ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Paper 2 (100 marks)

18 September 2019

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS | | | / | | |

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

This document consists of 20 printed pages and 4 blank pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1 Given that the curve $y = \dfrac{ax^2 + bx + c}{x-1} = \mathrm{f}(x)$ passes through the points $(-1, -3)$, $\left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$ and $\left(5, \dfrac{33}{2}\right)$, find the values of a, b and c. [3]

<u>Solution</u>

$$\frac{a(-1)^2 - b + c}{-1-1} = -3 \quad \Rightarrow \quad a - b + c = 6 \qquad \text{---(1)}$$

$$\frac{a\left(\frac{1}{2}\right)^2 + \frac{1}{2}b + c}{\frac{1}{2} - 1} = -\frac{3}{2} \quad \Rightarrow \quad \frac{1}{4}a + \frac{1}{2}b + c = \frac{3}{4} \qquad \text{---(2)}$$

$$\frac{a(5)^2 + 5b + c}{5-1} = \frac{33}{2} \quad \Rightarrow \quad 25a + 5b + c = 66 \qquad \text{---(3)}$$

Using GC, $a = 3, b = -2, c = 1$.

Hence find the exact range of values of x for which the gradient of $y = \mathrm{f}(x)$ is positive. [3]

<u>Solution</u>

$$\mathrm{f}'(x) > 0 \quad \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3x^2 - 2x + 1}{x-1}\right) > 0$$

$$\Rightarrow \frac{(x-1)(6x-2) - (3x^2 - 2x + 1)(1)}{(x-1)^2} > 0$$

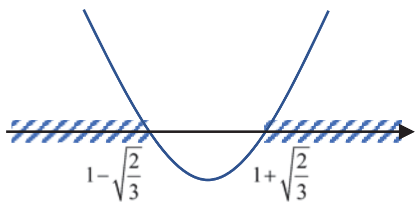
$$\Rightarrow \frac{3x^2 - 6x + 1}{(x-1)^2} > 0$$

Since $(x-1)^2 > 0$, $\quad 3x^2 - 6x + 1 > 0$

$$3(x-1)^2 - 2 > 0$$

$$\left(\sqrt{3}x - \sqrt{3} - \sqrt{2}\right)\left(\sqrt{3}x - \sqrt{3} + \sqrt{2}\right) > 0$$

$$x > 1 + \sqrt{\frac{2}{3}} \quad \text{or} \quad x < 1 - \sqrt{\frac{2}{3}}$$



2 The functions f and g are defined by:

$$\mathrm{f}: x \mapsto \left|ax - x^2\right|, \qquad x \in \mathbb{R}$$

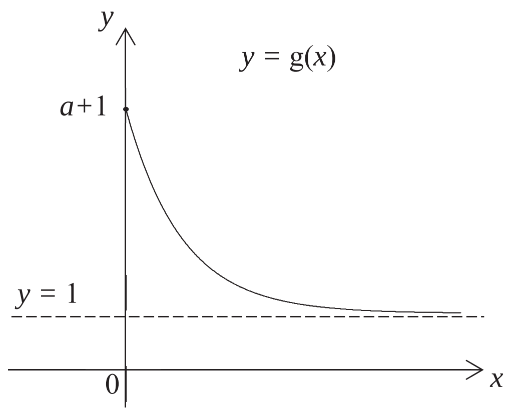
$$\mathrm{g}: x \mapsto ae^{-x} + 1, \qquad x \geq 0$$

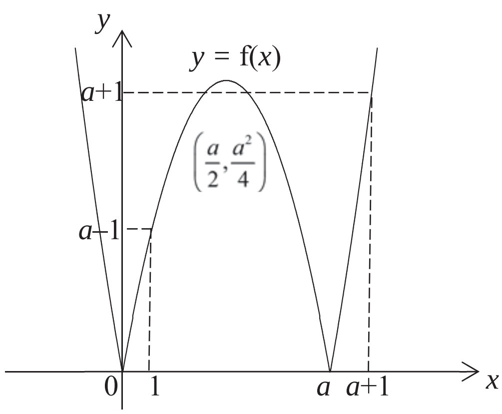
where a is a positive constant and $a \geq 5$.

(i) Sketch the graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{f}(x)$ on separate diagrams.

Hence find the range of the composite function fg, leaving your answer in exact form in terms of a. [5]

Solution





From the graphs,
$R_g = (1, a+1]$

Since $a \geq 5$, $\frac{a}{2} \geq \frac{5}{2} > 1 \Rightarrow 0 < 1 < \frac{a}{2} \Rightarrow \frac{a^2}{4} \geq a - 1$

and $a^2 \geq 5a \Rightarrow \frac{a^2}{4} \geq \frac{5a}{4} = a + \frac{1}{4}a \geq a + 1$

$[0, \infty) \xrightarrow{g} (1, \alpha + 1] \xrightarrow{f} R_{fg}$

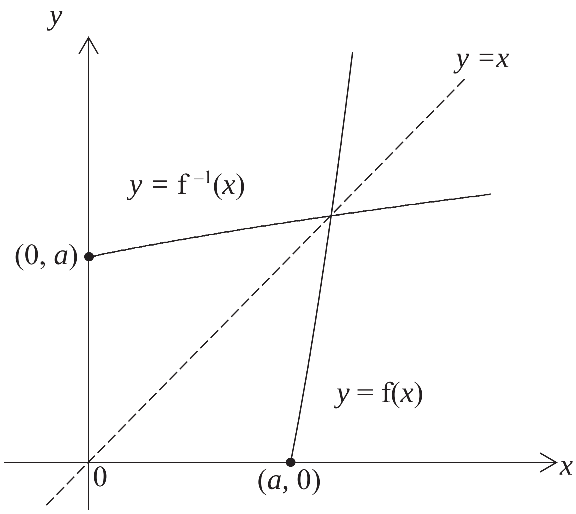
From the graphs, $R_{fg} = \left[0, \frac{a^2}{4}\right]$

[Turn Over

(ii) Given that the domain of f is restricted to the subset of $\mathbb{R}$ for which $x \geq k$, find the smallest value of k, in terms of a, for which f^{-1} exists. Hence, without finding the expression for $\mathrm{f}^{-1}(x)$, sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on a single diagram, stating the exact coordinates of any points of intersection with the axes. [3]

Solution

The smallest value of k for which f^{-1} exists = a.



(iii) With the restricted domain of f in part (ii), find with clear working, the set of values of x such that $\mathrm{f}\,\mathrm{f}^{-1}(x) = \mathrm{f}^{-1}\,\mathrm{f}(x)$. [2]

Solution

$\mathrm{f}^{-1}\mathrm{f}(x) = x$ for $x \in D_{\mathrm{f}}$, i.e $x \in [a, \infty)$

$\mathrm{f}\mathrm{f}^{-1}(x) = x$ for $x \in D_{\mathrm{f}^{-1}}$, i.e $x \in [0, \infty)$

For $\mathrm{f}\,\mathrm{f}^{-1}(x) = \mathrm{f}^{-1}\,\mathrm{f}(x)$, $[0, \infty) \cap [a, \infty) = [a, \infty)$

Set of values of x required = $[a, \infty)$

3 A student is investigating a chemical reaction between two chemicals A and B. In his experiments, he found that the reaction produces a new product C which does not react with A and B after being formed.

His experiments also suggest that the rate at which the product, C is formed is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (a-x)\sqrt{b-x},$$

where a and b are the initial concentrations (gram/litre) of A and B just before the reaction started and x is the concentration (gram/litre) of C at time t (mins).

(i) If $a = b = 9$, solve the differential equation to show that

$$x = 9 - \frac{36}{(3t+2)^2}.$$ [4]

<u>Solution</u>

(i) $\frac{\mathrm{d}x}{\mathrm{d}t} = (9-x)(9-x)^{\frac{1}{2}}$

$\int (9-x)^{-\frac{3}{2}}\,\mathrm{d}x = \int 1\,\mathrm{d}t$

$-2(-1)(9-x)^{-\frac{1}{2}} = t + c$

When $t = 0$, $x = 0$, $2(9-0)^{-\frac{1}{2}} = 0 + c$

$\therefore\ c = \frac{2}{3}$

$\Rightarrow \frac{2}{\sqrt{9-x}} = t + \frac{2}{3} = \frac{3t+2}{3}$

$\frac{4}{9-x} = \frac{(3t+2)^2}{9}$

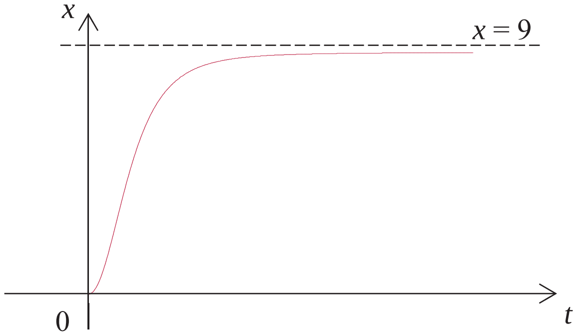
$9 - x = \frac{36}{(3t+2)^2}$

$\therefore\ x = 9 - \frac{36}{(3t+2)^2}$

[Turn Over

(ii) Sketch the graph of x and describe the behaviour of x as t increases. Find the amount of time needed for the concentration of C to reach 4.5 gram/litre. [3]

Solution



[G1 – shape with asymptote (no need for correct value of limit)]

As t increase, x increases and approaches the value of 9 gram/litre.

From GC or otherwise, $t = 0.276$ mins or 16.6 s (3sf) or $\dfrac{2\left(\sqrt{2}-1\right)}{3}$

It is now given that $a > b$ instead.

(iii) Solve the differential equation to find t in terms of x, a and b by using the substitution $u = \sqrt{b-x}$ for $0 \le x < b$. [6]

Solution

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-1}{2\sqrt{b-x}} = -\frac{1}{2u}$$

$$u = \sqrt{b-x} \Rightarrow x = b-u^2, \therefore \frac{\mathrm{d}u}{\mathrm{d}t} \times \frac{\mathrm{d}x}{\mathrm{d}u} = \left(a-b+u^2\right)u$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} \times (-2u) = \left(a-b+u^2\right)u$$

$$-2\int \frac{1}{(a-b)+u^2}\,\mathrm{d}u = \int 1\,\mathrm{d}t$$

$$-\frac{2}{\sqrt{a-b}}\tan^{-1}\frac{u}{\sqrt{a-b}} + C = t$$

When $t = 0$, $x = 0$, $u = \sqrt{b}$, $\therefore 0 = C - \dfrac{2}{\sqrt{a-b}}\tan^{-1}\sqrt{\dfrac{b}{a-b}}$

$$\therefore C = \frac{2}{\sqrt{a-b}}\tan^{-1}\sqrt{\frac{b}{a-b}}$$

$$\Rightarrow t = \frac{2}{\sqrt{a-b}}\tan^{-1}\sqrt{\frac{b}{a-b}} - \frac{2}{\sqrt{a-b}}\tan^{-1}\sqrt{\frac{b-x}{a-b}}$$

4 The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

A small object, P, is placed on the straight line passing through two light sources A and B that are eight metres apart. It is known that the light sources are similar except that S, the strength of light source A, is a positive constant and is three times that of light source B. It can be assumed that I, the total illumination of the object P by the two light source is the sum of the illumination due to each light source.

The distance between the object P and light source A is denoted by x.

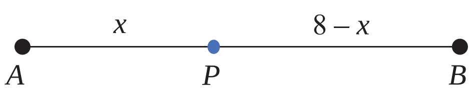
(i) Show that when P is between A and B,

$$I = kS\left[\frac{1}{x^2} + \frac{1}{\alpha(\beta - x)^2}\right]$$ where k is a positive constant and,

α and β are constants to be determined. [1]

<u>Solution</u>

$$I = \frac{kS}{x^2} + \frac{k\left(\frac{1}{3}S\right)}{(8-x)^2} \text{ where } k > 0$$



$$= kS\left[\frac{1}{x^2} + \frac{1}{3(8-x)^2}\right] \quad \text{so } \alpha = 3 \text{ and } \beta = 8$$

(ii) Show that there is only one value of x that will give a stationary value of I. Determine this value and the nature of the stationary value without the use of a graphic calculator. [5]

<u>Solution</u>

$$\frac{dI}{dx} = kS\left[-\frac{2}{x^3} - 2(-1)\frac{1}{3(8-x)^3}\right] = 2kS\left[-\frac{1}{x^3} + \frac{1}{3(8-x)^3}\right]$$

Let $\frac{dI}{dx} = 0 \quad \Rightarrow \quad -\frac{1}{x^3} + \frac{1}{3(8-x)^3} = 0$

$$x^3 = 3(8-x)^3$$

$$x = \sqrt[3]{3}(8-x)$$

$$x = \frac{8\sqrt[3]{3}}{1+\sqrt[3]{3}}$$

Hence I will have a stationary value only when $x = \frac{8\sqrt[3]{3}}{1+\sqrt[3]{3}}$. (Shown)

[Turn Over

$$\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = 2kS\left[\frac{3}{x^4} + \frac{(-3)(-1)}{3(8-x)^4}\right]$$

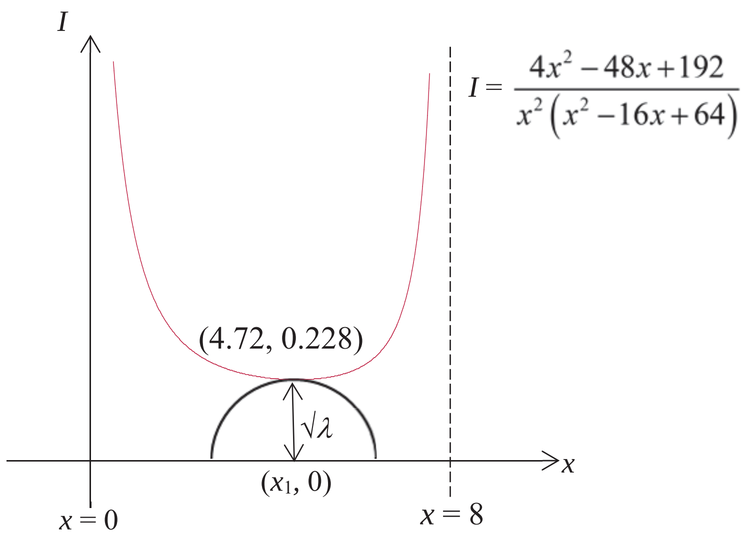
$$= 2kS\left[\frac{3}{x^4} + \frac{1}{(8-x)^4}\right] > 0 \text{ for all } x.$$

$\Rightarrow I$ is minimum when $x = \dfrac{8\sqrt[3]{3}}{1+\sqrt[3]{3}}$.

It is now given that $I = \dfrac{4x^2 - 48x + 192}{x^2\left(x^2 - 16x + 64\right)}$ for $0 < x < 8$.

(iii) Sketch the graph of I, indicating clearly any equations of asymptote and coordinates of the stationary point. [3]

Solution



(iv) It is desired to place P such that $I = \sqrt{\lambda - (x - x_1)^2}$ where x_1 is the value of x found in part **(ii)** and λ is a positive constant. Find the value of λ if there is only one possible position to place P. [2]

Solution

$I = \sqrt{\lambda - (x - x_1)^2}$ is the equation of a semi-circle centred at $(x_1, 0)$ and radius $\sqrt{\lambda}$.

Hence when there is only 1 position to place P, it must happen when $\sqrt{\lambda} = 0.2276094$

i.e. $\lambda = 0.0518$ (3sf)

Section B: Statistics [60 marks]

5 Greenhouse gases generated by Singapore comes mainly from the burning of fossil fuels to generate energy for industries, buildings, households and transportation.

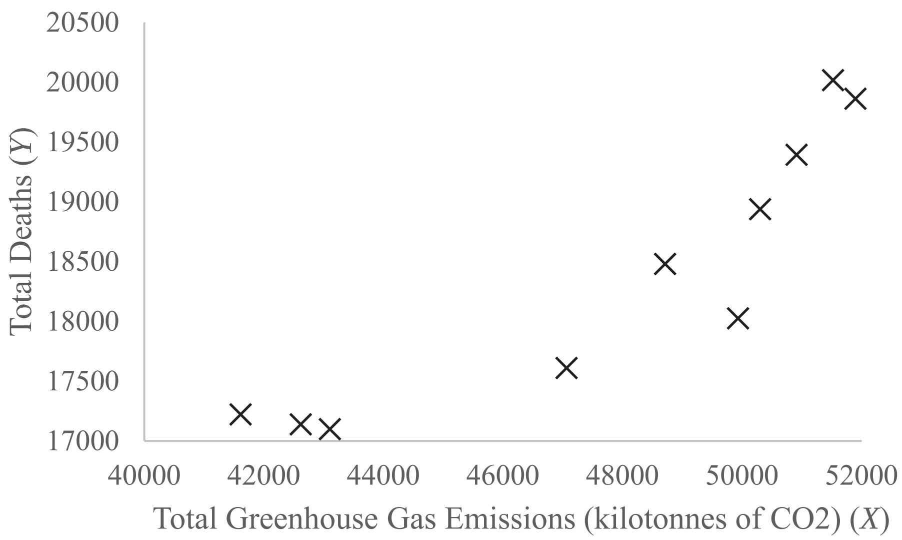
The following table* shows the total greenhouse gas emissions, X (kilotonnes of CO_2), and the total deaths in Singapore, Y from 2007 to 2016.

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Total Greenhouse Gas Emissions (X)	42613	41606	43100	47063	49930	48712	50299	50908	51896	51519
Total Deaths (Y)	17140	17222	17101	17610	18027	18481	18938	19393	19862	20017

**Extracted from www.singstat.gov.sg*

(i) Draw a scatter diagram for these values, labelling the axes and state the product moment correlation between X and Y. Use your diagram to explain why the best model for the relationship between X and Y may not be given by $Y = aX + b$, where a and b are constants. [3]

Solution



From the GC, r = 0.90527 = 0.905 (3 sf)

From the scatter diagram, the points follow a curvilinear trend rather than a linear trend, hence $Y = aX + b$ may not be the best model.

[Turn Over

(ii) Comment on the correctness of the following statement.

"Since there exists a high positive linear correlation between total greenhouse gas emission and total deaths, the deaths are a result of the greenhouse gas emissions." [1]

Solution

High positive linear correlation does not indicate causation, hence it is incorrect to state that the deaths are a result of the greenhouse gas emission. Moreover, there are other factors affecting the total deaths in Singapore.

(iii) Explain clearly which of the following equations, where a and b are constants, provides the more accurate model of the relationship between X and Y.

(A) $Y = aX^4 + b$

(B) $\lg Y = aX + b$ [3]

Solution

From GC, r = 0.928 for $Y = aX^4 + b$

r = 0.912 for $\lg Y = aX + b$

Since the product moment correlation coefficient for $Y = aX^4 + b$ is closer to 1 as compared to the one for $\lg Y = aX + b$, $Y = aX^4 + b$ is the better model.

The Government has pledged that Singapore's greenhouse gas emissions will peak around 2030 at the equivalent of 65 million tonnes of carbon dioxide.

(iv) Using the model you have chosen in part **(iii)**, write down the equation for the relationship, giving the regression coefficients correct to 5 significant figures. Hence find the estimated total deaths in the year 2030 if Singapore's pledge is fulfilled and comment on the reliability of the estimate. [3]

Solution

The required equation is $Y = 6.4326\times10^{-16}X^4 + 14911$

When $X = 65000$, $Y = 6.4326\times10^{-16}(65000)^4 + 14911$

$= 26393.6$

$= 26400$ (3 sf)

The estimate is not reliable as extrapolation is carried out where the linear correlation outside the range of X, i.e. [41606, 51896] may not be valid.

6 Birth weight can be used to predict short and long-term health complications for babies. Studies show that the birth weight of babies born to mothers who do not smoke in a certain hospital can be assumed to follow a normal distribution with mean 3.05 kg and variance σ^2 kg^2.

The hospital classifies babies based on their birth weight as shown in the table below.

Birth weight	Classification
Less than 1.5 kg	Very low birth weight
1.5 kg to 2.5 kg	Low birth weight
2.5 kg to 4.0 kg	Normal birth weight
More than 4.0 kg	High birth weight

(i) A sample showed that 20.2% of the babies born to mothers who do not smoke have low birth weight. If this is true for the entire population, find two possible values of σ, corrected to 2 decimal places. Explain clearly why one of the values of σ found should be rejected. [4]

<u>Solution</u>

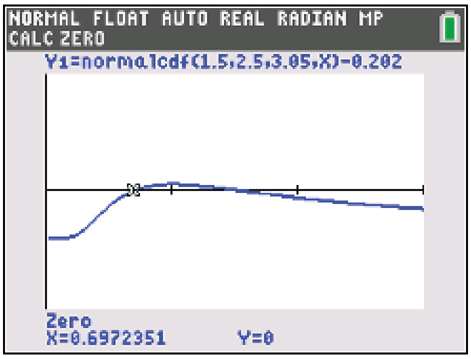
Let X be random variable representing the birth weight of a baby in kg.

$\therefore X \sim \mathrm{N}(3.05,\ \sigma^2)$

$\mathrm{P}(1.5 < X < 2.5) = 0.202$

By plotting the graph of $y = \mathrm{P}(1.5 < X < 2.5) - 0.202$ using GC,

we have $\sigma = 0.70$ (2 dp) or $\sigma = 1.56$ (2 dp)



If $\sigma = 0.70$, $\mathrm{P}(2.5 < X < 4) = 0.70$

If $\sigma = 1.56$, $\mathrm{P}(2.5 < X < 4) = 0.37$

Reject $\sigma = 1.56$ as this would mean that it is very much more likely that a baby will be born with "abnormal" birth weight than normal birth weight.

(Or when $\sigma = 1.56$, $\mathrm{P}(X < 0) = 0.03$ compared with $\mathrm{P}(X < 0) = 6.6\times10^{-6}$ for $\sigma = 0.7$)

Studies also show that babies born to mothers who smoke have a lower mean birth weight of 2.80 kg.

For the remaining of the question, you may assume the birth weight of babies born to mothers who smoke is also normally distributed and that the standard deviations for the birth weight of babies born to mothers who do not smoke and mothers who smoke are both 0.86 kg.

(ii) Three pregnant mothers-to-be who smoke are randomly chosen. Find the probability that all their babies will not be of normal birth weight. State an assumption that is needed for your working. [3]

[Turn Over

Solution

Let Y be r.v. denoting the birth weight of a baby in kg from a mother who smokes.
$Y \sim \mathrm{N}(2.8,\ 0.86^2)$

$$\text{Required probability} = [1-\mathrm{P}(2.5<Y<4)]^3$$
$$=(1-0.55493)^3$$
$$=0.088157$$
$$=0.0882 \text{ (3 sf)}$$

Assumption: The birth weights of babies are independent of one another.

(iii) Find the probability that the average birth weight of two babies from mothers who do not smoke differs from twice the birth weight of a baby from a mother who smokes by less than 2 kg. [3]

Solution

Let $A=\dfrac{X_1+X_2}{2}-2Y$

$\mathrm{E}(A)=3.05-2(2.80)=-2.55$

$\mathrm{Var}(A)=0.5^2\left(0.86^2\times 2\right)+2^2(0.86^2)=3.3282$

$\therefore\ A\sim \mathrm{N}(-2.55,\ 3.3282)$

Required probability = $\mathrm{P}\left(|A|<2\right) = \mathrm{P}\left(-2<A<2\right)=0.37521$
$=0.375$ (3 s.f.)

Babies whose birth weight not classified as normal will have to remain in hospital for further observation until their condition stabilises. Depending on the treatment received and length of stay, the mean hospitalisation cost per baby is \$2800 and standard deviation is \$500.

(iv) Find the probability that the average hospitalisation cost from a random sample of 50 babies, whose birth weight not classified as normal, exceeds \$3000. [2]

Solution

Let W be random variable representing the hospitalisation charge of a baby.
$\mathrm{E}(W)=2800$ and $\mathrm{Var}(W)=500^2=250000$

Since $n=50$ is large, by Central Limit Theorem,

$$\overline{W}=\frac{W_1+W_2+\ldots+W_{50}}{50}\sim \mathrm{N}\left(2800,\ \frac{500^2}{50}\right) \text{ approximately}$$

i.e. $\overline{W}\sim \mathrm{N}(2800,\ 5000)$ approximately [M1]

$\Rightarrow\ \mathrm{P}\left(\overline{W}>3000\right)=0.00234$ (3.s.f) [A1]

7 Three cards are to be drawn at random without replacement from a pack of six cards numbered 0, 0, 1, 1, 1, 1.

The random variables X_1, X_2 and X_3 denotes the numbers on the first, second and third card respectively.

(i) If $Y = X_1 - X_2 + X_3$, show that $P(Y = 1) = \frac{1}{3}$ and find the probability distribution of Y. [3]

Solution

$P(Y = 1) = P(0, 0, 1) + P(1, 0, 0) + P(1, 1, 1)$

$= \frac{2}{6}\times\frac{1}{5}\times\frac{4}{4}+\frac{4}{6}\times\frac{2}{5}\times\frac{1}{4}+\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}$

$= \frac{1}{3}$ (Shown)

y	-1	0	1	2
$P(Y = y)$	$P(0, 1, 0) = \frac{2}{6}\times\frac{4}{5}\times\frac{1}{4}$ $= \frac{1}{15}$	$P(1, 1, 0) + P(0, 1, 1)$ $= \frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}+\frac{2}{6}\times\frac{4}{5}\times\frac{3}{4}$ $= \frac{2}{5}$	$\frac{1}{3}$	$P(1, 0, 1) = \frac{4}{6}\times\frac{2}{5}\times\frac{3}{4}$ $= \frac{1}{5}$

(ii) Find the values of $E(Y)$ and $Var(Y)$. [2]

Solution

$E(Y) = -\frac{1}{15}+0+\frac{1}{3}+\frac{2}{5} = \frac{2}{3} = 0.667$ (3sf)

$Var(Y) = \frac{1}{15}+0+\frac{1}{3}+\frac{4}{5}-\left(\frac{2}{3}\right)^2 = \frac{34}{45} = 0.756$ (3sf)

(iii) Find the probability that the difference between Y and $E(Y)$ is less than one standard deviation of Y. [2]

Solution

$P\left(\left|Y-\frac{2}{3}\right| < \sqrt{\frac{34}{45}}\right) = P(Y = 0) + P(Y = 1)$

$= \frac{2}{5} + \frac{1}{3} = \frac{11}{15} = 0.733$ (3 sf)

[Turn Over

8 A factory makes a certain type of muesli bar which are packed in boxes of 20. The factory claims that each muesli bar weigh at least 200 grams. Muesli bars that weigh less than 200 grams are sub-standard. It is known that the probability that a muesli bar is sub-standard is 0.02 and the weight of a muesli bar is independent of other muesli bars.

(i) Find the probability that, in a randomly chosen box of muesli bars, there are at least one but no more than five sub-standard muesli bars. [2]

Solution

Let X be the random variable denoting the number of sub-standard muesli bars, out of 20. $\therefore$ $X \sim \mathrm{B}(20, 0.02)$

$\Rightarrow$ Probability required $= \mathrm{P}(1 \le X \le 5)$

$= \mathrm{P}(X \le 5) - \mathrm{P}(X = 0)$

$= 0.33239$

$= 0.332$ (3 sf)

The boxes of muesli bars are sold in cartons of 12 boxes each.

(ii) Find the probability that, in a randomly chosen carton, at least 75% of the boxes will have at most one sub-standard muesli bar. [2]

Solution

Let Y be the random variable denoting the number of boxes of muesli bars that have at most one sub-standard muesli bars, out of 12 boxes.

$\mathrm{P}(X \le 1) = 0.94010$

$\therefore$ $Y \sim \mathrm{B}(12, 0.94010)$

Probability required $= \mathrm{P}(Y \ge 0.75 \times 12)$

$= \mathrm{P}(Y \ge 9)$

$= 1 - \mathrm{P}(Y \le 8)$

$= 0.99568$

$= 0.996$

As part of quality control for each batch of muesli bars produced daily, one box of muesli bars is randomly selected to be tested. If all the muesli bars in the box weigh at least 200 grams, the batch will be released for sales. If there are more than one sub-standard muesli bar, the batch will not be released. If there is exactly one sub-standard muesli bar, a second box of muesli bars will be tested. If all the muesli bars in the second box weigh at least 200 grams, the batch will be released. Otherwise, the batch will not be released.

(iii) Find the probability that the batch is released on a randomly selected day. [2]

Solution

Probability required $= P(X_1 = 0) + P(X_1 = 1)\,P(X_2 = 0)$

$= 0.84953$

$= 0.850$ (3 sf)

(iv) If the factory operates 365 days in a year, state the expected number of days in the year when the batch will not be released. [1]

Solution

Expected number of days when the batch will not be released

$= 365 - 365 \times 0.84953$

$= 54.9$ (3 sf)

To entice more people to buy their muesli bars, a scratch card is inserted into each box. The probability that a scratch card will win a prize is p. It may be assumed that whether a scratch card will win a prize or not, is independent of the outcomes of the other scratch cards.

Jane buys a carton of muesli bars.

(v) Write in terms of p, the probability that Jane win 2 prizes. [1]

Solution

Let W be the random variable denoting the number of prizes Jane wins, out of 12.

$\therefore W \sim B(12, p)$

$$P(W = 2) = \binom{12}{2} p^2 (1-p)^{10}$$

$$= 66 p^2 (1-p)^{10}$$

[Turn Over

The probability that Jane win 2 prizes is more than twice the probability that she win 3 prizes.

(vi) Find in exact terms, the range of values that p can take. [2]

<u>Solution</u>

$\mathrm{P}(W=2) > 2\,\mathrm{P}(W=3)$

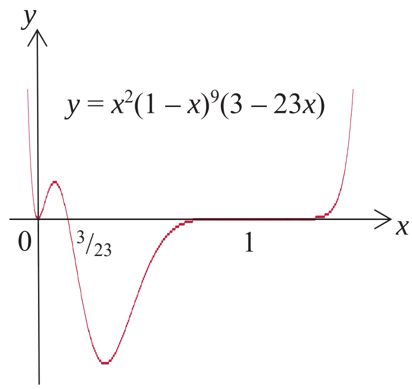
$66p^2(1-p)^{10} > 2\binom{12}{3}p^3(1-p)^9$

$66p^2(1-p)^{10} - 440p^3(1-p)^9 > 0$

$p^2(1-p)^9\left[3(1-p)-20p\right] > 0$

$p^2(1-p)^9\left[3-23p\right] > 0$

Since $0 \le p \le 1$, $0 < p < \dfrac{3}{23}$

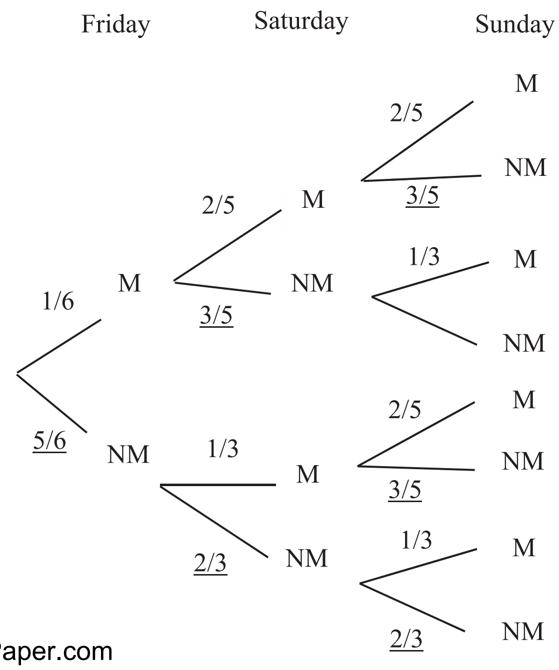


9 In preparation for an upcoming event, a student management team is considering having meetings on this Friday, Saturday and Sunday. The probability that the meeting is conducted on Friday is $\dfrac{1}{6}$. On each of the other days, the probability that a meeting is conducted when a meeting has already been conducted on the previous day, is $\dfrac{2}{5}$ and the probability that a meeting is conducted, when a meeting has not been conducted on the previous day, is $\dfrac{1}{3}$.

Find the probability that

(i) a meeting is conducted on Sunday, [3]

<u>Solution</u>



P(meeting is conducted on Sunday)

$= \dfrac{1}{6}\left[\dfrac{2}{5}\cdot\dfrac{2}{5}+\dfrac{3}{5}\cdot\dfrac{1}{3}\right]+\dfrac{5}{6}\left[\dfrac{1}{3}\cdot\dfrac{2}{5}+\dfrac{2}{3}\cdot\dfrac{1}{3}\right]$

$= \dfrac{481}{1350}$ (or 0.356)

(ii) a meeting is not conducted on Friday given that there is a meeting on Sunday. [3]

<u>Solution</u>

P(no meeting on Friday | meeting on Sunday)

$$= \frac{\text{P}\left(\text{no meeting on Friday and meeting on Sunday }\right)}{\text{P}\left(\text{no meeting on Friday}\right)}$$

$$= \frac{\frac{5}{6}\left[\frac{1}{3}\cdot\frac{2}{5}+\frac{2}{3}\cdot\frac{1}{3}\right]}{\left(\frac{481}{1350}\right)}$$

$$= \frac{400}{481}$$

$$= 0.83160$$

$$= 0.832 \text{ (correct to 3 s.f.)}$$

The student management team consists of 7 girls and 3 boys. At one of the meetings, they sit at a round table with 10 chairs.

(iii) Find the probability that the girls are seated together. [2]

<u>Solution</u>

$$\text{P}\left(\text{girls are seated together}\right) = \left[\frac{(4-1)!7!}{(10-1)!}\right]$$

$$= \frac{1}{12}$$

Two particular boys are absent from the meeting.

(iv) Find the probability that two particular girls do not sit next to each other. [3]

<u>Solution</u>

$$\text{Probability required} = \frac{(8-1)!\,{}^{8}\text{P}_2\,\frac{1}{2!}}{(10-1)!\,\frac{1}{2!}}$$

$$= \frac{7}{9}$$

[Turn Over

10 Along a 3km stretch of a road, the speed in km/h of a vehicle is a normally distributed random variable T. Over a long period of time, it is known that the mean speed of vehicles traveling along that stretch of the road is 90.0 km/h. To deter speeding, the traffic governing body introduced a speed monitoring camera. Subsequently, the speeds of a random sample of 60 vehicles are recorded. The results are summarised as follows.

$$\sum t = 5325, \qquad \sum \left(t-\bar{t}\right)^2 = 2000.$$

(i) Find unbiased estimates of the population mean and variance, giving your answers to 2 decimal places. [2]

Solution

Unbiased estimate of the population mean = $\bar{t} = \dfrac{5325}{60} = 88.75$

Unbiased estimate of the population variance = $s^2 = \dfrac{\sum\left(t-\bar{t}\right)^2}{59} = \dfrac{2000}{59}$

$= 33.89830508$

$= 33.90$

(ii) Test, at the 5% significance level, whether the speed-monitoring camera is effective in deterring the speeding of vehicles on the stretch of road. [4]

Solution

Let μ denote the population mean speed of the vehicles traveling along the stretch of the road.

To test H_0: $\mu = 90.0$

Against H_1: $\mu < 90.0$

Conduct a one-tail test at 5% level of significance, i.e., $\alpha = 0.05$

Under H_0, since $n = 60$ and is sufficiently large, by Central Limit Theorem,

$\overline{T} \sim N\left(90.0, \dfrac{33.89830508}{60}\right)$ approximately.

Using GC, p-value = 0.0481545117

Since p-value < 0.05, we reject H_0. There is sufficient evidence at 5% level of significance to conclude that that the mean speed along the 3km stretch of road has been reduced.

(iii) In another sample of size n ($n > 30$) that was collected independently, it is given that $\bar{t}$ = 89.0. The result of the subsequent test using this information and the unbiased estimate of the population variance in **part (i)** is that the null hypothesis is not rejected. Obtain an inequality involving n, and hence find the largest possible value n can take. [4]

Solution

If the null hypothesis is not rejected, z_{calc} must lie outside the critical region.

Critical Region: $z \leq -1.644853626$

Test Statistics, $Z = \dfrac{\overline{T} - 90.0}{\sqrt{\dfrac{33.89830508}{n}}} \sim \text{N}(0, 1)$

$$\therefore\ z_{\text{calc}} = \frac{89 - 90.0}{\sqrt{\dfrac{33.89830508}{n}}} > -1.644853626$$

$$\frac{-\sqrt{n}}{\sqrt{33.89830508}} > -1.644853626$$

$$\sqrt{n} < 9.576708062$$

$$n < 91.713$$

Since n is an integer, the largest possible value n can take is 91.