## Section A: Pure Mathematics [40 marks]

**1** The function f is defined by

$$\mathrm{f}: x \mapsto \frac{x^2}{2-x},\ x \in \mathbb{R},\ 0 \le x < 2\ .$$

**(i)** Find $\mathrm{f}^{-1}(x)$ and write down the domain of $\mathrm{f}^{-1}$. [4]

It is given that

$$\mathrm{g}: x \mapsto \frac{1}{1+\mathrm{e}^{-x}},\ x \in \mathbb{R},\ x \ge 0\ .$$

**(ii)** Show that fg exists. [2]

**(iii)** Find the range of fg. [2]

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| **1(i)** | Let $y = \dfrac{x^2}{2-x}$.<br>$2y - xy = x^2 \quad \Rightarrow \quad x^2 + xy - 2y = 0$<br>Then $x = \dfrac{-y \pm \sqrt{y^2+8y}}{2}$<br>Alternatively,<br>$2y - xy = x^2 \quad \Rightarrow \quad x^2 + xy - 2y = 0$<br>$\Rightarrow \left(x+\dfrac{y}{2}\right)^2 - \left(\dfrac{y}{2}\right)^2 - 2y = 0$<br>$\Rightarrow \left(x+\dfrac{y}{2}\right)^2 = \dfrac{y^2}{4} + 2y$<br>$\Rightarrow x + \dfrac{y}{2} = \pm\sqrt{\dfrac{y^2}{4} + 2y}$<br>$\Rightarrow x + \dfrac{y}{2} = \pm\sqrt{\dfrac{y^2}{4} + 2y}$<br>$\Rightarrow x = \dfrac{-y \pm \sqrt{y^2+8y}}{2}$<br>Since $0 \le x < 2$, $x = \dfrac{-y + \sqrt{y^2+8y}}{2}$.<br>$\mathrm{f}^{-1}(x) = \dfrac{-x + \sqrt{x^2+8x}}{2}$<br>$\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = [0, \infty)$ | Many students are uncertain how to make *x* the subject. Note that this can be done by either using the quadratic formula or completing square.<br><br>Students must note that it is essential to explain why $x = \dfrac{-y - \sqrt{y^2+8y}}{2}$ is rejected.<br><br>Some students did not provide the final answer. While they able to obtain $x = \dfrac{-y + \sqrt{y^2+8y}}{2}$, they fail to write down the expression for $\mathrm{f}^{-1}(x)$ and $\mathrm{D}_{\mathrm{f}^{-1}}$ to answer the question completely. |
| **1(ii)** | $\mathrm{R}_{\mathrm{g}} = [\tfrac{1}{2}, 1)$<br>$\mathrm{D}_{\mathrm{f}} = [0, 2)$ | It is necessary to write down the $\mathrm{R}_{\mathrm{g}}$ and $\mathrm{D}_{\mathrm{f}}$ to answer this |

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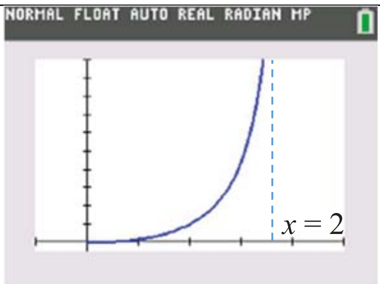
| | Since $R_g \subset D_f$, the function fg exists. | part of the question. Simply stating $R_g \subset D_f$ is insufficient. |
| **1(iii)** | <br>Observe that $y = f(x)$ is a strictly increasing function; so we only need to find $f(\frac{1}{2}) = \frac{1}{6}$ and $f(1) = 1$.<br>$R_{fg} = [\frac{1}{6}, 1)$ | Many students did not indicate that f is an increasing function. Simply writing $R_g = [\frac{1}{2}, 1)$ leading to $R_{fg} = [\frac{1}{6}, 1)$ is insufficient to obtain the full credit. |

**2** Express $\dfrac{6r+7}{r(r+1)}$ as partial fractions. [1]

**(i)** Hence use method of differences to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ in terms of $N$. (There is no need to express your answer as a single algebraic fraction.) [3]

**(ii)** Give a reason why the series $\displaystyle\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ converges, and write down its value. [2]

**(iii)** Use your answer in part **(i)** to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)$. [3]

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| **2** | $\dfrac{6r+7}{r(r+1)} = \dfrac{7}{r} - \dfrac{1}{r+1}$ | Most students got this right. |

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| (i) | $\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)=\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\left(\frac{7}{r}-\frac{1}{r+1}\right)\right)$<br>$=\sum_{r=1}^{N}\left(\left(\frac{7^{-r}}{r}-\frac{7^{-r-1}}{r+1}\right)\right)$<br>$=(\frac{7^{-1}}{1}-\frac{7^{-2}}{2}$<br>$+\frac{7^{-2}}{2}-\frac{7^{-3}}{3}$<br>$+\frac{7^{-3}}{3}-\frac{7^{-4}}{4}$<br>$+\cdots$<br>$+\frac{7^{-N+1}}{N-1}-\frac{7^{-N}}{N}$<br>$+\frac{7^{-N}}{N}-\frac{7^{-N-1}}{N+1})$<br>$=\frac{7^{-1}}{1}-\frac{7^{-N-1}}{N+1}=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$ | A common error here is to express $\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\left(\frac{7}{r}-\frac{1}{r+1}\right)\right)$ as $\sum_{r=1}^{N}\left(\frac{1}{7}\right)^{r+1}\cdot\sum_{r=1}^{N}\left(\frac{7}{r}-\frac{1}{r+1}\right)$. Students must take note that $\sum_{r=1}^{N}\left(f(r)\cdot g(r)\right)$ is not equal to $\sum_{r=1}^{N}f(r)\cdot\sum_{r=1}^{N}g(r)$. |
| (ii) | As $N\to\infty,\left(\frac{1}{7}\right)^{N+1}\to 0,\frac{1}{N+1}\to 0$<br>Hence as $N\to\infty$, $\frac{7^{-N-1}}{N+1}=\left(\frac{1}{7}\right)^{N+1}\left(\frac{1}{N+1}\right)\to 0$<br>Then $\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)=\frac{1}{7}$,<br>$\therefore\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ converges and it converges to $\frac{1}{7}$. | Students must know that the highlighted statement is the reason why the series converges, and therefore must be clearly written down in your answer. |
| (iii) | Method 1:<br>$\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)$<br>replace $r$ by $r$-1 $=\sum_{r-1=1}^{r-1=N}\left(\left(\frac{1}{7}\right)^{r-1+1}\frac{6(r-1)+13}{(r-1+1)(r-1+2)}\right)$<br>$=\sum_{r=2}^{N+1}\left(\left(\frac{1}{7}\right)^{r}\frac{6r+7}{(r)(r+1)}\right)=7\sum_{r=2}^{N+1}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{(r)(r+1)}\right)$<br>$=7\left[\sum_{r=1}^{N+1}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{(r)(r+1)}\right)-\left(\frac{1}{7}\right)^{2}\frac{13}{(1)(2)}\right]$ | Many students are aware that they must replace $r$ by $r$–1 in Method 1 or replace r by $r$+1 in Method 2. However, the plan is poorly executed with many making errors in its replacement. Some who have correctly replaced the $r$, are unsure how to proceed from there. |

$$=7\left[\sum_{r=1}^{N+1}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{(r)(r+1)}\right)-\left(\frac{1}{7}\right)^{2}\frac{13}{(1)(2)}\right]$$

$$=7\left[\frac{1}{7}-\frac{7^{-(N+1)-1}}{(N+1)+1}-\frac{13}{49(2)}\right]=\frac{1}{14}-\frac{7^{-N-1}}{N+2}=\frac{1}{14}-\frac{1}{7^{N+1}(N+2)}$$

Method 2:
From (i),

$$\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$$

Replace r by r+1, we have

$$\sum_{r+1=1}^{r+1=N}\left(\left(\frac{1}{7}\right)^{(r+1)+1}\frac{6(r+1)+7}{(r+1)((r+1)+1)}\right)=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$$

$$\sum_{r=0}^{N-1}\left(\left(\frac{1}{7}\right)^{r+2}\frac{6r+13}{(r+1)(r+2)}\right)=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$$

Replace $N$ by $N$+1, we have

$$\sum_{r=0}^{N}\left(\left(\frac{1}{7}\right)^{r+2}\frac{6r+13}{(r+1)(r+2)}\right)=\frac{1}{7}-\frac{1}{7^{N+2}(N+2)}$$

$$\left[\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+2}\frac{6r+13}{(r+1)(r+2)}\right)\right]+\underbrace{\left(\frac{1}{7}\right)^{2}\frac{13}{2}}_{\text{Term when } r=0}=\frac{1}{7}-\frac{1}{7^{N+2}(N+2)}$$

$$\frac{1}{7}\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)=\frac{1}{7}-\frac{1}{7^{N+2}(N+2)}-\frac{13}{98}$$

$$\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)=7\left(\frac{1}{98}-\frac{1}{7^{N+2}(N+2)}\right)$$

$$=\frac{1}{14}-\frac{1}{7^{N+1}(N+2)}$$

**3** A curve $C$ with equation $y = \mathrm{f}(x)$ satisfies the equation

$$(x^2+2x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

and passes through the point $(0, \pi)$.

**(i)** By further differentiation, find the Maclaurin expansion of $\mathrm{f}(x)$ in ascending powers of $x$ up to and including the term $x^3$. [5]

**(ii)** Solve the differential equation $(x^2+2x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$, given that $y = \pi$ when $x = 0$, leaving $y$ in terms of $x$.
Hence show that

$$\tan^{-1}(x+1) \approx \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3$$

for small values of $x$. [4]

**(iii)** With the aid of a sketch, explain why $\int_0^2 \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 \,\mathrm{d}x$ gives a more accurate approximation of $\int_0^2 \tan^{-1}(x+1)\,\mathrm{d}x$ than $\int_0^2 \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 \,\mathrm{d}x$. [2]

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| **3(i)** | (i) $(x^2+2x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$<br>Differentiating wrt $x$,<br>$(x^2+2x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$<br>Differentiating wrt $x$,<br>$(x^2+2x+2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (2x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$<br>$\Rightarrow (x^2+2x+2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (4x+4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$<br>When $x = 0$, $y = \pi$ since curve passes through $(0, \pi)$.<br>$2\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1$<br>$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(1) = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1$<br>$2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 4(-1) + 2(1) = 0 \Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 1$<br>Thus the Maclaurin expansion is<br>$y = \mathrm{f}(x) = \pi + x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$<br>i.e., $y = \mathrm{f}(x) = \pi + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$ | Question was very well done. Only a handful of students expressed $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x^2+2x+2}$ and performed direct differentiation.<br><br>Many students integrate the expression to find $y$, which is the question for (ii). This is an indicator that they do not know what the question is asking for. |

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<td>(ii)</td>
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(ii) $\quad (x^2+2x+2)\dfrac{dy}{dx}=2$<br>
$\Rightarrow \quad \int dy = 2\int \dfrac{1}{x^2+2x+2}\,dx$<br>
$\Rightarrow \quad y = 2\int \dfrac{1}{(x+1)^2+1^2}\,dx$<br>
$\Rightarrow \quad y = 2\tan^{-1}(x+1)+c$<br>
Sub $x=0$, $y=\pi$: $\quad \pi = 2\tan^{-1}(1)+c$<br>
$\Rightarrow \quad c = \pi - 2\left(\dfrac{\pi}{4}\right)=\dfrac{\pi}{2}$<br>
Thus, $y = 2\tan^{-1}(x+1)+\dfrac{\pi}{2}$.<br>
$\Rightarrow \quad 2\tan^{-1}(x+1)+\dfrac{\pi}{2} = \pi + x - \dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\ldots$<br>
$\Rightarrow \quad \tan^{-1}(x+1) = \dfrac{1}{2}\left(\pi - \dfrac{\pi}{2} + x - \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3+\ldots\right)$<br>
$\Rightarrow \quad \tan^{-1}(x+1) \approx \dfrac{1}{4}\pi + \dfrac{1}{2}x - \dfrac{1}{4}x^2 + \dfrac{1}{12}x^3$ (Shown)<br>
for small values of $x$
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Apart from algebraic errors, students were able to find the particular solution.<br><br>
Quite a number of students forgot to "$+C$", which affected the rest of their answers.<br><br>
Most students were able to see the relationship between their answers in (i) and (ii).
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<td>(iii)</td>
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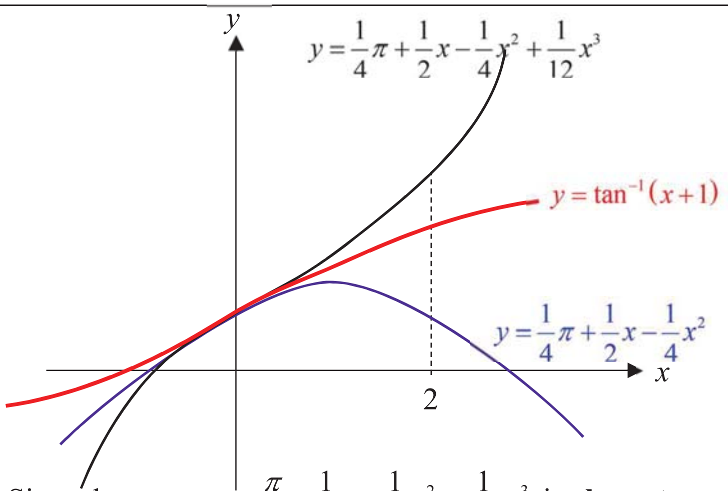

Since the curve $y=\dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2+\dfrac{1}{12}x^3$ is closer to $y=\tan^{-1}(x+1)$ than the curve $y=\dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2$ for $0\le x\le 2$, the area under the cubic curve, $\int_0^2 \dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2+\dfrac{1}{12}x^3\,dx$, will give a <u>better</u> approximation to $\int_0^2 \tan^{-1}(x+1)\,dx$, the area under the curve $y=\tan^{-1}(x+1)$, <u>than</u> the area under the quadratic curve, $\int_0^2 \dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2\,dx$.<br>

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This part was poorly performed. Students should realise that they must sketch $y=\tan^{-1}(x+1)$ in order to make any comparison.<br>
For the sketches, many students did not pay attention to the details which led to loss of marks. The more common ones are:<br>
1. not realizing that they have the same y-intercept,<br>
2. Not showing that the quadratic curve has a turning point before $x=2$, and cuts the $x$-axis after $x=2$.<br>
3. Not showing that the cubic curve is concave up.
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| | Students plotted / labelled the graphs as (for e.g.) $\int_0^2 \frac{\pi}{4}+\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{12}x^3 \, dx$ which is a definite integral (i.e. just a value). This has no meaning for comparison.<br>A number of students drew rectangles under the graph, which clearly shows regurgitation without understanding.<br>For the qualitative explanation, students should make it clear that the comparison of how close the curves are is limited from $x=0$ to $x=2$. Also, the link between the definite integral to the area must be clear. |

**4** The points *A*, *B*, *C* and *D* have coordinates $(1,0,3)$, $(-1,0,1)$, $(1,1,3)$ and $(1,k,0)$ respectively, where *k* is a positive real number. The plane $p_1$ contains *A*, *B* and *C* while the plane $p_2$ contains *A*, *B* and *D*.

Given that $p_1$ makes an angle of $\frac{\pi}{3}$ with $p_2$, show that $k=\frac{\sqrt{6}}{2}$. [5]

The point *X* lies on $p_2$ such that the vector $\overrightarrow{XC}$ is perpendicular to $p_1$. Find $\overrightarrow{XC}$. [5]

Hence find the exact area of the triangle *AXC*. [2]

**Solution:**

A standard question which can provide a good source of marks, but students seemed to be unprepared for the third question on Vectors in Paper 2.

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| **4** | $\overrightarrow{AB}=\begin{pmatrix}-1\\0\\1\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=-2\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}1\\1\\3\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$,<br>$\overrightarrow{AD}=\begin{pmatrix}1\\k\\0\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}0\\k\\-3\end{pmatrix}$<br>Normal vector $\mathbf{n}_1$ of $p_1=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$<br>Normal vector $\mathbf{n}_2$ of $p_2=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\k\\-3\end{pmatrix}=\begin{pmatrix}-k\\3\\k\end{pmatrix}$<br>Since $p_1$ makes an angle of $\frac{\pi}{3}$ with $p_2$, $\frac{\lvert\mathbf{n}_1.\mathbf{n}_2\rvert}{\lvert\mathbf{n}_1\rvert\lvert\mathbf{n}_2\rvert}=\cos\frac{\pi}{3}$ | Many students used the correct method but made numerous errors seen in vector product operation to obtain the normal to planes $p_1$ and $p_2$. This had a knock-on effect on the other parts of the question.<br><br>This part required the direct application of formula to obtain the angle between planes in order to show that $k=\frac{\sqrt{6}}{2}$. However students made errors in obtaining normals to planes $p_1$ and $p_2$. Also some students wrongly |

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| | $\dfrac{\left\lvert\begin{pmatrix}-1\\0\\1\end{pmatrix}.\begin{pmatrix}-k\\3\\k\end{pmatrix}\right\rvert}{\sqrt{2}\sqrt{2k^2+9}}=\dfrac{1}{2}$<br>$\dfrac{\lvert 2k\rvert}{\sqrt{2}\sqrt{2k^2+9}}=\dfrac{1}{2}$<br>Squaring both sides, we have $\dfrac{4k^2}{2\left(2k^2+9\right)}=\dfrac{1}{4}$<br>$8k^2=2k^2+9$<br>$k^2=\dfrac{3}{2}$<br>Since $k$ is a positive real number, we have $k=\sqrt{\dfrac{3}{2}}=\dfrac{\sqrt{6}}{2}$ (shown) | stated that the formula gave the sine of the angle.<br>A handful students were not able to find the correct pair of vectors to "cross-product"<br>Example, to find normal to $p_1$ which contains *A, B and C*:<br>$\overrightarrow{OA}\times\overrightarrow{OB}$ [WRONG]<br><br>A mark was deducted for not including $k=-\dfrac{\sqrt{6}}{2}$ and rejecting the answer accordingly. |
| | $\overrightarrow{XC}$ is perpendicular to $p_1$ $\Rightarrow$ $\overrightarrow{XC}$ is parallel to $\mathbf{n}_1$<br>$\Rightarrow \overrightarrow{XC}=m\begin{pmatrix}-1\\0\\1\end{pmatrix}$<br>$\Rightarrow \overrightarrow{OX}=\overrightarrow{OC}-m\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\3\end{pmatrix}-m\begin{pmatrix}-1\\0\\1\end{pmatrix}$ | This part was badly attempted. There was a lack of clarity in presentation for this part.<br>Many students were able to write out the equation of line<br>*XC*: **r** = $\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}$, but<br>instead of<br>$\overrightarrow{OX}=\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}$ [RIGHT],<br>many wrote<br>$\overrightarrow{XC}=\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}$ [WRONG]<br>This was an indication of poor understanding of the equation of line! |

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| | Equation of $p_2$ is<br><br>$\mathbf{r}.\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\begin{pmatrix}1\\0\\3\end{pmatrix}.\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\sqrt{6}$ **i.e.** $\mathbf{r}.\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\sqrt{6}$<br><br>Since $X$ lies on $p_2$, we have $\left[\begin{pmatrix}1\\1\\3\end{pmatrix}-m\begin{pmatrix}-1\\0\\1\end{pmatrix}\right].\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\sqrt{6}$<br><br>$\Rightarrow\left(-\frac{\sqrt{6}}{2}+3+\frac{3\sqrt{6}}{2}\right)-m\left(\frac{\sqrt{6}}{2}+0+\frac{\sqrt{6}}{2}\right)=\sqrt{6}$<br>$\Rightarrow\sqrt{6}+3-m\left(\sqrt{6}\right)=\sqrt{6}$<br>$\Rightarrow m\left(\sqrt{6}\right)=3$<br>$\therefore m=\frac{3}{\sqrt{6}}$<br>$\Rightarrow\ \overrightarrow{XC}=\frac{3}{\sqrt{6}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$ | |
| | $\left\lvert\overrightarrow{XC}\right\rvert=\left\lvert\sqrt{\frac{3}{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\rvert=\sqrt{\frac{3}{2}}\sqrt{2}=\sqrt{3}$<br><br>$\left\lvert\overrightarrow{AC}\right\rvert=\left\lvert\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rvert=1$<br><br>Since $XC$ is perpendicular to $p_1$ and $AC$ lies on $p_1$, we have $XC$ perpendicular to $AC$.<br><br>Therefore, area of triangle $AXC$ is $\frac{1}{2}\left\lvert\overrightarrow{XC}\right\rvert\left\lvert\overrightarrow{AC}\right\rvert=\frac{1}{2}\left(\sqrt{3}\right)(1)=\frac{\sqrt{3}}{2}$ | Many students failed to see that triangle $AXC$ is a right-angled triangle and the area can be quite easily obtained by $\frac{1}{2}(base)(\perp height)$ i.e. $\frac{1}{2}(AC)(XC)$. Instead many students found the area by finding $\frac{1}{2}\left\lvert\overrightarrow{AC}\times\overrightarrow{XC}\right\rvert$. |

## Section B: Probability and Statistics [60 marks]

**5** Anand, Beng, Charlie, Dayanah and 6 other people attend a banquet dinner, and are to sit at a round table.

**(i)** Dayanah will only sit next to Anand, Beng or Charlie (and no one else), and Anand, Beng and Charlie do not want to sit next to each other. Find the number of ways the 10 people can seat themselves around the table. [4]

**(ii)** As part of dinner entertainment, 4 people from the table are chosen to participate in a game.

Among Anand, Beng and Charlie, if any one of them is chosen, the other two will refuse to participate in the game. Furthermore, Dayanah refuses to participate unless at least one of Anand, Beng or Charlie is also chosen.

Find the number of ways the 4 people can be chosen for the game. [3]

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| **5(i)** | No. of ways to arrange 2 of A/B/C next to D, to form a unit<br>$= {}^3C_2 \times 2!$<br>No. of ways to arrange the unit and 6 other people in a circle<br>$= (7-1)!$<br>No. of ways to insert the remaining person from A/B/C $= {}^5C_1$<br>$\therefore$ Total no. of ways =<br>${}^3C_2 \times 2! \times (7-1)! \times {}^5C_1 = 21600$ | **In general, students lost marks for this whole question because they failed to understand the question.**<br>For this part, many students interpreted "sit next to" as applying to one side (should be both sides) and therefore were incorrect.<br>**Students should also take the effort to describe/explain their steps** to possibly gain partial credit for their methodology. |
| **(ii)** | Case 1: $A$, $B$ or $C$ not chosen<br>Number of ways $= {}^6C_4 = 15$<br>Case 2: Exactly one of $A$, $B$ or $C$ chosen<br>Number of ways $= {}^3C_1 \times {}^7C_3 = 105$<br><br>Total number of ways = 15 + 105 = 120<br><br>[Note: Case 2 may be split into:<br>Case 2a: Exactly one of $A$, $B$ or $C$ chosen but not Dayanah,<br>(Number of ways $= {}^3C_1 \times {}^6C_3 = 60$)<br>And Case 2b: Dayanah and exactly one of $A$, $B$ or $C$ chosen<br>(Number of ways $= {}^3C_1 \times {}^6C_2 = 45$).] | A large number of students assumed that when $A$, $B$ or $C$ were chosen, $D$ must be chosen when this is not the case.<br>Otherwise this part was the better done of the two. |

**6** A bag contains four identical counters labelled with the digits 0, 1, 2, and 3. In a game, Amira chooses one counter randomly from the bag and then tosses a fair coin. If the coin shows a Head, her score in the game is the digit labelled on the counter chosen. If the coin shows a Tail, her score in the game is the negative of the digit labelled on the counter chosen. $T$ denotes the score in a game.

**(i)** Find the probability distribution of $T$. [2]

**(ii)** Amira tosses the coin and it shows a Tail. Find the probability that $T < -1$. [3]

**(iii)** Amira plays the game twice. Find the probability that the sum of her two scores is positive. [3]

| | | |
|---|---|---|
| **6(i)** | $\begin{array}{c\|c\|c\|c\|c\|c\|c\|c} t & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ \hline \mathrm{P}(T=t) & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{array}$ | Most students could do this part correctly. |
| **(ii)** | $\mathrm{P}(T<-1 \mid \text{coin shows a tail})$<br>$= \dfrac{\mathrm{P}(\text{counter shows 2 or 3 and coin shows a Tail})}{\mathrm{P}(\text{coin shows a Tail})}$<br>$\dfrac{\left(\frac{2}{4}\right)\times\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \dfrac{1}{2}$ | Many students could not identify that this was a conditional probability question. Details were also insufficient among the rest of the answers. |
| **(iii)** | Required probability $= \frac{1}{2}[1-\mathrm{P}(\text{sum of scores}=0)]$<br>$= \frac{1}{2}\left[1-\left(\frac{1}{4}\times\frac{1}{4}+\frac{1}{8}\times\frac{1}{8}\times 6\right)\right]$<br>$= \frac{27}{64}$<br><br>Alternative:<br>Required probability<br>$= 2[\mathrm{P}(0,1)+\mathrm{P}(0,2)+\mathrm{P}(0,3)]+2[\mathrm{P}(-1,2)+\mathrm{P}(-1,3)]+2\mathrm{P}(-2,3)$<br>$+2[\mathrm{P}(1,2)+\mathrm{P}(1,3)+\mathrm{P}(2,3)]+[\mathrm{P}(1,1)+\mathrm{P}(2,2)+\mathrm{P}(3,3)]$<br>$= 2\left[\left(\frac{1}{4}\times\frac{1}{8}\times 3\right)\right]+2\left[\left(\frac{1}{8}\times\frac{1}{8}\times 2\right)\right]+2\left(\frac{1}{8}\times\frac{1}{8}\right)+2\left(\frac{1}{8}\times\frac{1}{8}\times 3\right)+3\left(\frac{1}{8}\times\frac{1}{8}\right)$<br>$= \frac{27}{64}$ | Most students attempted the second method, but there were many common errors:<br>(1) not considering permutations for each case,<br>(2) missing out cases (such as 1 + 1)<br>(3) forgetting that $\mathrm{P}(T=0) = \frac{1}{4}$.<br><br>A smaller number tried to use a table of outcomes, but were not very successful as they did not notice that not all outcomes had equal probabilities. |

**7** It is generally accepted that a person's diet and cardiorespiratory fitness affects his cholesterol levels. The results of a study on the relationship between the cholesterol levels, $C$ mmol/L, and cardiorespiratory fitness, $F$, measured in suitable units, on 8 individuals with similar diets are given in the following table.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Cardiorespiratory Fitness ($F$ units) | 55.0 | 50.7 | 45.3 | 40.2 | 34.7 | 31.9 | 27.9 | 26.0 |
| Cholesterol ($C$ mmol/L) | 4.70 | 4.98 | 5.30 | 5.64 | 6.04 | 6.30 | 6.99 | 6.79 |

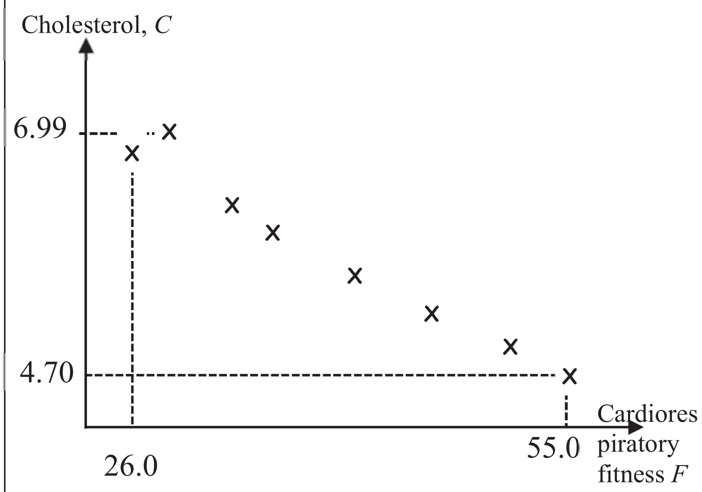
**(i)** Draw a scatter diagram of these data. Suppose that the relationship between $F$ and $C$ is modelled by an equation of the form $\ln C = aF + b$, where $a$ and $b$ are constants. Use your diagram to explain whether $a$ is positive or negative. [4]

**(ii)** Find the product moment correlation coefficient between ln $C$ and $F$, and the constants $a$ and $b$ for the model in part **(i)**. [3]

**(iii)** Bronz is a fitness instructor. His cardiorespiratory fitness is 52.0 units. Estimate Bronz's cholesterol level using the model in **(i)** and the values of $a$ and $b$ in part **(ii)**. Comment on the reliability of the estimate. [2]

**(iv)** Bronz then had a medical checkup and found his actual cholesterol level to be 6.2 mmol/L. Assuming his cholesterol level is measured accurately, explain why there is a great difference between Bronz's cholesterol level and the estimated value in **(iii)**. [1]

**(i)**




$\ln C = aF + b \quad \Rightarrow \quad C = e^{aF+b}$

From the scatter diagram, as $F$ increases, <u>$C$ decreases at a decreasing rate</u>, hence $a < 0$.



- Many students gave a scatter diagram of ln$C$ vs $F$. However, the question asked for a scatter diagram of *these data* & the given data is $C$ and $F$.
- Please ensure that the **relative positions** of the 8 data points (and not 7 or 9 points) are correct.
- Many students did not label the max/min values of $C$ and $F$. Amongst those who did, a large majority did not label the max $C$ (6.99) and/or min $F$ (26) correctly.

This part was poorly attempted.

- Many students say that the gradient of the points in the scatter diagram ($C$ against $F$) are negatively correlated so it has a negative gradient and thus $a$ is negative. BUT "$a$" is the gradient of the regression line of ln$C$ on $F$ and <u>not</u> $C$ on $F$.

| | | |
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| | | • <u>Alternative solution given by some students</u><br>ln$C$ decreases as $C$ decreases.<br>Thus as $F$ increases, ln$C$ decreases.<br>This means that the gradient of the regression line of ln$C$ on $F$ is negative.<br>However, they failed to **explain <u>clearly</u>** that "$a$" is the gradient of the regression line <u>before concluding</u> that "$a$" is negative. |
| **(ii)** | Product moment correlation $r = -0.992$ (3 sf)<br><br>Model is $\ln C = -0.013371F + 2.2772$ (5 sf)<br>Thus, $a = -0.0134$ (3 sf)<br>and $b = 2.28$ (3 sf) | Some students identified the values of $a$ and $b$ wrongly, i.e., wrote $a = 2.28$ in spite of the fact they had claimed that $a < 0$ in (i). |
| **(iii)** | $\ln C = -0.013371(52.0) + 2.2772$<br>$C = \mathrm{e}^{1.581908} = 4.86$<br>Estimate of Bronz's cholesterol level is 4.86 mmol/L<br><br>Since $r = -0.992$ is close to $-1$ which suggests that the linear model is a good one <u>and</u> Bronz's cardiorespiratory fitness level, $F = 52.0$ lies within the data range of $[26.0, 55.0]$, the estimate is reliable. | Some students used the inaccurate values of $a$ and $b$, i.e., used $\ln C = -0.0134F + 2.28$ to compute $C$ and got an inaccurate value.<br><br>Many students mentioned only one of the 2 conditions needed for a reliable estimate with the regression line. |
| **(iv)** | Bronz's **diet** could be very high in cholesterol **compared to the 8 individuals** which resulted in his actual cholesterol level of 6.20 to be higher than the value of 4.86 mmol/L estimated using the values in (iv) which are based on the 8 individuals. | This part was very poorly attempted.<br>Students need to check if there is a reason provided by the context before giving any other reasonable explanation.<br>In this question, a reason was given (diet) and so this is the only answer accepted. The regression line that was used to find the estimate was based on the data from the 8 people with a similar diet. So the estimate of 4.86 mmol/L would be Bronz's cholesterol level if he was on the same diet. |

**8** A research laboratory uses a data probe to collect data for its experiments. There is a probability of 0.04 that the probe will give an incorrect reading. In a particular experiment, the probe is used to take 80 readings, and $X$ denotes the number of times the probe gives an incorrect reading.

**(i)** State, in context, two assumptions (not conditions) necessary for $X$ to be well modelled by a binomial distribution. [2]

**(ii)** Find the probability that between 5 and 10 (inclusive) incorrect readings are obtained in the experiment. [3]

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| **8(i)** | 1. The **probability** of the probe giving an incorrect reading is **constant** at 0.04 for each reading.<br><br>2. Trial of whether a reading by the probe is incorrect is **independent** of other readings. | Not well answered. Despite the many reminders, common errors are:<br>1. Probability of getting an incorrect reading is independent of …<br>2. Two outcomes: correct or incorrect reading (given in question, not assumption)<br>3. Fixed number (80) of readings (given in question, not assumption)<br>4. Trial is on "reading correct or incorrect", not the probe (only one probe) nor the experiment (only one experiment). |
| **(ii)** | $X \sim \mathrm{B}(80, 0.05)$<br><br>$\mathrm{P}(5 \leq X \leq 10) = \mathrm{P}(X \leq 10) - \mathrm{P}(X \leq 4)$<br>$= 0.216$ (3sf) | Generally well done for this part.<br><br>Common error is interpreting "inclusive" to apply only to 10 and not 5, i.e. wrongly gave $\mathrm{P}(5 < X \leq 10)$.<br><br>Some students wrongly gave $\mathrm{P}(5 \leq X \leq 10) = \mathrm{P}(X \leq 10) \times \mathrm{P}(X \geq 5)$ which is not true as the events $X \leq 10$ and $X \geq 5$ are not independent. |

When the probe gives an incorrect reading, it will give a reading that is 5% greater than the actual value.

**(iii)** Suppose the 80 readings are multiplied together to obtain a Calculated Value. Find the probability that the Calculated Value is **at least** 50% more than the product of the 80 actual values. [5]

**Solution**

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| **(iii)** | Let the actual $i^{\text{th}}$ value be $V_i$.<br>Product of 80 actual values = $V_1 \times V_2 \times \ldots \times V_{80}$<br><br>For an incorrect reading, $V_i$ is read as $(1.05)V_i$ (5% greater)<br>If there are $x$ incorrect readings,<br>Calculated Value = $V_1 \times V_2 \times \ldots \times V_{80}\,(1.05)^x$.<br><br>$V_1 \times V_2 \times \ldots \times V_{80}\left(1.05\right)^x \geq 1.5\,(V_1 \times V_2 \times \ldots \times V_{80})$<br>$\Rightarrow (1.05)^x \geq 1.5$<br>$\Rightarrow x\ln(1.05) \geq \ln 1.5$<br>$\Rightarrow x \geq \dfrac{\ln 1.5}{\ln 1.05} = 8.31$<br>i.e. at least 9 incorrect readings<br><br>Required probability = $P(X \geq 9)$<br>$= 1 - P(X \leq 8)$<br>$= 0.00468$ (3sf) | Most students did not know how to interpret the question.<br><br>Many students tried to apply Central Limit Theorem for sample sum – note that CLT does not apply to product of sample.<br><br>Of those who could interpret the question, many did not explain how they obtained $(1.05)^x \geq 1.5$<br><br>Some students presented 'product of the 80 actual values' wrongly as $V^{80}$ instead of $V_1 \times V_2 \times \ldots \times V_{80}$ (80 different readings) |

**9** A Wheel Set refers to a set of wheel rim and tyre. The three types of wheel sets are the Clincher Bike Wheel Set, Tubular Bike Wheel Set and Mountain Bike Wheel Set.
The weight of a rim of a Clincher Bike Wheel Set follows a normal distribution with mean 1.5 kg and standard deviation 0.01 kg. The weight of its tyre follows a normal distribution with mean 110 g and standard deviation 5 g.

**(i)** Let $C$ be the total weight in grams of a randomly chosen Clincher Bike Wheel Set in grams. Find $\mathrm{P}\left(C > 1620\right)$. [3]

**(ii)** State, in the context of the question, an assumption required in your calculation in **(i)**. [1]

Let $T$ be the total weight in grams of a Tubular Bike Wheel Set, where $T \sim \mathrm{N}\left(\mu,\ 15^2\right)$.

**(iii)** The probability that the weight of a randomly chosen Clincher Bike Wheel Set exceeds a randomly chosen Tubular Bike Wheel Set by more than 150 g is smaller than 0.70351 correct to 5 decimal places. Find the range of values that $\mu$ can take. [5]

Let $M$ be the total weight in grams of a randomly chosen Mountain Bike Wheel Set with mean 1800 g and standard deviation 20 g.

**(iv)** Find the probability that the mean weight of 50 randomly chosen Mountain Bike Wheel Sets is more than 1795 g. [3]

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| **9(i)** | $C \sim \mathrm{N}(1500 + 110,\ 10^2 + 5^2)$<br>$\mathrm{P}\left(A > 1620\right) = 0.185546 \approx 0.186$ (3sf) | Well attempted. Only a few students did not convert kilograms to grams. Note that $10^2 +\ 5^2 \neq \left(10+5\right)^2$ |
| **(ii)** | Assume that the weight of a randomly chosen Clincher Bike rim and tyre are independent of each other.<br>(This is required for the calculation of Var($C$))<br>(Recall Var($X$+$Y$) = Var($X$) + Var($Y$) if $X$ and $Y$ are independent) | Poorly done. Many gave assumptions such as:<br>- The weight of a randomly chosen Clincher Bike Wheelset is independent of another set. (There's only one set in (i))<br>- Assume that a set has only one rim and one wheel (This was already stated in the question) |

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| **(iii)** | $C-T \sim N\left(1610-\mu,\ 125+15^2\right) \Rightarrow C-T \sim N\left(1610-\mu,\ 350\right)$<br>$\mathrm{P}\left(C-T>150\right)<0.70351$<br>$\mathrm{P}\left(Z \geq \dfrac{150-1610+\mu}{\sqrt{350}}\right)<0.70351$<br>$\mathrm{P}\left(Z \geq \dfrac{-1460+\mu}{\sqrt{350}}\right)<0.70351$<br>$\dfrac{-1460+\mu}{\sqrt{350}} \geq -0.534523$<br>$\mu \geq 1449.9999$<br>$\mu \geq 1450$ | Note that GC table is not allowed as $\mu$ is not an integer value.<br>Many students failed to standardise correctly.<br>Many wrote 0.534523 instead of – 0.534523, indicating that they are still not able to identify the correct area for the invnorm command in GC. |
| **(iv)** | Since $n=50$ is large, by Central Limit Theorem,<br>$\overline{M} \sim N\left(1800,\ \dfrac{400}{50}\right)$ approximately<br>$\overline{M} \sim N\left(1800,\ 8\right)$ approximately<br>$P\left(\overline{M}>1795\right)=0.96146 \approx 0.961\,(3\text{sf})$ | Many students wrote<br>$M \sim N\left(1800,\ 20^2\right)$<br>which is incorrect as it is not given in the question that M is normally distributed.<br>Not many students knew that CLT has to be used for this part.<br>Some applied CLT to *M*. |
| Students are able to find the variance and expectation of the random variables but would need to work on problems which require them to standardise (especially those that involve inequality) | | |

**10 (a)** Two random samples of different sample sizes of households in the town of Aimek were taken to find out the mean number of computers per household there. The first sample of 50 households gave the following results.

| Number of computers | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Number of households | 5 | 12 | 18 | 10 | 5 |



The results of the second sample of 60 households were summarised as follows.

$$\sum y = 118 \qquad \sum y^2 = 314,$$

where $y$ is the number of computers in a household.

**(i)** By combining the two samples, find unbiased estimates of the population mean and variance of the number of computers per household in the town. [4]

**(ii)** Describe what you understand by 'population' in the context of this question. [1]

**(b)** Past data has shown that the working hours of teachers in a city are normally distributed with mean 48 hours per week. In a recent study, a large random sample of $n$ teachers in the city was surveyed and the number of working hours per week was recorded. The sample mean was 46 hours and the sample variance was 131.1 hours$^2$. A hypothesis test is carried out to determine whether the mean working hours per week of teachers has been reduced.

**(i)** State appropriate hypotheses for the test. [1]

The calculated value of the test statistic is $z = -1.78133$ correct to 5 decimal places.

**(ii)** Deduce the conclusion of the test at the 2.5 % level of significance. [2]

**(iii)** Find the value of $n$. [3]

**(iv)** In another test, using the same sample, there is significant evidence at the $\alpha\%$ level that there is a change in the mean working hours per week of teachers in that city. Find the smallest possible integral value of $\alpha$. [2]

| | | |
|---|---|---|
| This question is poorly done in general. Students display a lack of understanding of hypothesis testing concepts and procedures, coupled with a perpetual incompetency in obtaining the unbiased estimates from given data. Many of the solutions presented also reveal a lack in comprehension by students on what the questions are asking for. | | |
| **10(a)(i)** | Let $X$ be the number of computers in each household in the first sample.<br>Using GC, $\sum fx = 98, \sum fx^2 = 254$<br>Unbiased estimate of the population mean<br>$= \dfrac{98+118}{50+60} = \dfrac{108}{55} (\approx 1.96)$<br>Unbiased estimate of the population variance<br>$= \dfrac{1}{110-1}\left[(254+314) - \dfrac{(98+118)^2}{110}\right] = \dfrac{7912}{5995} (\approx 1.32)$ | The most common error for the unbiased estimate for the population mean from the combined sample is taking the averages of the unbiased estimates for the respective sample sizes, i.e. $\dfrac{1}{2}\left(\dfrac{98}{50}+\dfrac{118}{60}\right)$.<br><br>The most common error for the unbiased estimate for the population variance is using the formula for pooled variance given in the MF26. |
| **(a)(ii)** | The population in this question refers to all the households in the town. | Majority of students are able to correctly identify what the population is. Common errors include failing to state that the households are from the town (and not anywhere else) and for referring the population to computers instead. A small handful gave the definition of a random sample here, which indicates a lack in comprehension what the question is asking for. |
| **(b)(i)** | Let $\mu$ be the population mean working hours of teachers in the city.<br>$H_0 : \mu = 48$<br>$H_1 : \mu < 48$ | This part is generally well done. |
| **(ii)** | Critical region $= \{z : z \le -1.960\}$<br>Since $z_{cal} = -1.781 > -1.960$, we do not reject $H_0$<br>**OR** $p$-value $= P(Z \le -1.781) = 0.0375 > 0.025$, we do not reject $H_0$<br>There is insufficient evidence at 2.5% level of significance that the mean working hours per week of teachers in the city has been reduced. | Most students know that the finding the critical region is key to stating the correct conclusion, and most are able to obtain the correct critical value (not the region though). This leads to an overall confusion with whether to reject/accept $H_0$, |

| | | |
|---|---|---|
| | | or whether to reject/accept $H_1$. Subsequently, the confusion with whether there is sufficient/insufficient evidence persists as well. There is a sizable number of students who are of the impression that the decision for rejection of $H_0$ is the conclusion. |
| **(iii)** | $s^2 = \frac{n}{n-1}(131.1)$<br>Given z= $-1.781 \Rightarrow \frac{46-48}{\left(\sqrt{\frac{131.1}{n-1}}\right)} = -1.781$<br>$-2 = -1.781\sqrt{\frac{131.1}{n-1}}$<br>$\sqrt{n-1} = \frac{-1.781\sqrt{131.1}}{-2}$<br>$n = 105$ | Most students use 131.1 as the population variance, therefore losing marks for accuracy. |
| **(iv)** | $p$-value for the two tailed test = $2 \times \mathrm{P}(Z \le -1.781) = 0.0749$ (accept 0.075)<br>For $H_0$ to be rejected,<br>$p\text{-value} \le \frac{\alpha}{100}$<br>$\alpha \ge 7.49$<br>Smallest value of $\alpha = 8$ | Students who rightly approach this part using a 2-tailed test are able to obtain the correct result. There is a small handful who mistakenly compared the test statistic $-1.78133$ to the level of significance. |