqrt{2}\sqrt{4-2} = 20\mathrm{m}^3$</td>
<td>Most students were able to verify the maximum volume but the presentation was quite bad for some scripts.

Some students lost one mark as they forgot to find the max volume.</td>
</tr>
<tr>
<td>Rate of change</td>
<td>(ii) $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.001\ \mathrm{cm}^3/\mathrm{s}$
After 29 min, $V = 0.001(60)(29) = 1.74\,\mathrm{m}^3$
Volume of larger cylinder, $V_L = \pi r^2 h = \pi(1)^2(0.5) = 1.570796\,\mathrm{m}^3$
Thus the water level is at the smaller cylinder after 29 minutes.

$V_S = \pi(0.6)^2 h$
$\frac{\mathrm{d}V_S}{\mathrm{d}h} = \pi(0.6)^2$
$\frac{\mathrm{d}V_S}{\mathrm{d}t} = \frac{\mathrm{d}V_S}{\mathrm{d}h_S} \cdot \frac{\mathrm{d}h_S}{\mathrm{d}t}$</td>
<td>Generally quite OK. Most students attempted this part using a variety of methods to obtain the answer.

Common mistake: students forgot to check the water level at 29 minutes and just made their own assumption while majority attempted to form the expression of the volume by considering both large and small containers.</td>
</tr>
</table>

<table>
<tr><td></td><td>$\frac{\mathrm{d}h_s}{\mathrm{d}t} = \frac{1}{\pi(0.6)^2} \cdot 0.001$ for $r = 0.6$
$= \frac{1}{360\pi}$ m/s</td><td></td></tr>
<tr><td>Sketch graph</td><td>(iii) For the larger cylinder:
$V_L = \pi(1)^2 h$
$\frac{\mathrm{d}V_L}{\mathrm{d}h} = \pi(1)^2$
$\frac{\mathrm{d}V_L}{\mathrm{d}t} = \frac{\mathrm{d}V_L}{\mathrm{d}h_L} \cdot \frac{\mathrm{d}h_L}{\mathrm{d}t}$
$\frac{\mathrm{d}h_L}{\mathrm{d}t} = \frac{1}{\pi(1)^2} \cdot 0.001$ for $r = 1$
$= \frac{1}{1000\pi}$ m/s
For the smaller cylinder:
$\frac{\mathrm{d}h_s}{\mathrm{d}t} = \frac{1}{360\pi}$
Time taken to fill the larger cylinder
$= \frac{\pi(1)^2(0.5)}{0.001} = 1570.796$ s=26.179938 min=26.2 min
Time taken to fill up the smaller cylinder
$= \frac{\pi(0.6)^2(0.5)}{0.001} = 565.4867$ s $= 9.42478$ min
Total time taken to fill up the container
= 26.179938+9.42478 min
= 35.6 min</td><td>Many students attempted this part and also able to obtained at least 1 or 2 out of 3 marks.</td></tr>
</table>

	 	Common mistake: (extra vertical line)  Many lost the 1 mark as they didn't manage to find this.

Q8. Complex Numbers		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Solving of simultaneous equations	**(a)** $3z-\mathrm{i}w=12+19\mathrm{i} \quad \cdots\cdots(1)$ $2z^*+3w=16-23\mathrm{i} \quad \cdots\cdots(2)$ $(1)\times 3$: $9z-3\mathrm{i}w=36+57\mathrm{i}\cdots\cdots(3)$ $(2)\times \mathrm{i}$: $2\mathrm{i}z^*+3\mathrm{i}w=16\mathrm{i}+23\cdots\cdots(4)$ $(3)+(4): 9z+2\mathrm{i}z^*=73\mathrm{i}+59$ Let $z=x+\mathrm{i}y$, so $z^*=x-\mathrm{i}y$ $9(x+\mathrm{i}y)+2\mathrm{i}(x-\mathrm{i}y)=73\mathrm{i}+59$ **Comparing real and imaginary parts,** $9x+2y=59$ and $9y+2x=73$ **Solving,** $x=5$ and $y=7$ $\therefore \quad z=5+7\mathrm{i}$ **From (2),** $2z^*+3w=16-23\mathrm{i}$ $w=\frac{1}{3}(16-23\mathrm{i}-2z^*)=\frac{1}{3}(16-23\mathrm{i}-2(5-7\mathrm{i}))$ $w=2-3\mathrm{i}$	Majority of the candidates lost their final mark because they simply cannot copy the question properly. Many wrote $9\mathrm{i}$ instead of $19\mathrm{i}$, this is unacceptable! Some candidates miss out on the instruction "Do not use calculator in answering the question", they still write "using GC" while solving the simultaneous equation after comparing real and imaginary parts. Some candidates took a longer way by letting $z=a+b\mathrm{i}$ and $w=c+d\mathrm{i}$ and solved for 4 unknowns. It is a tedious process and they ended up using GC to solve. This is not recommended. Some candidates made a conceptual error. After making z the subject from equation (1): $z=\frac{1}{3}(12+(19+w)\mathrm{i})$, they went on to concude that $z^*=\frac{1}{3}(12-(19+w)\mathrm{i})$ **which is incorrect!** They cannot do this step here as w is a complex number.
Complex conjugate roots	**(b)(i)** Since all coefficients are real, the other root is $(-1-\sqrt{3}\mathrm{i})$. $\left[z-(-1+\sqrt{3}\mathrm{i})\right]\left[z-(-1-\sqrt{3}\mathrm{i})\right]=z^2+2z+4$ $z^4+3z^3+4z^2-8=(z^2+2z+4)(z^2+az+b)$	Many candidates are unsure about factors and roots. Some candidates gave the conjugate as $1+\sqrt{3}\mathrm{i}$ (negate the real part) which is wrong.

	Comparing coefficient of z^3: $3=a+2 \quad \Rightarrow a=1$ Comparing coefficient of z^2: $4=b+2a+4 \quad \Rightarrow b=-2$ Comparing coefficient of z: $0=2b+4a \quad \Rightarrow b=-2$ $z^4+3z^3+4z^2-8=\left(z^2+2z+4\right)\left(z^2+z-2\right)$	Candidates who did not get full credit struggled at algebraic manipulations when comparing coefficients/ long division. There is no need to solve for the roots.
Complex numbers in exponential form	**(b)(ii)** $\mathrm{e}^{p+\mathrm{i}q}=z_1^{\ 5}$ $z_1=-1+\sqrt{3}\mathrm{i}=2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$ $\mathrm{e}^p\mathrm{e}^{\mathrm{i}q}=\left(2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}\right)^5$ $\mathrm{e}^p\mathrm{e}^{\mathrm{i}q}=32\mathrm{e}^{\mathrm{i}\left(\frac{10\pi}{3}\right)}$ $\mathrm{e}^p=32 \quad \Rightarrow p=\ln 32$ $q=\frac{10\pi}{3}-\frac{12\pi}{3}=-\frac{2\pi}{3}$	Many left this blank. Candidates who expanded this using Cartesian form often struggled, they are recommended to convert to modulus-argument form. Some forgot about the power 5. Candidates need to practice more on finding arguments of complex number. $z_1=-1+\sqrt{3}\mathrm{i}$ lies in the 2^{nd} quadrant and hence, $\arg\left(z_1\right)=\pi-\tan^{-1}\left\|\frac{\sqrt{3}}{-1}\right\|$ (when finding basic angle, always $\tan^{-1}$ the positive real/ im parts) Some did not know how to convert q into the principal range. They need to subtract multiples of 2π

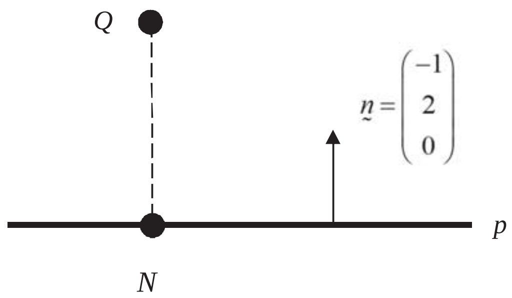
Q9. Vectors (Lines & Planes)

Assessment Objectives	Solution	Examiner's Feedback
Line lies on plane	**(i)** $l_1 : \mathbf{r} = \begin{pmatrix} -1 \\ 8 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1+2\lambda \\ 8+\lambda \\ 3+5\lambda \end{pmatrix}$ $p : \mathbf{r} \cdot \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 17$ Since l_1 lies completely on p, $\begin{pmatrix} -1 \\ 8 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 17$ $-a + 8b = 17 \cdots\cdots (1)$ Since l_1 lies in p, $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ are perpendicular, $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = 0$ $2a + b = 0$ $b = -2a$ Substitute $b = -2a$ into (1), $-a - 16a = 17$ $-17a = 17$ $\therefore a = -1$ and $b = -2a = 2$ (Shown)	Most students were able to use the information provided to formulate 2 equations in a and b, and proceeded to solve simultaneously. Alternative method of obtaining the two equations was to substitute different values of λ in order to obtain different points on l_1.

<table>
<tr>
<td>Foot of perpendicular</td>
<td>(ii)

$\overrightarrow{OQ}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}$

Let N be the foot of perpendicular from point Q to p.



Equation of the line passing though N and Q is

$\underset{\sim}{r}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\0\end{pmatrix},\mu\in\mathbb{R}$

$=\begin{pmatrix}-1-\mu\\-2+2\mu\\3\end{pmatrix}$

When the line passes through N on p

$\begin{pmatrix}-1-\mu\\-2+2\mu\\3\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\0\end{pmatrix}=17$

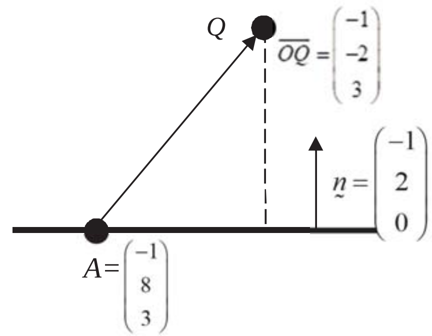
$(1+\mu)+(-4+4\mu)=17$

$5\mu-3=17$

$5\mu=20$

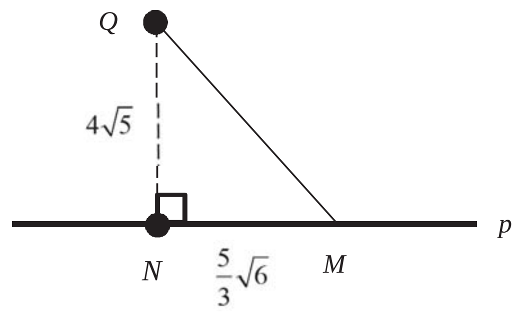
$\mu=4$</td>
<td>Common mistake was for students to erroneously assume that point Q lies on the plane. A quick check would have showed otherwise.

A handful of students were able to visualize the relationship between l_1, p and Q to discover that the perpendicular distance can be found efficiently using the length of projection formula.

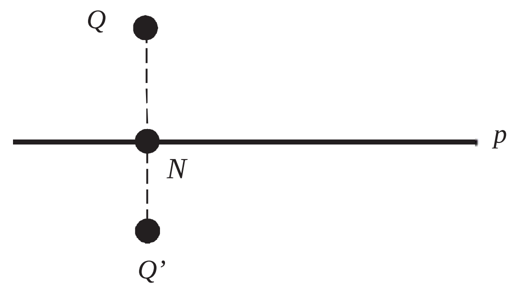
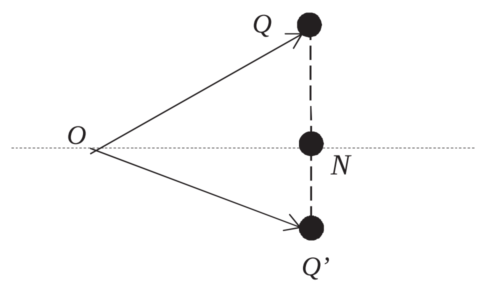


$\overrightarrow{AQ}=\overrightarrow{OQ}-\overrightarrow{OA}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}-\begin{pmatrix}-1\\8\\3\end{pmatrix}=\begin{pmatrix}0\\-10\\0\end{pmatrix}$

$\left|\overrightarrow{AQ}\bullet\hat{\underset{\sim}{n}}\right|=\left|\begin{pmatrix}0\\-10\\0\end{pmatrix}\bullet\frac{\begin{pmatrix}-1\\2\\0\end{pmatrix}}{\sqrt{1^2+2^2}}\right|=\frac{20}{\sqrt{5}}=4\sqrt{5}$</td>
</tr>
</table>

	$\therefore \overrightarrow{ON} = \begin{pmatrix} -1-\mu \\ -2+2\mu \\ 3 \end{pmatrix} = \begin{pmatrix} -1-4 \\ -2+8 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ 3 \end{pmatrix}$ Shortest Distance $= \left\|\overrightarrow{QN}\right\| = \left\|\overrightarrow{ON} - \overrightarrow{OQ}\right\|$ $= \left\| \begin{pmatrix} -5 \\ 6 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \right\|$ $= \left\| \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix} \right\|$ $= \sqrt{4^2 + 8^2}$ $= \sqrt{80}$ $= 4\sqrt{5}$	
Shortest distance	**(iii)**  Shortest distance between Q and l_1	There were some students who identified the lengths to be found for (ii) and (iii) wrongly (i.e. gave (iii) answer for (ii) and vice versa). They were still given the credit. Alternative methods (e.g. using length of projection) were accepted.

	$=\sqrt{\left(4\sqrt{5}\right)^2+\left(\frac{5}{3}\sqrt{6}\right)^2}$ $=\sqrt{\left(\sqrt{80}\right)^2+\left(\frac{25}{9}\right)(6)}$ $=\sqrt{(80)+\left(\frac{150}{9}\right)}$ $=\sqrt{\frac{290}{3}}$ or $=\frac{1}{3}\sqrt{870}$ or 9.83 (3 s.f)	
Equation of line	**(iv)** Direction vector of l_2 $=\begin{pmatrix}2\\1\\5\end{pmatrix}\times\begin{pmatrix}-1\\2\\0\end{pmatrix}$ $=\begin{pmatrix}(0)-(2\times5)\\-\left[(0)-(-1\times5)\right]\\(4)-(-1)\end{pmatrix}$ $=\begin{pmatrix}-10\\-5\\5\end{pmatrix}=-5\begin{pmatrix}2\\1\\-1\end{pmatrix}$ Vector equation of the line l_2 is $\underset{\sim}{r}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}, t\in\mathbb{R}$	Most students were unable to visualize the relationship between l_2, l_1 and p . They tried to work backwards using the given Cartesian equation to obtain the vector equation of l_2 in an attempt to fulfill the question requirement of showing the process to find l_2.

	$x=-1+2t \quad t=\frac{x+1}{2}$ $\Rightarrow\ y=-2+t \Rightarrow t=y+2$ $z=3-t \quad t=3-z$ Cartesian equation of the line l_2 is $\frac{x+1}{2}=y+2=3-z$	
Reflection of line about a plane	**(v)**  Using Ratio Theorem  Since N is the mid-point of $\overrightarrow{OQ}$ and $\overrightarrow{OQ'}$, $\overrightarrow{ON}=\frac{1}{2}\left(\overrightarrow{OQ}+\overrightarrow{OQ'}\right)$ $\overrightarrow{OQ'}=2\overrightarrow{ON}-\overrightarrow{OQ}$	Most students were unable to visualize the situation when a line parallel to a plane is reflected in the plane. They did not realise that the line and reflected lines both share the same direction vector. Students who made mistakes in previous parts (e.g. position vector of foot of perpendicular) were given credit for their attempt to apply Ratio Theorem.

	$\overrightarrow{OQ'} = 2\begin{pmatrix}-5\\6\\3\end{pmatrix} - \begin{pmatrix}-1\\-2\\3\end{pmatrix} = \begin{pmatrix}-9\\14\\3\end{pmatrix}$ Vector equation of the line l_3 is $\underset{\sim}{r} = \begin{pmatrix}-9\\14\\3\end{pmatrix} + s\begin{pmatrix}2\\1\\-1\end{pmatrix}, s \in \mathbb{R}$	

Q10. Differential Equations

Assessment Objectives	Solution		Examiner's Feedback
Formulate of differential equation	**(i)** Growth rate $\propto \sqrt{L}$ Destruction rate $\propto L$ $\frac{\mathrm{d}L}{\mathrm{d}t} = a\sqrt{L} - bL$ When $L = 1$, $\frac{\mathrm{d}L}{\mathrm{d}t} = 1$ $a - b = 1$ $\quad -(1)$ When $L = 4$, $\frac{\mathrm{d}L}{\mathrm{d}t} = -2$ $a\sqrt{4} - 4b = -2$ $2a - 4b = -2$ $\quad -(2)$ Solving, $a = 3, b = 2$ $\frac{\mathrm{d}L}{\mathrm{d}t} = 3\sqrt{L} - 2L$ (shown)		Some students did not manage to formulate two equations with two unknowns as they set $a = b$ Most of those who managed to get the two equations with two unknowns proceeded to get the correct values of a and b. Many students did not attempt this part.
Solving differential equation	**(ii)** Using substitution: $L = y^2$ Differentiate with respect to t: $\frac{\mathrm{d}L}{\mathrm{d}t} = 2y\frac{\mathrm{d}y}{\mathrm{d}t}$ $2y\frac{\mathrm{d}y}{\mathrm{d}t} = 3y - 2y^2$ $2\frac{\mathrm{d}y}{\mathrm{d}t} = 3 - 2y$ $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3-2y}{2}$	Using substitution: $y = \sqrt{L}$ Differentiate with respect to t: $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}L^{-\frac{1}{2}}\frac{\mathrm{d}L}{\mathrm{d}t}$ $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}L^{-\frac{1}{2}}\left(3\sqrt{L} - 2L\right)$ $= \frac{3}{2} - \sqrt{L} = \frac{3}{2} - y$ $= \frac{3-2y}{2}$	Most students used $L = y^2$. The minority who used $y = \sqrt{L}$ are also able to get the correct equation.

	$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{3-2y}{2}$ $\frac{2}{3-2y}\frac{\mathrm{d}y}{\mathrm{d}t}=1$	
	$\int \frac{2}{3-2y}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t=\int 1\,\mathrm{d}t$ $-\int \frac{-2}{3-2y}\,\mathrm{d}y=t+C$	Most students are able to identify this as a separation of variables DE.
	$-\ln\lvert 3-2y\rvert=t+C$ $\lvert 3-2y\rvert=\mathrm{e}^{-t-C}$ $3-2y=\pm\mathrm{e}^{-C}\mathrm{e}^{-t}$ $3-2y=A\mathrm{e}^{-t}$, where $A=\pm\mathrm{e}^{-C}$ $2y=3-A\mathrm{e}^{-t}$ $y=\frac{3-A\mathrm{e}^{-t}}{2}$ $y=\sqrt{L}=\frac{3-A\mathrm{e}^{-t}}{2}$	Many students did not include the modulus sign for the logarithm. Among those who did have the modulus sign, many removed the modulus and did not put $\pm$
	$L=\left(\frac{3-A\mathrm{e}^{-t}}{2}\right)^2$ When $t=0$, $L=1$, $1=\left(\frac{3-A\mathrm{e}^{-0}}{2}\right)^2$ Since $y>0$, $\frac{3-A}{2}=1\Rightarrow A=1$ $L=\left(\frac{3-\mathrm{e}^{-t}}{2}\right)^2$	Many students stopped here without finding A .

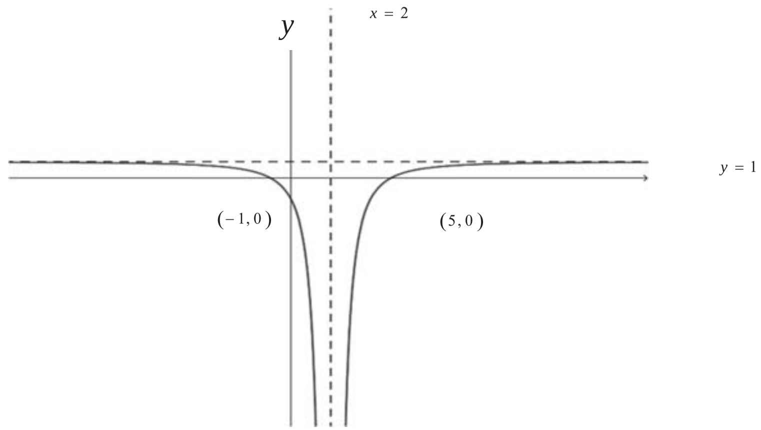
	(iii) $\frac{\mathrm{d}^2L}{\mathrm{d}t^2}=\frac{-2t}{\left(1+t^2\right)^2}$ Integrate both sides with respect to *t*, $\frac{\mathrm{d}L}{\mathrm{d}t}=\int\frac{-2t}{\left(1+t^2\right)^2}\,\mathrm{d}t$ $=-\int 2t\left(1+t^2\right)^{-2}\,\mathrm{d}t$ $=-\left[\frac{\left(1+t^2\right)^{-1}}{-1}\right]+C$ $=\frac{1}{1+t^2}+C$ When $t=0,\ L=1,\ \frac{\mathrm{d}L}{\mathrm{d}t}=1$, $1=\frac{1}{1+0^2}+C$ $C=0$ $\therefore\frac{\mathrm{d}L}{\mathrm{d}t}=\frac{1}{1+t^2}$ Integrate again both sides with respect to *t*, $L=\int\frac{1}{1+t^2}\,\mathrm{d}t$ $=\tan^{-1}t+D$ When $t=0,\ L=1$, $1=\tan^{-1}0+D$ $D=1$ $\therefore L=\tan^{-1}t+1$	Most students could not identify the form $\mathrm{f}'(x)\mathrm{f}(x)^n$ and thus could not integrate. Some students who proceeded either missed out or did not understand the phrase 'particular solution', thus they did not substitute in the initial values.

Limits	**(iv)** As $t \to \infty$, For Model A: $\mathrm{e}^{-t} \to 0, L \to \left(\frac{3}{2}\right)^2 = \frac{9}{4}$ For Model B: $\tan^{-1} t \to \frac{\pi}{2}, L \to 1 + \frac{\pi}{2}$	Common mistake was using 90 instead of $\frac{\pi}{2}$. Angles should be in radians for calculus questions.

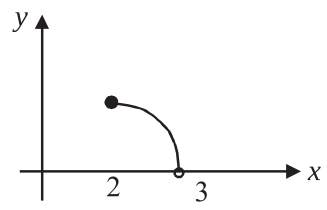
Q1. System of Linear Equations

Assessment Objectives	Solution	Examiner's Feedback
Solve system of linear equations.	$\frac{13}{4}=a(1)+b(1)+\frac{1}{c}$ $97=a(64)+b(16)+\frac{4}{c}$ $642=a(512)+b(64)+\frac{8}{c}$ From G.C., $a=1,\quad b=2,\quad \frac{1}{c}=0.25$ $\therefore a=1,\quad b=2,\quad c=4$	Most candidates were successful with the question except those who did not know how to handle the transformed coordinates or the replacement of 1/*c* as another variable. There were also a lot of algebraic manipulation errors which reflected the lack in numeracy skills of the candidature. There is also a significant group of candidates who attempted to solve the equations manually instead of using the GC, and most of them were unsuccessful.

Q2. Vectors (Basic)		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Concept of parallel vectors	**(i)** $k\mathbf{p} = m\mathbf{q}$ where $m = (\mathbf{p}\cdot\mathbf{q}) \in \mathbb{R}\setminus\{0\}$ $\mathbf{p} = \frac{m}{k}\mathbf{q} = n\mathbf{q}$ where $n = \frac{m}{k} \in \mathbb{R}\setminus\{0\}$ Since $\mathbf{p} = n\mathbf{q}$, $\mathbf{p}$ and $\mathbf{q}$ are parallel vectors.	This basic question proved to be difficult for most candidates, where only a handful provided a complete solution with explanation. Majority of the accepted answers simply contain the key word "parallel" with ambiguous statements and were given the benefit of doubt.
Use of scalar product	**(ii)** $k\lvert\mathbf{p}\rvert = \lvert\mathbf{p}\bullet\mathbf{q}\rvert\lvert\mathbf{q}\rvert$ Since **p** and **q** are parallel vectors from part **(i)**, $\theta = 0°$ or $\theta = 180°$ so $\cos\theta = 1$, so $\lvert\mathbf{p}\bullet\mathbf{q}\rvert = \lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert$. $k\lvert\mathbf{p}\rvert = (\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert)\lvert\mathbf{q}\rvert$ $k = \lvert\mathbf{q}\rvert^2$ since $\lvert\mathbf{p}\rvert \neq 0$ $\lvert\mathbf{q}\rvert^2 = k$ $\lvert\mathbf{q}\rvert = \sqrt{k}$ since $k > 0$.	This question was either not attempted or very badly attempted by the candidature. Almost every candidate performed division on the vectors directly, which is conceptually wrong, instead of applying modulus throughout to reduce the equation to that consisting of only scalars and the usual algebraic manipulation can be applied. Other errors include assuming that the vectors are in the same direction (excluding opposite as a possibility) and comparing coefficients.

Q3. Applications of Differentiation (f' graph)		
Assessment Objectives	**Solution**	**Examiner's Feedback**
Sketching graph of $f'(x)$	**(i)** 	Generally students did quite ok for this part. Some did not present the turning points in coordinates form.
Graphical interpretation of $f'(x)<0$	**(ii)(a)** $-1<x<2$ or $2<x<5$	Quite badly attempted.
Graphical interpretation of $f''(x)>0$	**(ii)(b)** $x>2$	Quite badly attempted.

Q4. Functions

<table>
<tr><th>Assessment Objectives</th><th>Solution</th><th>Examiner’s Feedback</th></tr>
<tr><td>Condition for existence of inverse function</td><td>(i)


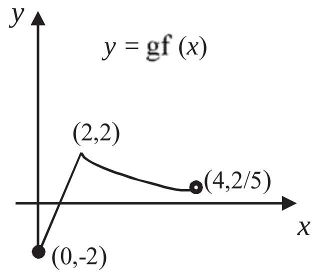
Any horizontal line cuts the graph of f at most once, f is a one-one function, f^{-1} exists.</td><td>Badly attempted.

Common mistakes:

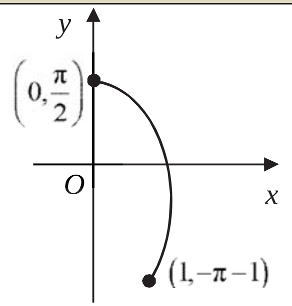
Many used only one counter example to show that f is one-one.</td></tr>
<tr><td>Able to resolve the modulus based on the domain and able to find the inverse function.</td><td>(ii)
$y = x(3-x)$
$y = 3x - x^2$
$x^2 - 3x + y = 0$
$\left(x-\frac{3}{2}\right)^2 - \frac{9}{4} + y = 0$
$x = \frac{3}{2} \pm \sqrt{\frac{9}{4} - y}$
Since $x > 2$, $x = \frac{3}{2} + \sqrt{\frac{9}{4} - y}$
$f^{-1}(x) = \frac{3}{2} + \sqrt{\frac{9}{4} - x}$, $D_{f^{-1}} = (0, 2]$</td><td>Badly done. Many did not know how to resolve the modulus by looking at the domain.

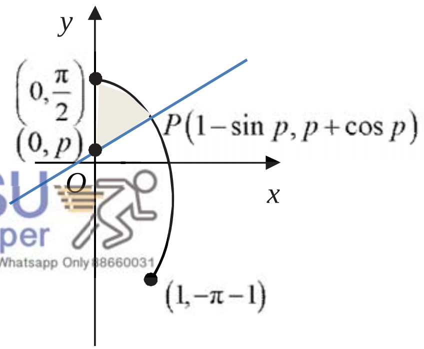
Some made algebraic slip and hence not able to perform completing the square to make x the subject.

However majority were able to find the domain.</td></tr>
</table>

Sketch of piece wise functions	**(iii)** 	Most students attempted this part and were able to get either 1 or 2 marks. Students need to pay more attention to the presentation such as to label the coordinates of the end points, inclusive or exclusive.
Composite function with piece wise.	**(iv)** $\text{gf}(x) = 2(x(3-x)) - 2$ $= 2(3x - x^2 - 1)$	Badly attempted. Many students did not seem to understand the condition of the existence of composite function.
Range of composite function.	**(v)** $R_{\text{gf}} = (-2, 2]$	Badly done. Students who used the mapping method were able to get the answer easily.

Q5. Parametric Equations

Assessment Objectives	Solution	Examiner's Feedback
Sketch parametric graphs	**(i)**  At $\theta=-\pi$, $x=1-\sin(-\pi)=1$, $y=-\pi+\cos(-\pi)=-\pi-1$ $\therefore (1,-\pi-1)$ At $\theta=\frac{\pi}{2}$, $x=1-\sin\left(\frac{\pi}{2}\right)=0$, $y=\frac{\pi}{2}+\cos\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$ $\therefore \left(0,\frac{\pi}{2}\right)$	Many candidates failed to give the coordinates in the **exact** form, as required by the question. They are also unable to sketch the curve in the specified domain. Endpoints must be clearly labelled with solid circles since both endpoints are included.
Gradient of parametric equations	(ii) $x=1-\sin\theta$, $y=\theta+\cos\theta$, $\frac{dx}{d\theta}=-\cos\theta$, $\frac{dy}{d\theta}=1-\sin\theta$ $\frac{dy}{dx}=\frac{1-\sin\theta}{-\cos\theta}=\frac{\sin\theta-1}{\cos\theta}$ As $\theta\to-\frac{\pi}{2}$, $\cos\left(-\frac{\pi}{2}\right)\to 0$, $\frac{dy}{dx}=\frac{1-\sin\theta}{\cos\theta}\to\infty$. The tangent is vertical. **OR** The tangent is parallel to y-axis.	Finding $\frac{dy}{dx}$ was generally well-attempted. However some candidates are still confused over differentiation of simple trigo terms and carelessness. The last part was badly attempted. Candidates found the value of $\frac{dy}{dx}$ but did not proceed to explain "what happened to the tangent". Those who wrote "gradient of tangent tends to infinity" were given the credit. However, "tangent tends to infinity" is not accepted as tangent is a line!

		Candidates should learn how to spell “vertical” properly.
Equation of normal	**(iii)** At P, $\theta = p$, $(1-\sin p, p+\cos p)$ Gradient of normal $= \dfrac{\cos p}{1-\sin p}$ Equation of normal at P: $y-(p+\cos p)=\dfrac{\cos p}{1-\sin p}(x-(1-\sin p))$ At y-axis, $x=0$: $y-(p+\cos p)=\dfrac{\cos p}{1-\sin p}(0-(1-\sin p))$ $y-p-\cos p=-\cos p$ $y=p$ $\therefore Q(0, p)$	Majority of the candidates could give the gradient of normal based on their results in (ii), while some still struggled. Many went on to assume values of p to be 0, $\frac{\pi}{4}$ or $\frac{\pi}{2}$ which is not needed. They merely used the results $(0, p)$ to verify instead of showing. Algebraic manipulation needs to be further strengthened.
Area involving parametric equations	**(iv)** Integrating abt y-axis: 	This was badly attempted, showing that candidates have very weak foundation in finding areas involving parametric equations. Diagram has to be aided to solve this part.

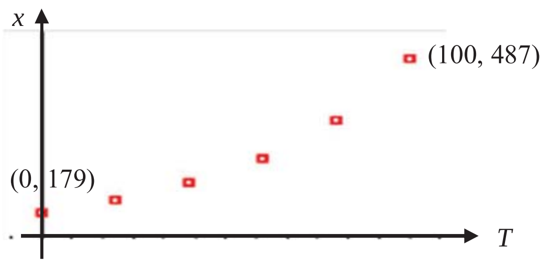
	area $=\frac{1}{2}(p+\cos p-p)(1-\sin p)+\int_{p+\cos p}^{\frac{\pi}{2}} x\ \mathrm{d}y$ $=\frac{1}{2}(\cos p)(1-\sin p)+\int_{p}^{\frac{\pi}{2}} x\ \frac{\mathrm{d}y}{\mathrm{d}\theta}\ \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin p\cos p}{2}+\int_{p}^{\frac{\pi}{2}}(1-\sin\theta)^2\ \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\int_{p}^{\frac{\pi}{2}}1-2\sin\theta+\sin^2\theta\ \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\int_{p}^{\frac{\pi}{2}}1-2\sin\theta+\frac{1-\cos 2\theta}{2}\ \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\int_{p}^{\frac{\pi}{2}}\frac{3}{2}-2\sin\theta-\frac{\cos 2\theta}{2}\ \mathrm{d}\theta$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\left[\frac{3}{2}\theta+2\cos\theta-\frac{\sin 2\theta}{4}\right]_{p}^{\frac{\pi}{2}}$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\left[\frac{3}{2}\cdot\frac{\pi}{2}+2\cos\frac{\pi}{2}-\frac{\sin\pi}{4}-\left(\frac{3}{2}p+2\cos p-\frac{\sin 2p}{4}\right)\right]$ $=\frac{1}{2}\cos p-\frac{\sin 2p}{4}+\frac{3\pi}{4}-\frac{3}{2}p-2\cos p+\frac{\sin 2p}{4}$ $=\frac{3\pi}{4}-\frac{3}{2}p-\frac{3}{2}\cos p$	For those who attempted, they wrote the limits wrongly: $\int_{p}^{\frac{\pi}{2}} x\ \mathrm{d}y$ instead of $\int_{p+\cos p}^{\frac{\pi}{2}} x\ \mathrm{d}y$ due to confusion between y-limits and θ-limits. Some did not include the area of the triangle. Do not change the parametric equation into Cartesian equation unless you are specifically told to do so!

	Integrating abt x-axis: area $= \int_0^{1-\sin p} y \, \mathrm{d}x - \frac{1}{2}(p + \cos p + p)(1 - \sin p)$ $= \int_{\frac{\pi}{2}}^{p} y \, \frac{\mathrm{d}x}{\mathrm{d}\theta} \, \mathrm{d}\theta - \frac{1}{2}(2p + \cos p)(1 - \sin p)$ $= \int_{\frac{\pi}{2}}^{p} (\theta + \cos\theta)(-\cos\theta) \, \mathrm{d}\theta - p - \frac{1}{2}\cos p + p\sin p + \frac{1}{2}\sin p \cos p$ $= -\int_{\frac{\pi}{2}}^{p} (\theta\cos\theta + \cos^2\theta) \, \mathrm{d}\theta - p - \frac{1}{2}\cos p + p\sin p + \frac{\sin 2p}{4}$ $= -\left[[\theta\sin\theta]_{\frac{\pi}{2}}^{p} - \int_{\frac{\pi}{2}}^{p} \sin\theta \, \mathrm{d}\theta \right] - \int_{\frac{\pi}{2}}^{p} \cos^2\theta \, \mathrm{d}\theta - p - \frac{1}{2}\cos p + p\sin p + \frac{\sin 2p}{4}$ $= -\left[p\sin p - \frac{\pi}{2}\sin\frac{\pi}{2} + [\cos\theta]_{\frac{\pi}{2}}^{p} \right] - \int_{\frac{\pi}{2}}^{p} \frac{1 + \cos 2\theta}{2} \, \mathrm{d}\theta - p - \frac{1}{2}\cos p + p\sin p + \frac{\sin 2p}{4}$ $= -\left[p\sin p - \frac{\pi}{2} + \cos p - \cos\frac{\pi}{2} \right] - \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4} \right]_{\frac{\pi}{2}}^{p} - p - \frac{1}{2}\cos p + p\sin p + \frac{\sin 2p}{4}$ $= -p\sin p + \frac{\pi}{2} - \cos p - \left[\frac{1}{2}p + \frac{\sin 2p}{4} - \left(\frac{1}{2}\cdot\frac{\pi}{2} + \frac{\sin\pi}{4} \right) \right] - p - \frac{1}{2}\cos p + p\sin p + \frac{\sin 2p}{4}$ $= \frac{\pi}{2} - \frac{3}{2}\cos p - \frac{1}{2}p - \frac{\sin 2p}{4} + \frac{\pi}{4} - p + \frac{\sin 2p}{4}$ $= \frac{3\pi}{4} - \frac{3}{2}p - \frac{3}{2}\cos p$	Candidates who chose to integrate about the x-axis often forgot that they need to subtract away a “trapezium”. Again, they wrote their limits wrongly.

Locus	**(v)** At R, $y=0$, $0-(p+\cos p)=\dfrac{\cos p}{1-\sin p}\left(x-(1-\sin p)\right)$ $-p-\cos p=\dfrac{\cos p}{1-\sin p}x-\cos p$ $\dfrac{\cos p}{1-\sin p}x=-p$ $x=\dfrac{p(\sin p-1)}{\cos p}$ $\therefore R\left(\dfrac{p(\sin p-1)}{\cos p},0\right)$ Midpoint of QR: $\left(\dfrac{0+\dfrac{p(\sin p-1)}{\cos p}}{2},\dfrac{p+0}{2}\right)$ $\left(\dfrac{p(\sin p-1)}{2\cos p},\dfrac{p}{2}\right)$ $x=\dfrac{p(\sin p-1)}{2\cos p}$, $y=\dfrac{p}{2}\Rightarrow p=2y$ $x=\dfrac{y(\sin 2y-1)}{\cos 2y}$	Many left this part blank, simply because they are unsure of the word “locus”. However, they are still encouraged to continue finding the midpoint and gain some credit. There is no need to bring in vectors here to find the midpoint. The cohort needs to go and revise what is meant by “locus”!

Q6. Discrete Random Variables

Assessment Objectives		Solution	Examiner's Feedback
The ability to list all possible outcomes and find their respective probabilities	(i)	<table><tr><td>x</td><td>\$10</td><td>\$20</td><td>\$40</td><td>\$80</td><td>\$160</td></tr><tr><td>$P(X = x)$</td><td>$\frac{1}{9}$</td><td>$\frac{2}{9}$</td><td>$\frac{1}{3}$</td><td>$\frac{2}{9}$</td><td>$\frac{1}{9}$</td></tr></table>	A handful of students are confused over table of outcomes and probability distribution table.
HOT – to identify and use the concept of expectation as long-term average loss in this context	(ii)	$E(X)=10\left(\frac{1}{9}\right)+20\left(\frac{2}{9}\right)+40\left(\frac{1}{3}\right)+80\left(\frac{2}{9}\right)+160\left(\frac{1}{9}\right)=\$54.44$ Since \$54.44 should be the profit, which is 40% of $\$R$, Therefore, $0.4R = 54.444$, Hence, need to set R to be \$136	Most students were able to find expectation but evaluated it wrongly due to sheer carelessness in pressing the calculator.
Finding the variance of a d.r.v	(iii)	$\text{Var}(X)$ $=E(X^2)-[E(X)]^2$ $=10^2\left(\frac{1}{9}\right)+20^2\left(\frac{2}{9}\right)+40^2\left(\frac{1}{3}\right)+80^2\left(\frac{2}{9}\right)+160^2\left(\frac{1}{9}\right)-\left(\frac{490}{9}\right)^2$ $=1935.8$ Hence, $\sigma=\sqrt{1935.8}=44.0$ $\therefore P(X<\sigma)=P(X<44.0)=\frac{2}{3}$	Students recalled the variance formula wrongly. For those who did so correctly, the numerical value was often wrong. Many students assumed normal distribution for X and went on to find $P(X<\sigma)$. They failed to recognize that this question involves discrete random variable.

Q7. Correlation and Regression			
Assessment Objectives	**Solution**		**Examiner's Feedback**
Understand that $(\overline{T},\overline{x})$ always lie on the regression line		$x = 2.94857T + 146.238$ $\frac{1474+k}{6} = 2.94857(50)+146.238$ $k = 288 \quad (3\text{sf})$	Common mistake is to find k by substituting $T = 60$ into the given regression line.
Scatter diagram.	(i)	 	Common mistake is to take x as the horizontal axis (independent variable). A lot of students also did not label the axes nor the greatest and smallest x and y values.
Linear transformation	(ii)	Least square estimate of a is 5.12898 = 5.13 (3 sf) Least square estimate of b is 0.0098734 = 0.00987 (3 sf) r between T and ln x is 0.990 (3 sf)	Quite a number of students did not realised that they can/must use $k = 288$ to find the least square estimates of a and b, as well as the value of r between T and ln$x$. Hence not being able to get the correct answers.
Able to compare the models based on the scatter diagram and the r value.	(iii)	In (i), as T increases, x increases at an increasing rate instead of constant rate. In part (ii), the r value between T and ln x as compared to the r value between T and x is closer to 1. Hence $\ln x = 0.0098734T + 5.12898$ is the better model.	A lot of students only managed to get the part on comparing the r values between T and ln$x$ and between T and x. While only some managed to describe that "As T increases, x increases at an increasing rate instead of constant rate." on the graph of (i).

Able to use the appropriate line to do the prediction and knowing the factors of the reliability.	(iv)	$\ln(300) = 0.0098734T + 5.12898$ $T = 58.2$ (3sf) The prediction is reliable as $x = 300$ is within the data range of x and r value is close to 1.	Quite a number of students uses 3s.f. for their intermediate steps for calculations of the estimate for T and end up with a less accurate answer. A number of students wrote "both x and T are in the data range hence it is a reliable prediction." Instead of "$x = 300$ is within the data range of x". A lot of students also missed out "r value is close to 1" as part of the reasons for reliability of the prediction.

<table>
<tr><th colspan="4">Q8. Binomial Distribution</th></tr>
<tr><th>Assessment Objectives</th><th colspan="2">Solution</th><th>Examiner's Feedback</th></tr>
<tr><td>Identifying binomial distribution</td><td>(i)</td><td>Let X be the number of bullseyes in a series.
$X \sim \mathrm{B}(15, 0.7)$

$\mathrm{P}(\text{good series})$
$= \mathrm{P}(X \geq 11)$
$= 1 - \mathrm{P}(X \leq 10)$
$= 0.515$</td><td>This part was generally well attempted.

Some students did not know how to define a binomial random variable. Some had defined it wrongly as the probability of getting a bullseyes.

Some students mixed up equivalent events with complementary events. The probability of getting at least 11 bullseyes is equal to the probability of getting at most 4 non-bullseyes. In other words, $\mathrm{P}(X \geq 11) \neq 1 - \mathrm{P}(X \leq 4)$.

Some students considered the complementary events wrongly. A common mistake is $\mathrm{P}(X \geq 11) = 1 - \mathrm{P}(X \leq 11)$.

Some students considered the probabilities for the outcomes of the shots but failed to consider the ${}^{n}\mathrm{C}_{r}$.</td></tr>
<tr><td>Ability to interpret question correctly and setting up the relevant mathematical representation of the situation.

Efficient use of GC in obtaining answer</td><td>(ii)</td><td>Let Y be the number of good series in n attempts
$Y \sim \mathrm{B}(n, 0.51549)$

$\mathrm{P}(Y \geq 6) \geq 0.99$
$1 - \mathrm{P}(Y < 6) \geq 0.99$
$\mathrm{P}(Y \leq 5) \leq 0.01$
Using GC,
<table><tr><th>n</th><th colspan="2">$\mathrm{P}(Y \geq 6)$</th></tr><tr><td>20</td><td>0.9853</td><td>< 0.99</td></tr><tr><td>21</td><td>0.9908</td><td>> 0.99</td></tr><tr><td>22</td><td>0.9943</td><td>> 0.99</td></tr></table></td><td>This part was generally well attempted.

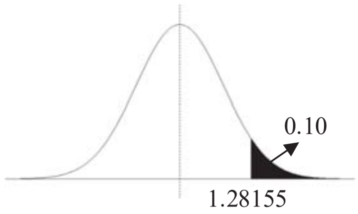
Students should state the mathematical inequality clearly which they should solve. This, in turn, would help them to write the correct headers for GC table. A significant percentage of students were not able to do write appropriate headers in GC table.

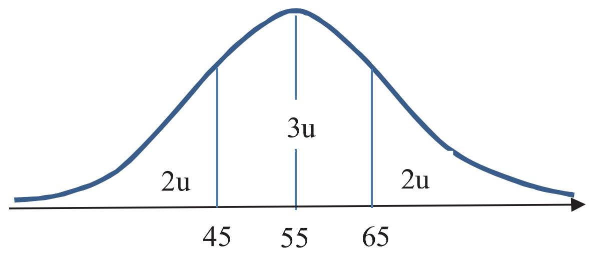
Students should also present a GC table to support their workings so that credit might be given when the wrong value of n was found.</td></tr>
</table>

		Hence, least $n = 21$	Some looked at the wrong range of n as they had compared the probabilities against 0.001 instead of 0.01.
Identifying the need to apply the central limit theorem and obtain the correct parameters	(iii)	From (i), Let X be the number of bullseyes in a series. **Method 1:** Let $\overline{X}$ be the average number of bullseyes per series. Since sample size $n = 40$ is sufficiently large, $\overline{X} \sim \mathrm{N}\left(15 \times 0.7, \dfrac{15 \times 0.7 \times 0.3}{40}\right)$ approximately by Central Limit Theorem Required probability $= \mathrm{P}(\overline{X} > 10)$ $= 0.963$ **Method 2:** Let $X_1 + X_2 + \ldots + X_{40}$ be the total number of bullseyes for 40 series. Since sample size $n = 40$ is sufficiently large, $X_1 + X_2 + \ldots + X_{40} \sim \mathrm{N}(40(15 \times 0.7), 40(15 \times 0.7 \times 0.3))$ approximately by Central Limit Theorem Required probability $= \mathrm{P}(X_1 + X_2 + \ldots + X_{40} > 40 \times 10)$ $= 0.963$	This part was badly done. Students should pick out the keywords 'estimate' and 'on average' in the question. A large percentage of students failed to • recognize the use of Central Limit Theorem for the distribution of the sample mean, • identify the sample size 40 correctly, • identify the success event (getting a bullseye) and its probability of 0.7 correctly. A significant number of students wrote $X \sim \mathrm{N}(10.5, 3.15)$. Students should recognize that the random variable X does not follow a normal distribution. The Central Limit Theorem is used to approximate the distribution of sample mean $\overline{X}$, not X. A minority of students identified the total number of bullseyes as a binomial distribution and proceeded to find the required probability. They overlooked the keyword 'estimate' in this question part and thus were not given full credit.

Q9. Hypothesis Testing

Assessment Objectives	Solution		Examiner's Feedback
Unbiased estimate of population mean and variance	(i)	Let X be the random variable denoting the speed of a car in the school compound. Unbiased estimate of population mean $= \frac{1325}{50} = 26.5$ Unbiased estimate of population variance $= \frac{50}{49}\left[7.75^2\right] = 61.288 = 61.3 \text{ (3sf)}$	Almost all candidates could calculate unbiased est. of pop. Mean Most candidates could not recall the formula to link sample variance to unbiased ets. Of pop. Variance.
Apply to carry out the hypothesis testing with the concept of CLT.	(ii)	$H_0 : \mu = 25$ $H_1 : \mu > 25$, where μ is the population mean of X. Under H_0, Since n is large, by CLT, $\bar{X} \sim N\left(25, \frac{61.288}{50}\right)$ approx. $z_{test} = \frac{26.5-25}{\sqrt{61.288/50}}$ Since p-value $= 0.0877345 > 0.05$, we do not reject H_0 and conclude that we have insufficient evidence at 5% level of significance that the mean speed of the cars is more than 25.	Many candidates used 26.5 instead of 25 for the population mean. Many candidates did not get the correct phrasing of the conclusion, missing out key points such as 5% level of significance/ do not reject H_0 etc
Able to use the critical approach to find the unknown n given the conclusion of the test.	(iii)	$X \sim N(25, 36)$ $\bar{X} \sim N\left(25, \frac{36}{n}\right)$ $z_{test} = \frac{26.5-25}{\sqrt{36/n}} = 0.25\sqrt{n}$	Many candidates are able to obtain critical value 1.28155, but could not form correct inequality

		 Since H_0 is rejected at 10 % level of significance, $z_{test} \geq 1.28155$, $0.25\sqrt{n} \geq 1.28155$ $n \geq 26.278$ $\{n : n \in \mathbb{Z}, n \geq 27\}$	Candidates need to note that n is an integer, and also the answer should be given in a set of values
Able to interpret the question and able to find the unknown given the probability.	(iv)	$X \sim \mathrm{N}(25,36)$ $\mathrm{P}(X > t) = 0.75$ $t = 21.0$ (3 s.f.)	Many candidates were not able to interpret the question.

Q10. Normal Distribution			
Assessment Objectives	**Solution**		**Examiner's Feedback**
Use of symmetrical properties of the normal distribution curve Finding unknown parameters by means of standardization	(i)	 $2\text{P}(X > 45) = 5\text{P}(X > 65)$ $\frac{\text{P}(X > 45)}{\text{P}(X > 65)} = \frac{5}{2}$ $\therefore$ From diagram, $\text{P}(X < 45) = \frac{2}{7}$ $\text{P}(Z < \frac{45-55}{\sigma}) = \frac{2}{7}$ $\frac{-10}{\sigma} = -0.56595$ $\sigma = 17.7$	Many students were unable to observe the relationship and symmetry between $\text{P}(X > 65)$ and $\text{P}(X < 45)$ A significant number of students were not able to present the standardization steps correctly with errors and omissions in presentation. Students need to learn to use graphical representation for probabilities under Normal Distribution.
Proper combining of random variables, and finding the associated parameters Identifying the need to use the modulus in the setup and manipulate it properly in the context of finding probability	(ii)	$(X_1 + X_2 + X_3 - 4Y) \sim \text{N}(\ 3\times 55 - 4\times 45\ ,\ 3\times 17.669^2 + 4^2\times 10^2)$ Required probability $= \text{P}(\lvert X_1 + X_2 + X_3 - 4Y\rvert \le 10)$ $= \text{P}(-10 \le (X_1 + X_2 + X_3 - 4Y) \le 10)$ $= 0.151$	Most common mistake by students was the inability to interpret the question and distinguish between $X_1 + X_2 + X_3$ and $3X$ which led to the wrong calculation of variance. Also, majority of students did not take the modulus as required by the question.

HOT- cannot simply just quote standard distribution of sample mean because now 2 different random variables are involved in the sample	(iii)	$\frac{X_1+X_2+X_3+Y_1+Y_2+Y_3+Y_4}{7} \sim \mathrm{N}\left(\frac{3\times 55+4\times 45}{7}, \frac{3\times 17.669^2+4\times 10^2}{7^2}\right)$ Required probability $=\mathrm{P}\left(\frac{X_1+X_2+X_3+Y_1+Y_2+Y_3+Y_4}{7}\geq 50\right)$ $=0.446$	Many students were unable to interpret the question to obtain the random variable denoting the "mean score": $\frac{X_1+X_2+X_3+Y_1+Y_2+Y_3+Y_4}{7}$
Theory	(iv)	The random variables X and Y are independent of each other.	Many students were not aware that the conditions for summing random variables that are Normally distributed. There were some students who listed the conditions for a Binomial distribution instead.
Theory	(v)	Let C be the score of a randomly chosen student in Group C. If normal distribution is assumed, $\mathrm{P}(C<0)=0.0548$ A significant proportion would fall under an inadmissible area, hence normal distribution is not suitable as a model.	Most students gave vague answers about the large variance without substantiation.

Q11. Probability

Assessment Objectives	Solution		Examiner's Feedback
Systematic consideration/application of P&C techniques in given situation	(i)	No of ways to form the string unrestricted = $4^5-4=1020$ No of ways to form the string using 2 letters = ${}^4C_2\times(2^5-2)=180$ Hence $P(A)=\frac{180}{1020}=\frac{3}{17}=0.176$ No of ways to have a palindrome = $4^3-4=60$ Hence $P(B)=\frac{60}{1020}=\frac{1}{17}=0.0588$	Poorly attempted. Many candidates forgot that they are supposed to find a probability, not the number of ways, so end up with answers like $P(A)=24768737542$ or $P(B)=-0.446$
Probability of intersection of two events is not necessarily the product of their probabilities Mathematical proof of independence	(ii)	No of ways to form palindrome of 2 letters = ${}^4C_2\times(2^3-2)$ or ${}^4C_2\times\frac{3!}{2!}\times2=36$ Hence $P(A\cap B)=\frac{36}{1020}=\frac{3}{85}=0.0353$ $P(A)\times P(B)=0.0104$ Since $P(A\cap B)\neq P(A)\times P(B)$, A and B are not independent.	Some candidates have incorrect conceptual understanding. The tests they used were $P(A\cap B)\neq 0$ $P(A\cap B)\neq P(A)+P(B)$
Ability to identify that the question is describing the union of two events, and use the relevant formula to find the probability	(iii)	$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{3}{17}+\frac{1}{17}-\frac{3}{85}=\frac{1}{5}=0.2$	Some candidates have incorrect conceptual understanding. They used $P(A\cup B)=P(A)+P(B)+P(A\cap B)$ Some did not see that this is asking for the union in the first place.
Finding conditional probability	(iv)	$P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{3/85}{1/17}=\frac{3}{5}=0.6$	Some candidates have incorrect conceptual understanding. They assumed that $P(A\cap B)=P(A)\cdot P(B)$