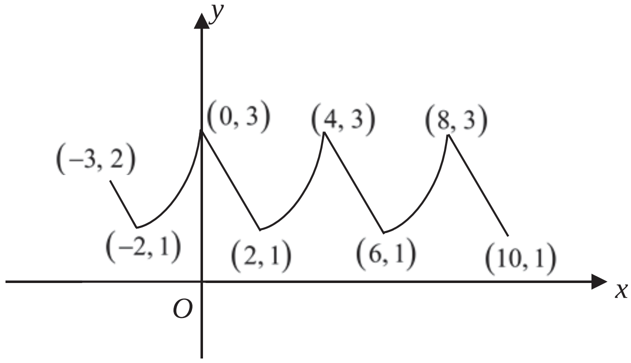
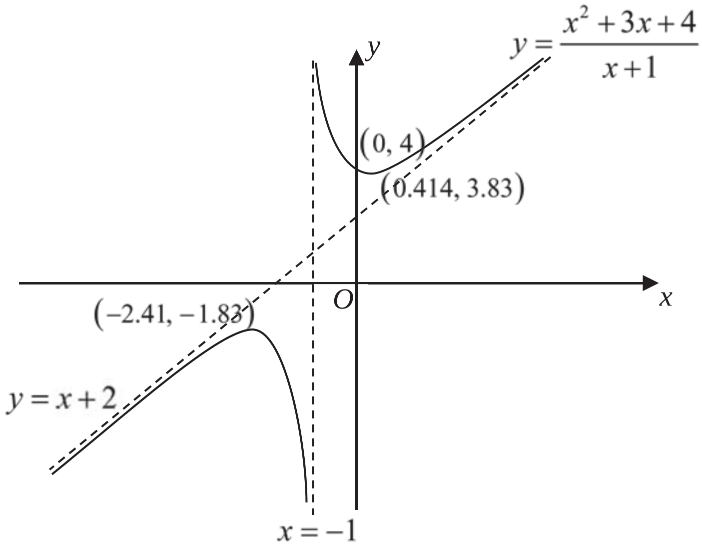
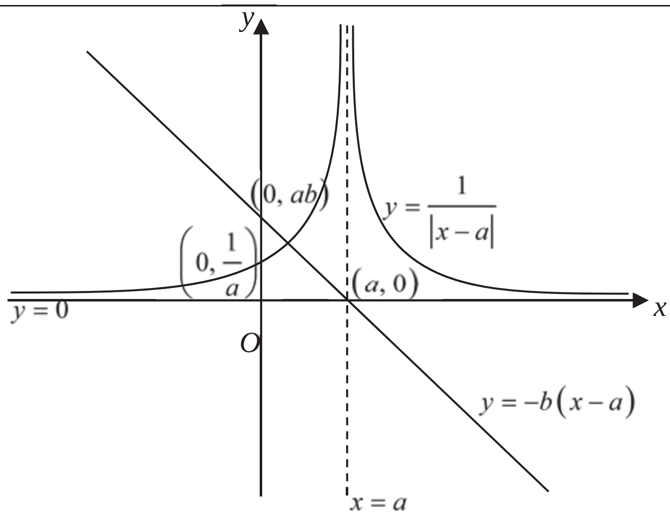
Solution to Paper 9758/01

Qn	Solution
1(i)	$u_n = n^2 + \frac{1}{n!}$ $\text{LHS} = u_n - u_{n-1}$ $= n^2 + \frac{1}{n!} - \left((n-1)^2 + \frac{1}{(n-1)!} \right)$ $= n^2 + \frac{1}{n!} - n^2 + 2n - 1 - \frac{1}{(n-1)!}$ $= 2n - 1 + \frac{1}{n!} - \frac{n}{(n-1)! \times n}$ $= 2n - 1 + \frac{1-n}{n!}$ $= \text{RHS}$
1(ii)	$\sum_{r=2}^{2n} 2r - 1 + \frac{1-r}{r!}$ $= \sum_{r=2}^{2n} u_r - u_{r-1}$ $= \quad u_2 - u_1$ $+ \ u_3 - u_2$ $+ \ u_4 - u_3$ $+ \ \ldots$ $+ \ u_{2n-2} - u_{2n-3}$ $+ \ u_{2n-1} - u_{2n-2}$ $+ \ u_{2n} - u_{2n-1}$ $= u_{2n} - u_1$ $= (2n)^2 + \frac{1}{(2n)!} - (1+1)$ $= 4n^2 - 2 + \frac{1}{(2n)!}$
1(iii)	As $n \to \infty$, $4n^2 - 2 \to \infty$, $\frac{1}{(2n)!} \to 0$. Hence $\sum_{r=2}^{2n} 2r - 1 + \frac{1-r}{r!}$ does not converge. It diverges.

Qn	Solution
2(i)	
2(ii)	$\text{Volume} = \pi\int_0^2 (3-x)^2 \ \mathrm{d}x + \pi\int_3^4 \left(\frac{x^2+2}{6}\right)^2 \ \mathrm{d}x$ $= 45.5$
3(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\cos x^2 = -2x\sin x^2$
3(ii)	$\int x^3 \sin x^2 \ \mathrm{d}x$ $= -\frac{1}{2}\int\left(-2x\sin x^2\right)\left(x^2\right)\ \mathrm{d}x$ $= -\frac{1}{2}\left[x^2\cos x^2 - \int(2x)\left(\cos x^2\right)\ \mathrm{d}x\right]$ $= -\frac{x^2}{2}\cos x^2 + \int x\left(\cos x^2\right)\ \mathrm{d}x$ $= -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\int 2x\left(\cos x^2\right)\ \mathrm{d}x$ $= -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2 + c$ $u = x^2 \quad v' = -2x\sin x^2$ $u' = 2x \quad v = \cos x^2$ Note: $\frac{\mathrm{d}}{\mathrm{d}x}\sin x^2 = 2x\cos x^2$

Qn	Solution
4a(i)	$y=\dfrac{x^2+3x+4}{x+1}$ $=x+2+\dfrac{2}{x+1}$ Vertical asymptote: $x=-1$, Oblique asymptote: $y=x+2$ 
4a(ii)	$x<-2.41$ or $x>0.414$
4(b)	**Method 1a:** Applying transformations backward We are given the transformations in the order of A, B and C, and the resulting curve $y=\dfrac{x^2+3x+4}{x+1}$. In order to find the original curve $y=\mathrm{g}(x)$, we use our resulting curve, apply the C', B' then A', where A', B' and C' are the "opposites" of A, B and C respectively. $$\text{Original Curve} \underset{A'}{\overset{A}{\rightleftarrows}} \spadesuit \underset{B'}{\overset{B}{\rightleftarrows}} \heartsuit \underset{C'}{\overset{C}{\rightleftarrows}} \text{Resulting Curve},$$ where $\spadesuit$ and $\heartsuit$ represent intermediate curves in the entire process. Hence, we have C': Translate 1 unit in negative y-direction (Replace y by $y+1$); B': Translate 1 unit in positive x-direction (Replace x by $x-1$); A': Stretch with scale factor $\dfrac{1}{2}$ parallel to x-axis (Replace x by $2x$).

Qn	Solution
	$y=\dfrac{x^2+3x+4}{x+1} \xrightarrow[C']{\text{Replace } y \text{ by } y+1} y+1=\dfrac{x^2+3x+4}{x+1}$ (i.e. $y=\dfrac{x^2+2x+3}{x+1}$) $y=\dfrac{x^2+2x+3}{x+1} \xrightarrow[B']{\text{Replace } x \text{ by } x-1} y=\dfrac{(x-1)^2+2(x-1)+3}{(x-1)+1}$ $=\dfrac{x^2-2x+1+2x-2+3}{x}$ $=\dfrac{x^2+2}{x}$ $=x+\dfrac{2}{x}$ $y=x+\dfrac{2}{x} \xrightarrow[A']{\text{Replace } x \text{ by } 2x} y=2x+\dfrac{2}{2x}$ $=2x+\dfrac{1}{x}=\mathrm{g}(x)$ **Method 1b:** Applying transformations backward **Note that:** Hence, we have C': Translate 1 unit in negative *y*-direction (Replace *y* by $y+1$); B': Translate 1 unit in positive *x*-direction (Replace *x* by $x-1$); A': Stretch with scale factor $\dfrac{1}{2}$ parallel to *x*-axis (Replace *x* by $2x$). Note that $y=\dfrac{x^2+3x+4}{x+1}=x+2+\dfrac{2}{x+1}$ $y=x+2+\dfrac{2}{x+1} \xrightarrow[C']{\text{Replace } y \text{ by } y+1} y+1=x+2+\dfrac{2}{x+1}$ (i.e. $y=x+1+\dfrac{2}{x+1}$) $y=x+1+\dfrac{2}{x+1} \xrightarrow[B']{\text{Replace } x \text{ by } x-1} y=(x-1)+1+\dfrac{2}{(x-1)+1}$ $=x+\dfrac{2}{x}$ $y=x+\dfrac{2}{x} \xrightarrow[A']{\text{Replace } x \text{ by } 2x} y=2x+\dfrac{2}{2x}$ $=2x+\dfrac{1}{x}=\mathrm{g}(x)$ • **Method 2:** Start from the original curve $y=\mathrm{g}(x)$ Original Curve $y=\mathrm{g}(x) \xrightarrow{A} \spadesuit \xrightarrow{B} \heartsuit \xrightarrow{C}$ Resulting Curve $y=\dfrac{x^2+3x+4}{x+1}$

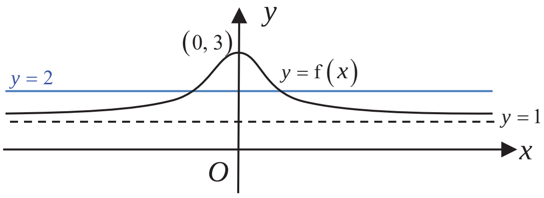
Qn	Solution
	A: Stretch with scale factor 2 parallel to x-axis (Replace x by $\frac{1}{2}x$); B: Translate 1 unit in negative x-direction (Replace x by $x+1$); C: Translate 1 unit in positive y-direction (Replace y by $y-1$). $y=\mathrm{g}(x)\xrightarrow[\text{by } \frac{1}{2}x]{\text{Replace } x} y=\mathrm{g}\left(\frac{x}{2}\right)\xrightarrow[\text{by } x+1]{\text{Replace } x} y=\mathrm{g}\left(\frac{x+1}{2}\right)\xrightarrow[\text{by } y\text{-}1]{\text{Replace } y} y-1=\mathrm{g}\left(\frac{x+1}{2}\right)$ $y=\mathrm{g}\left(\frac{x+1}{2}\right)+1$ Let $\mathrm{g}\left(\frac{x+1}{2}\right)+1=\frac{x^2+3x+4}{x+1}$ $\mathrm{g}\left(\frac{x+1}{2}\right)+1=x+2+\frac{2}{x+1}$ $\mathrm{g}\left(\frac{x+1}{2}\right)=x+1+\frac{2}{x+1}$ Note that we need to progress till we reach $\mathrm{g}(x)$. To do that, we need to replace x by $2x-1$ in $\mathrm{g}\left(\frac{x+1}{2}\right)=x+1+\frac{2}{x+1}$. $\mathrm{g}\left(\frac{(2x-1)+1}{2}\right)=(2x-1)+1+\frac{2}{(2x-1)+1}$ $\therefore \mathrm{g}(x)=2x+\frac{1}{x}$
5(i)	
5(ii)	**Method 1**: From graph in **(i)**, one of the solution to $\frac{1}{\lvert x-a\rvert}>-b(x-a)$ is $x>a$. By observation, since $x>1$, $a=1$

Qn	Solution
	When $x=\frac{1}{2}, y=\frac{1}{\left\|\frac{1}{2}-1\right\|}=2$ Subst $x=\frac{1}{2}$ and $y=2$ into $y=-b(x-1)$ $2=-b\left(\frac{1}{2}-1\right)$ $b=4$ ----- **<u>Method 2</u>**: From graph in **(i)**, one of the solution to $\frac{1}{\|x-a\|}>-b(x-a)$ is $x>a$. By observation, since $x>1$, $a=1$ To find the *x*-coordinate of the point of intersection: $\frac{1}{-(x-1)}=-b(x-1)$ $(x-1)^2=\frac{1}{b}$ $x-1=\pm\frac{1}{\sqrt{b}}$ When $x=\frac{1}{2}$, $\frac{1}{2}-1=\pm\frac{1}{\sqrt{b}}$ $-\frac{1}{2}=-\frac{1}{\sqrt{b}}$ or $-\frac{1}{2}=\frac{1}{\sqrt{b}}$ (rej. since $\sqrt{b}>0$) $\sqrt{b}=2$ $b=4$
5(iii)	$x>1$
6(i)	**<u>Method 1</u>**: $\mathrm{f}(x)=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ $\mathrm{f}'(x)=a\cos\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ $\mathrm{f}''(x)=a^2\cos^2\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}+a\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}\left[-a\sin\left(ax+\frac{\pi}{2}\right)\right]$ $\mathrm{f}''(x)=a^2\cos^2\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}-a^2\sin\left(ax+\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ ----

Qn	Solution
	Method 2: Let $y=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}$ $\Rightarrow \ln y=\sin\left(ax+\frac{\pi}{2}\right)$ Differentiate w.r.t. x: $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=a\cos\left(ax+\frac{\pi}{2}\right)$ Differentiate w.r.t. x: $\frac{1}{y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(-\frac{1}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-a^2\sin\left(ax+\frac{\pi}{2}\right)$ --- **Method 3:** $\mathrm{f}(x)=\mathrm{e}^{\sin\left(ax+\frac{\pi}{2}\right)}=\mathrm{e}^{\sin ax\cos\frac{\pi}{2}+\cos ax\sin\frac{\pi}{2}}=\mathrm{e}^{\cos ax}$ $\mathrm{f}'(x)=-(\sin ax)\mathrm{e}^{\cos ax}$ $\mathrm{f}''(x)=\mathrm{e}^{\cos ax}(-a\cos ax)+(\sin^2 ax)\mathrm{e}^{\cos ax}$ --- Hence $\mathrm{f}(0)=\mathrm{e}^{\sin\left(\frac{\pi}{2}\right)}=\mathrm{e}$ $\mathrm{f}'(0)=0$ $\mathrm{f}''(0)=-a^2\mathrm{e}$ By Maclaurin Series, $\mathrm{f}(x)=\mathrm{f}(0)+x\,\mathrm{f}'(0)+\frac{x^2}{2!}\mathrm{f}''(0)+\ldots$ $\mathrm{f}(x)=\mathrm{e}-\frac{a^2\mathrm{e}}{2}x^2+\ldots$ Or it can be presented as $\mathrm{f}(x)\approx\mathrm{e}-\frac{a^2\mathrm{e}}{2}x^2$
6(ii)	$\frac{1}{\sqrt{b+x^2}}=\left(b+x^2\right)^{-\frac{1}{2}}$ $=b^{-\frac{1}{2}}\left(1+\frac{x^2}{b}\right)^{-\frac{1}{2}}$ $=\frac{1}{\sqrt{b}}\left(1+\left(-\frac{1}{2}\right)\left(\frac{x^2}{b}\right)+\ldots\right)$ $=\frac{1}{\sqrt{b}}-\frac{x^2}{2b\sqrt{b}}+\ldots$

Qn	Solution
	Comparing the constant term, $\frac{1}{\sqrt{b}}=\mathrm{e} \quad \Rightarrow \quad b=\frac{1}{\mathrm{e}^2}$ Comparing coefficient of x^2, $-\frac{1}{2b\sqrt{b}}=\frac{-a^2\mathrm{e}}{2}$ $a^2\mathrm{e}=\frac{1}{\left(\frac{1}{\mathrm{e}^2}\right)\left(\frac{1}{\mathrm{e}}\right)}$ $a^2=\mathrm{e}^2$ $a=\pm\mathrm{e}$
7(a)	$\int_0^1 \frac{x}{2-x^2}\,\mathrm{d}x=-\frac{1}{2}\int_0^1 \frac{-2x}{2-x^2}\,\mathrm{d}x$ $=-\frac{1}{2}\left[\ln\left\|2-x^2\right\|\right]_0^1$ $=-\frac{1}{2}(\ln 1-\ln 2)$ $=-\frac{1}{2}\ln\frac{1}{2} \quad \text{or} \quad \frac{1}{2}\ln 2$
7(b)i)	$x^2=A(2-x)^2+B(2-x)+C$ $=Ax^2-4Ax+4A+x^2+2B-Bx+C$ $=Ax^2+(-4A-B)x+4A+2B+C$ By comparing coefficients, $A=1$, $-4(1)-B=0 \Rightarrow B=-4$, $4(1)+2(-4)+C=0 \Rightarrow C=4$
7(b)ii)	$\int_0^1 \frac{x^2}{(2-x)^2}\,\mathrm{d}x=\int_0^1 1-\frac{4}{2-x}+\frac{4}{(2-x)^2}\,\mathrm{d}x$ $=\left[x+4\ln\left\|2-x\right\|+\frac{4}{2-x}\right]_0^1$ $=1+4\ln(1)+\frac{4}{2-1}-4\ln 2-2$ $=3-4\ln 2$ $p=3,\ q=-4$

Qn	Solution
8(a)i)	$\overrightarrow{OQ}=\begin{pmatrix}2\\-\frac{3}{2}\\-\frac{1}{2}\end{pmatrix}$ Acute angle $=\cos^{-1}\dfrac{\left\lvert\begin{pmatrix}2\\-1.5\\-0.5\end{pmatrix}\bullet\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rvert}{\sqrt{6.5}\sqrt{1}}$ $=\cos^{-1}\left(\dfrac{1.5}{\sqrt{6.5}}\right)$ $=54.0^{\circ}$ (or 0.942 rad)
8(a)ii)	xy-plane: $\mathbf{r}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ Since $\mathbf{m}$ is perpendicular to xy-plane, $\mathbf{m}\parallel\begin{pmatrix}0\\0\\1\end{pmatrix}$. Since $\lvert\mathbf{m}\rvert=1$, $\mathbf{m}=\begin{pmatrix}0\\0\\1\end{pmatrix}$. $\lvert\mathbf{q}\bullet\mathbf{m}\rvert$ refers to the perpendicular distance from Q to xy-plane. $\lvert\mathbf{q}\bullet\mathbf{m}\rvert=\left\lvert\begin{pmatrix}2\\-\frac{3}{2}\\-\frac{1}{2}\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert=\frac{1}{2}$
8 (b)	$\mathbf{c}\times\mathbf{a}=\lambda\mathbf{b}\times\mathbf{c}$ $\mathbf{c}\times\mathbf{a}-\lambda\mathbf{b}\times\mathbf{c}=\mathbf{0}$ $\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\lambda\mathbf{b}=\mathbf{0}$ $\mathbf{c}\times(\mathbf{a}+\lambda\mathbf{b})=\mathbf{0}$ $\mathbf{c}\times\mathbf{r}=\mathbf{0}$ $\therefore\ \mathbf{r}\parallel\mathbf{c}$ (shown) Since $\mathbf{c}\parallel\mathbf{r}$, $\mu\mathbf{c}=\mathbf{r}$.

Qn	Solution
	$\mathbf{r} \bullet \mathbf{r} = \mu\mathbf{c} \bullet \mu\mathbf{c}$ $\mathbf{r} \bullet \mathbf{r} = \mu^2 \lvert\mathbf{c}\rvert^2$ Since a is a unit vector, $\lvert\mathbf{a}\rvert = 1$. $\mathbf{r} \bullet \mathbf{r} = (\mathbf{a} + \lambda\mathbf{b}) \bullet (\mathbf{a} + \lambda\mathbf{b})$ $= \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \lambda\mathbf{b} + \lambda\mathbf{b} \bullet \mathbf{a} + \lambda\mathbf{b} \bullet \lambda\mathbf{b}$ $= \lvert\mathbf{a}\rvert^2 + 2\lambda(\mathbf{a} \bullet \mathbf{b}) + \lambda^2 \lvert\mathbf{b}\rvert^2$ $= 1 + 4\lambda^2$ $\mu^2 \lvert\mathbf{c}\rvert^2 = 1 + 4\lambda^2$ $\lvert\mathbf{c}\rvert^2 = \frac{1}{\mu^2}(1 + 4\lambda^2)$ $\lvert\mathbf{c}\rvert = \sqrt{\frac{1}{\mu^2}(1 + 4\lambda^2)}$ (rej. $-\sqrt{\frac{1}{\mu^2}(1 + 4\lambda^2)}$ since $\lvert\mathbf{c}\rvert > 0$) $\lvert\mathbf{c}\rvert = \sqrt{\frac{1}{\mu^2}}\left(\sqrt{(1 + 4\lambda^2)}\right)$ $\lvert\mathbf{c}\rvert = k\sqrt{(1 + 4\lambda^2)}$ where $k = \sqrt{\frac{1}{\mu^2}}$ (or $k = \frac{1}{\lvert\mu\rvert}$)
9(i)	**<u>Method 1: Horizontal Line Test</u>** $\mathrm{f} : x \mapsto 1 + 2\mathrm{e}^{-x^2}$, $x \in \mathbb{R}$  Since there exists **a/the horizontal line** $y = 2$ that intersects the graph of $y = \mathrm{f}(x)$ **more than once**, f is **not one-one**, f does not have an inverse. **<u>Method 2: Counterexample</u>** Since $\mathrm{f}(-1) = \mathrm{f}(1) = 1 + \frac{2}{\mathrm{e}}$, f is **not one-one**, f does not have an inverse.
9(ii)	Largest value of k is 0.
9(iii)	Let $y = 1 + 2\mathrm{e}^{-x^2}$ $y - 1 = 2\mathrm{e}^{-x^2}$ $\mathrm{e}^{-x^2} = \frac{y-1}{2}$ $-x^2 = \ln\left(\frac{y-1}{2}\right)$ $x^2 = -\ln\left(\frac{y-1}{2}\right)$ $x = \pm\sqrt{-\ln\left(\frac{y-1}{2}\right)}$

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	Since $x \le 0$ (restricted domain of f), $x = -\sqrt{-\ln\left(\frac{y-1}{2}\right)}$ $\mathrm{f}^{-1}(x) = -\sqrt{-\ln\left(\frac{x-1}{2}\right)}$ OR $-\sqrt{\ln\left(\frac{2}{x-1}\right)}$
9(iv)	Range of $\mathrm{f} = (1,\ 3]$ Domain of g = Range of $\mathrm{g}^{-1} = (1,\ 3]$ Since Range of f $\subseteq$ Domain of g, gf exists.
9(v)	Domain of gf = Domain of f = $(-\infty,\ 0]$ Range of gf = $(-\infty,\ 0]$
9(vi)	Given that $\mathrm{gf}(x) = x$, $\mathrm{gf}(-2) = -2$ $\mathrm{g}^{-1}\mathrm{gf}(-2) = \mathrm{g}^{-1}(-2)$ $\mathrm{g}^{-1}(-2) = \mathrm{f}(-2)$ $= 1 + 2\mathrm{e}^{-4}$
10(i) a)	**<u>Method 1:</u>** $\frac{\mathrm{d}m}{\mathrm{d}t} = c$ where c is a <u>negative</u> constant because the mass of the raindrop is decreasing with time due to evaporation. **<u>Method 2:</u>** $\frac{\mathrm{d}m}{\mathrm{d}t} = -c$ where c is a <u>positive</u> constant because the mass of the raindrop is decreasing with time due to evaporation.
10(i) b)	$\frac{\mathrm{d}m}{\mathrm{d}t} = c$ $m = ct + D$ When $t = 0$, $m = 0.05$, $0.05 = c(0) + D \Rightarrow D = 0.05$ When $t = 60$, $m = 0.004$, $0.004 = c(60) + 0.05$ $c = -0.00076667$ (5s.f.) $c = -0.000767$ (3s.f.) $\therefore m = -0.000767t + 0.05$
10(ii) a)	$K = \frac{mg^2t^2}{2}$ From (i)(b), $m = -0.00076667t + 0.05$,

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	$K=\dfrac{\left(-0.00076667t+0.05\right)g^2t^2}{2}$ $=-0.00038333g^2t^3+0.025g^2t^2$ At stationary points, $\dfrac{\mathrm{d}K}{\mathrm{d}t}=-0.00115g^2t^2+0.05g^2t=0$ $t\left(-0.00115g^2t+0.05g^2\right)=0$ $t=0$ (rejected since $K=0$ when $t=0$) or $t=43.478$ $\dfrac{\mathrm{d}^2K}{\mathrm{d}t^2}=-0.0023g^2t+0.05g^2$ When $t=43.478$, $\dfrac{\mathrm{d}^2K}{\mathrm{d}t^2}=-0.0500g^2$ <0 (since $g^2>0$) $\therefore t=43.478$ gives maximum K. When $t=43.478$, $K=15.75g^2$, where $p=15.75$ (2d.p.)
10(ii) b)	$K=-0.00038333\left(10\right)^2t^3+0.025\left(10\right)^2t^2$ $=-0.038333t^3+2.5t^2$ At surface of ground, $1=-0.038333t^3+2.5t^2$ Using GC, $t=-0.629,\ 0.636$ or 65.2 Since $t>60$, $\therefore t=65.2$ (3 s.f.) Required time taken = 65.2 s. (3 s.f.)
11(i)	Let A be the point where it departs from the ground. $\mathbf{r}=\overrightarrow{OA}+t\begin{pmatrix}-2\\1\\1\end{pmatrix}$ $\begin{pmatrix}-9\\4\\6\end{pmatrix}=\overrightarrow{OA}+6\begin{pmatrix}-2\\1\\1\end{pmatrix}$ $\overrightarrow{OA}=\begin{pmatrix}-9\\4\\6\end{pmatrix}-6\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}3\\-2\\0\end{pmatrix}$ Coordinates: $A\left(3,\ -2,\ 0\right)$

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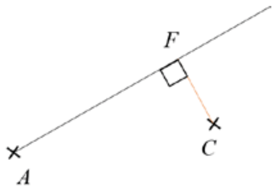

Shortest distance from C to the flight path (line) refers to the perpendicular distance from C to the flight path.

<u>Method 1 (via foot of perpendicular)</u>

Let F be the foot of perpendicular from C to the flight path

$$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad \text{for } t \in \mathbb{R},\ t \geq 0$$

$$\overrightarrow{OF} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad \text{for some } t \in \mathbb{R},\ t \geq 0$$

$$\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC} = \begin{pmatrix} 3-2t \\ -2+t \\ t \end{pmatrix} - \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 10-2t \\ -7+t \\ t-2 \end{pmatrix}$$

Since $\overrightarrow{CF} \perp$ flight path, $\begin{pmatrix} 10-2t \\ -7+t \\ t-2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 0$

$$-20+4t-7+t+t-2=0$$

$$t = \frac{29}{6} \quad (\text{or } 4.8333)$$

$$\overrightarrow{CF} = \begin{pmatrix} 10-2\left(\frac{29}{6}\right) \\ -7+\left(\frac{29}{6}\right) \\ \left(\frac{29}{6}\right)-2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{13}{6} \\ \frac{17}{6} \end{pmatrix}$$

$$\left|\overrightarrow{CF}\right| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{13}{6}\right)^2 + \left(\frac{17}{6}\right)^2} = \sqrt{\frac{77}{6}}$$

$$\left|\overrightarrow{CF}\right| = 3.58236 = 3.58 \text{ m} \ \ (3 \text{ sf})$$

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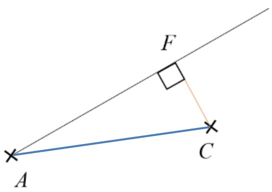
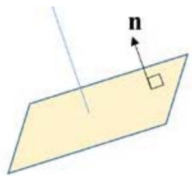
Qn	Solution
	Method 2 (Vector product) $\overrightarrow{OC}=\begin{pmatrix}-7\\5\\2\end{pmatrix}$ $\overrightarrow{AC}=\begin{pmatrix}-7\\5\\2\end{pmatrix}-\begin{pmatrix}3\\-2\\0\end{pmatrix}=\begin{pmatrix}-10\\7\\2\end{pmatrix}$ $\text{Perpendicular distance}=\dfrac{\left\lvert\overrightarrow{AC}\times\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert}$ $=\dfrac{\left\lvert\begin{pmatrix}-10\\7\\2\end{pmatrix}\times\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\rvert}{\sqrt{4+1+1}}$ $=\dfrac{\left\lvert\begin{pmatrix}5\\6\\4\end{pmatrix}\right\rvert}{\sqrt{6}}$ $=\dfrac{\sqrt{77}}{\sqrt{6}}$ $=3.58236=3.58\text{ m}\ \ (3\text{ sf})$ **Method 3 (Pythagoras Theorem)** $\overrightarrow{OC}=\begin{pmatrix}-7\\5\\2\end{pmatrix}$ $\overrightarrow{AC}=\begin{pmatrix}-7\\5\\2\end{pmatrix}-\begin{pmatrix}3\\-2\\0\end{pmatrix}=\begin{pmatrix}-10\\7\\2\end{pmatrix}$ 

Qn	Solution
	$\left\|\overrightarrow{AF}\right\| = \dfrac{\left\|\overrightarrow{AC} \bullet \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\|}{\left\|\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\|}$ $= \dfrac{\left\|\begin{pmatrix} -10 \\ 7 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\|}{\sqrt{4+1+1}}$ $= \dfrac{29}{\sqrt{6}}$ Perpendicular distance $= \sqrt{\left\|\overrightarrow{AC}\right\|^2 - \left\|\overrightarrow{AF}\right\|^2}$ $= \sqrt{\left((-10)^2 + 7^2 + 2^2\right) - \left(\dfrac{29}{\sqrt{6}}\right)^2}$ $= 3.58236 = 3.58 \text{ m} \ \ (3 \text{ sf})$
11(iii)	Let $\lambda = \dfrac{2x-1}{-6} = \dfrac{y+7}{4} = \dfrac{z-10}{k}$ $\dfrac{2x-1}{-6} = \lambda \ \Rightarrow \ x = \dfrac{1}{2} - 3\lambda$ $\dfrac{y+7}{4} = \lambda \ \Rightarrow \ y = -7 + 4\lambda$ $\dfrac{z-10}{k} = \lambda \ \Rightarrow \ z = 10 + k\lambda$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ k \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} \frac{1}{2} \\ -7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ k \end{pmatrix}$, where $\lambda \in \mathbb{R}, \lambda \geq 0$
11(iv)	Given that both flight paths (lines) intersect,

Qn	Solution
	$\begin{pmatrix} \frac{1}{2} \\ -7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ k \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{2} - 3\lambda \\ -7 + 4\lambda \\ 10 + k\lambda \end{pmatrix} = \begin{pmatrix} 3 - 2t \\ -2 + t \\ t \end{pmatrix}$ $\frac{1}{2} - 3\lambda = 3 - 2t \quad \Rightarrow \quad -3\lambda + 2t = \frac{5}{2} \quad$ ---(1) $-7 + 4\lambda = -2 + t \quad \Rightarrow \quad 4\lambda - t = 5 \quad$ ---(2) $10 + k\lambda = t$ Using GC to solve simultaneously, $\lambda = 2.5, \quad t = 5$ $10 + k(2.5) = 5$ $k = -2$
11(v)	Equation of line: $\mathbf{r} = \begin{pmatrix} -9 \\ 4 \\ 6 \end{pmatrix} + s \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ Equation of plane: $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 2$  If the new flight path is perpendicular to the slope, it will be parallel to the normal vector of the slope. **Method 1:** If **a** and **b** are parallel, then $\mathbf{a} = k\,\mathbf{b}$ for all $k \in \mathbb{R}$. Since $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$ for all $k \in \mathbb{R}$, $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$. Thus the new flight path and the inclined slope are not perpendicular to each other.

Qn	Solution
	Method 2: If **a** and **b** are parallel, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. **Note:** **0** means a zero vector, not a constant 0. If the new flight path (line) is perpendicular to the slope (plane), then this means the direction vector of the line and the normal vector of the plane is parallel. This is to then show that the vector (cross) product of the direction vector of the line and the normal vector of the plane is **0**. $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 13 \\ -22 \\ -5 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. This shows that $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$. Thus the new flight path and the inclined slope are not perpendicular to each other. **Method 3:** If **a** and **b** are parallel, then the angle between **a** and **b** is 0°. If the new flight path (line) is perpendicular to the slope (plane), then this means the direction vector of the line and the normal vector of the plane is parallel. This is to then show that the angle between the line and the plane is 0°. $\sin\theta = \dfrac{\left\| \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} \right\|}{\sqrt{14}\sqrt{51}} \Rightarrow \theta = 13.0^\circ \neq 0^\circ$ This shows that $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$. Thus the new flight path and the inclined slope are not perpendicular to each other.
11(vi)	Equation of line: $\mathbf{r} = \begin{pmatrix} -9 \\ 4 \\ 6 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ Equation of plane: $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 2$

Qn	Solution
	$\begin{pmatrix} -9+3s \\ 4+2s \\ 6-s \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 2$ $-9+3s-4-2s+42-7s=2$ $6s=27$ $s=4.5$ $\mathbf{r} = \begin{pmatrix} -9 \\ 4 \\ 6 \end{pmatrix} + 4.5\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ 13 \\ \frac{3}{2} \end{pmatrix}$ Coordinates: $\left(\frac{9}{2},\ 13,\ \frac{3}{2}\right)$

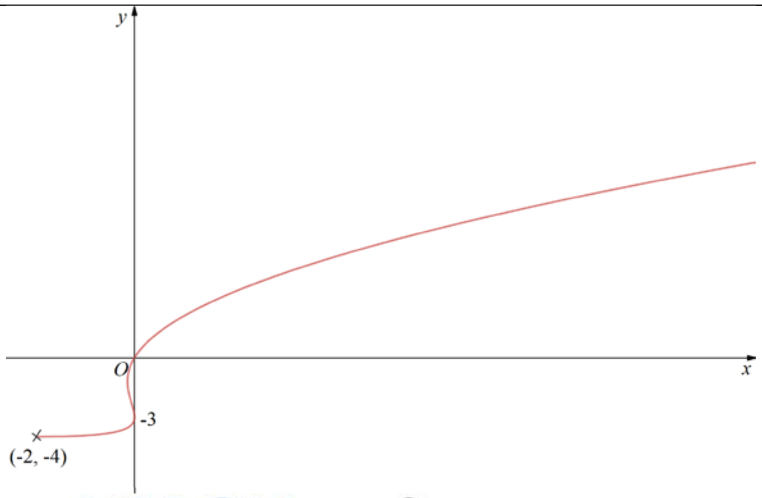
Solution to Paper 9758/02

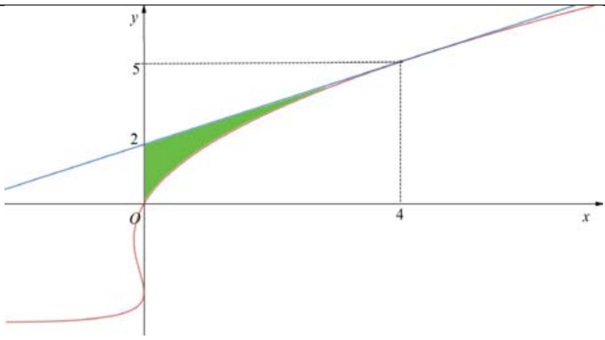
Section A: Pure Mathematics

Qn	Solution
1	$p_n = 3^{n-1} + a$ **Method 1** $\sum_{r=3}^{n} p_r = \sum_{r=3}^{n}\left(3^{r-1}+a\right) = \sum_{r=3}^{n}\left(3^{r-1}\right) + \sum_{r=3}^{n} a$ $= \left[3^2 + 3^3 + 3^4 + \ldots + 3^{n-1}\right] + (n-3+1)a$ $= \dfrac{3^2\left(3^{n-2}-1\right)}{3-1} + (n-3+1)a$ $= \dfrac{9}{2}\left(3^{n-2}-1\right) + (n-2)a$ **Method 2** $\sum_{r=3}^{n} p_r = \sum_{r=3}^{n}\left(3^{r-1}+a\right)$ $= \dfrac{1}{3}\sum_{r=3}^{n}\left(3^{r}\right) + \sum_{r=3}^{n} a$ $= \dfrac{1}{3}\left(3^3 + 3^4 + 3^5 + \ldots + 3^n\right) + (n-3+1)a$ $= \dfrac{1}{3}\left(\dfrac{3^3\left(3^{n-2}-1\right)}{3-1}\right) + (n-3+1)a$ $= \dfrac{9}{2}\left(3^{n-2}-1\right) + (n-2)a$

Qn	Solution
2(i)	Let T_n be the length of the nth plank Given that it is an AP: Let $a = 4$, $S_3 = 11.46$. Let d be the common difference. $S_3 = 11.46 = \frac{3}{2}\left(2(4)+2d\right)$ $d = -0.18$ $T_{18} = 4 + 17(-0.18)$ $\quad = 0.94$ Hence the length of the 18th plank is 0.94 m.
(ii)	The length of the remaining **7** planks follows a GP: first term of GP: $T_{19} = 0.94\left(\frac{5}{4}\right)$ and $r = \frac{5}{4}$. Length of last plank $= T_{25}$ $= ar^{n-1}$ $= T_{19}\left(\frac{5}{4}\right)^{7-1}$ $= 0.94\left(\frac{5}{4}\right)\left(\frac{5}{4}\right)^6$ $= 0.94\left(\frac{5}{4}\right)^7$ $= 4.4822$ $= 4.48$ m (3 sf)
(iii)	**<u>Method 1</u>** Total length of blue planks $= \underbrace{T_2 + T_4 + T_6 + \ldots + T_{18}}_{\text{Sum of AP with } a=T_2,\ d=-0.36,\ l=T_{18}} + \underbrace{T_{20} + T_{22} + T_{24}}_{\text{Sum of GP with } a=T_{20},\ r=\left(\frac{5}{4}\right)^2}$ $= \frac{9}{2}\left[(4-0.18)+0.94\right] + 0.94\left(\frac{5}{4}\right)^2\left(\frac{\left(\left(\frac{5}{4}\right)^2\right)^3 - 1}{\left(\frac{5}{4}\right)^2 - 1}\right)$ $= 21.42 + 7.34948$ $= 28.8$ m (3 sf) **<u>Method 2</u>** Total length of blue planks

Qn	Solution
	$=\frac{9}{2}\left[2(4-0.18)+(9-1)(2(-0.18))\right]$ $+0.94\left(\frac{5}{4}\right)^2\left(\frac{\left(\left(\frac{5}{4}\right)^2\right)^3-1}{\left(\frac{5}{4}\right)^2-1}\right)$ $=21.42+7.34948$ $=28.8$ m (3 sf)
3(a)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^{-5x+3}+\sin x$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\int\left(\mathrm{e}^{-5x+3}+\sin x\right)\,\mathrm{d}x=-\frac{1}{5}\mathrm{e}^{-5x+3}-\cos x+c$ $y=\int\left(-\frac{1}{5}\mathrm{e}^{-5x+3}-\cos x+c\right)\mathrm{d}x$ $y=\frac{1}{25}\mathrm{e}^{-5x+3}-\sin x+cx+d$ where $c,\ d$ are arbitrary constants.
3(b)	$z=x+\frac{\mathrm{d}y}{\mathrm{d}x}\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=1+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}-x+1=0\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+1=\left(\frac{\mathrm{d}y}{\mathrm{d}x}+x\right)$ Thus, replacing accordingly, $\frac{\mathrm{d}z}{\mathrm{d}x}=z$ (Shown) $\frac{\mathrm{d}z}{\mathrm{d}x}=z$ $\Rightarrow\int\frac{1}{z}\,\mathrm{d}z=\int 1\,\mathrm{d}x$ $\ln\lvert z\rvert=x+c$ **Method 1:** $\lvert z\rvert=\mathrm{e}^{x+c}$ $z=\pm\mathrm{e}^{x+c}$ $z=A\mathrm{e}^{x}$ where $A=\pm\mathrm{e}^{c}$ Since $z=x+\frac{\mathrm{d}y}{\mathrm{d}x}$, when $x=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=1\Rightarrow z=1$ $z=A\mathrm{e}^{x}\Rightarrow 1=A\mathrm{e}^{0}\Rightarrow A=1$ $z=\mathrm{e}^{x}$

Qn	Solution
	Thus, $z = e^x \Rightarrow x + \frac{dy}{dx} = e^x$ $\frac{dy}{dx} = e^x - x$ $\Rightarrow y = e^x - \frac{1}{2}x^2 + d$ When $x = 0$, $y = 1 \Rightarrow 1 = 1 + d \Rightarrow d = 0$ Hence, $y = e^x - \frac{1}{2}x^2$ **<u>Method 2:</u>** $\ln\lvert z\rvert = x + c$ $\Rightarrow \ln 1 = 0 + c \Rightarrow c = 0$ Thus, $z = \pm e^x \Rightarrow x + \frac{dy}{dx} = \pm e^x$ $\frac{dy}{dx} = e^x - x$ or $\frac{dy}{dx} = -e^x - x$ (rej since $\left.\frac{dy}{dx}\right\rvert_{x=0} \neq 1$) $\Rightarrow y = e^x - \frac{1}{2}x^2 + d$ When $x = 0$, $y = 1$ $\Rightarrow 1 = 1 + d \Rightarrow d = 0$ Hence, $y = e^x - \frac{1}{2}x^2$
4(i)	
(ii)	$\frac{dx}{dt} = 3t^2 - 2t$, $\frac{dy}{dt} = 2t + 2$ $\frac{dy}{dx} = \frac{2t+2}{3t^2 - 2t}$ At $t = 2$, $x = 4$, $y = 5$, $\frac{dy}{dx} = \frac{3}{4}$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td>Equation of tangent:
$y-5=\frac{3}{4}(x-4)$
$y=\frac{3}{4}x+2$</td></tr>
<tr><td>(iii)</td><td>

<u>Method 1: Integrate wrt y</u>
When $y=0 \Rightarrow t^2+2t-3=0$
$\Rightarrow t=-3$ (rej) or $t=1$
When $y=5 \Rightarrow t=2$ (from part (ii))
$\text{Area}=\int_0^5 x\,\mathrm{d}y-\frac{1}{2}(4)(5-2)$
$=\int_1^2 (t^3-t^2)(2t+2)\,\mathrm{d}t-6$
$=\int_1^2 2t^4-2t^2\,\mathrm{d}t-6$
$=\left[\frac{2t^5}{5}-\frac{2t^3}{3}\right]_1^2-6$
$=\frac{26}{15}$

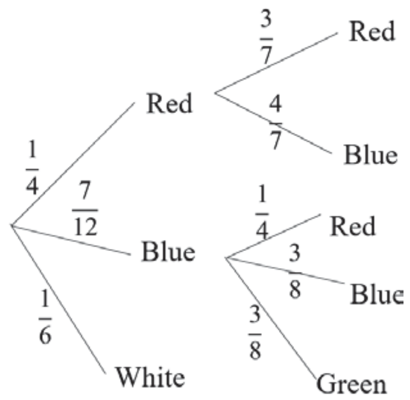
<u>Method 2: Integrate wrt x</u>
When $x=0 \Rightarrow t^3-t^2=0$
$\Rightarrow t=0$ or $t=1$
$\Rightarrow y=-3$ (not this point) or $y=0$
When $x=4 \Rightarrow t=2$ (from part (ii))
$\text{Area}=\frac{1}{2}(2+5)(4)-\int_0^4 y\,\mathrm{d}x$
$=14-\int_1^2 (t^2+2t-3)(3t^2-2t)\,\mathrm{d}t$
$=14-\int_1^2 3t^4+4t^3-13t^2+6t\,\mathrm{d}t$
$=14-\left[\frac{3t^5}{5}+\frac{4t^4}{4}-\frac{13t^3}{3}+\frac{6t^2}{2}\right]_1^2$
$=\frac{26}{15}$</td></tr>
</table>

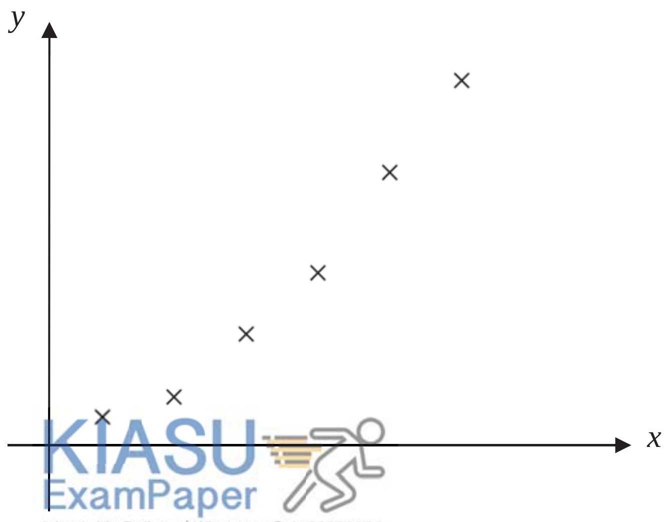
Qn	Solution
5(a) (i)	**Method 1** $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ Since bi is a root, $(b\text{i})^4 - 2(b\text{i})^3 + a(b\text{i})^2 - 8(b\text{i}) + 40 = 0$ $b^4 + 2b^3\text{i} - ab^2 - 8b\text{i} + 40 = 0$ By comparing real and imaginary parts, $2b^3 - 8b = 0$ $2b(b^2 - 4) = 0$ Since $b \neq 0$, $b = -2$ (rej) or $b = 2$ When $b = 2$, $b^4 - ab^2 + 40 = 0$ $16 - 4a + 40 = 0$ $a = 14$ $z^4 - 2z^3 + 14z^2 - 8z + 40 = 0$ Using GC, Other roots are $z = 1 - 3\text{i},\ 1 + 3\text{i},\ -2\text{i}$. **Method 2** $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ Since all coefficients are real, $-b$i is also a root. Quadratic factor: $(z - b\text{i})(z + b\text{i}) = z^2 + b^2$ $z^4 - 2z^3 + az^2 - 8z + 40 = (z^2 + b^2)(z^2 + cz + d)$ $z^4 - 2z^3 + az^2 - 8z + 40 = z^4 + cz^3 + (d + b^2)z^2 + cb^2z + b^2d$ By comparison of: Coefficient of z^3 : $c = -2$ Coefficient of z : $-8 = (-2)b^2 \Rightarrow b = 2$ (since $b > 0$) Constant: $40 = 4d \Rightarrow d = 10$ Coefficient of z^2 : $a = 10 + 4 = 14$ $z^4 - 2z^3 + 14z^2 - 8z + 40 = 0$ Using GC, Other roots are $z = 1 - 3\text{i},\ 1 + 3\text{i},\ -2\text{i}$ **Method 3** $z^4 - 2z^3 + az^2 - 8z + 40 = 0$ Since bi is a root, $(b\text{i})^4 - 2(b\text{i})^3 + a(b\text{i})^2 - 8(b\text{i}) + 40 = 0$ $b^4 + 2b^3\text{i} - ab^2 - 8b\text{i} + 40 = 0$

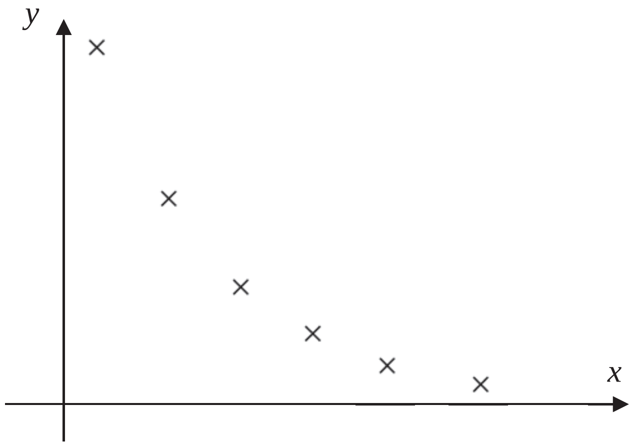
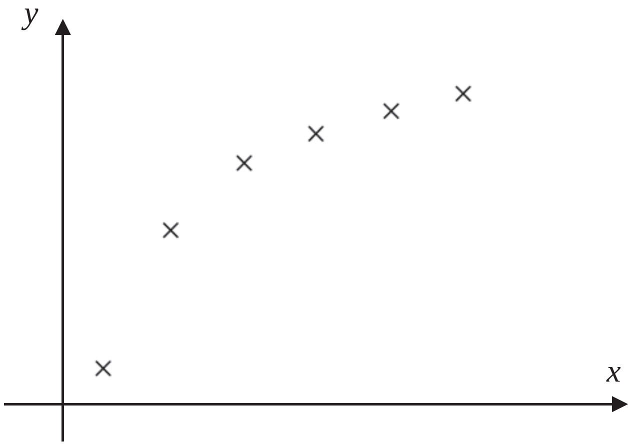
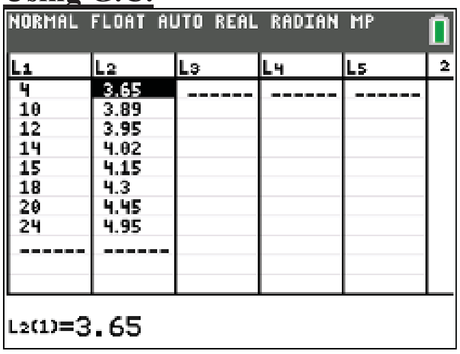
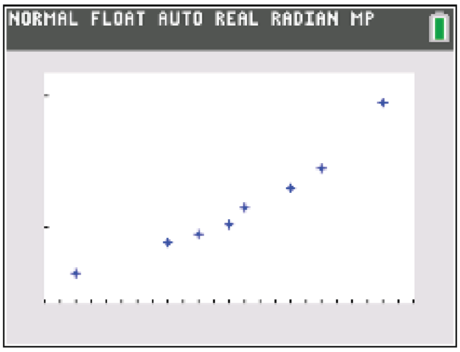
Qn	Solution
	By comparing real and imaginary parts, $2b^3-8b=0$ $2b(b^2-4)=0$ Since $b>0$, $b=-2$ (rejected) or $b=2$ When $b=2$, $b^4-ab^2+40=0$ $16-4a+40=0$ $a=14$ $z^4-2z^3+14z^2-8z+40=0$ Since all coefficients are real, –2i is also a root. Quadratic factor: $(z-2\mathrm{i})(z+2\mathrm{i})=z^2+4$ $z^4-2z^3+az^2-8z+40=(z^2+4)(z^2-2z+10)=0$ $z^2-2z+10=0$ $z=\dfrac{2\pm\sqrt{4-4(10)}}{2}=\dfrac{2\pm\sqrt{36\mathrm{i}^2}}{2}=1\pm3\mathrm{i}$ Hence, other roots are $z=1-3\mathrm{i},\ 1+3\mathrm{i},\ -2\mathrm{i}$.
5(a) (ii)	$w^4+2w^3+aw^2+8w+40=0$ Let $z=-w$ For $z^4-2z^3+14z^2-8z+40=0$ $\Rightarrow\ (-w)^4-2(-w)^3+14(-w)^2-8(-w)+40=0$ $w^4+2w^3+14w^2+8w+40=0$ $-w=1-3\mathrm{i},\ 1+3\mathrm{i},\ -2\mathrm{i}$ or $2\mathrm{i}$ $w=-1+3\mathrm{i},\ -1-3\mathrm{i},\ 2\mathrm{i}$ or $-2\mathrm{i}$
5(b) (i)	$w=-6+(2\sqrt{3})\mathrm{i}$ $\lvert w\rvert=\sqrt{(-6)^2+(2\sqrt{3})^2}=\sqrt{48}$ (or $4\sqrt{3}$) $\arg(w)=\pi-\tan^{-1}\left(\dfrac{2\sqrt{3}}{6}\right)=\dfrac{5\pi}{6}$ $\lvert w^n\rvert=\lvert w\rvert^n=(\sqrt{48})^n=48^{\frac{n}{2}}$ (or $(4\sqrt{3})^n$) $\arg(w^n)=n\arg(w)=\dfrac{5n\pi}{6}$ $w^n=48^{\frac{n}{2}}\mathrm{e}^{\mathrm{i}\left(\frac{5n\pi}{6}\right)}=(4\sqrt{3})^n\mathrm{e}^{\mathrm{i}\left(\frac{5n\pi}{6}\right)}$ Or $w^n=48^{\frac{n}{2}}\left[\cos\left(\dfrac{5n\pi}{6}\right)+\mathrm{i}\sin\left(\dfrac{5n\pi}{6}\right)\right]$

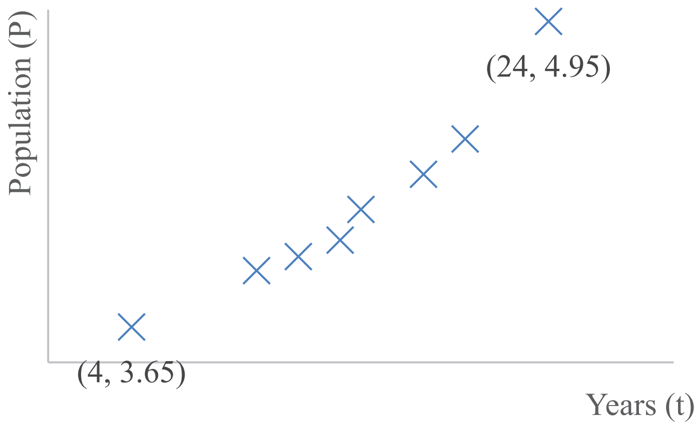
Qn	Solution
5(b) (ii)	$\arg\left(w^n w^*\right)=\frac{5n\pi}{6}-\frac{5\pi}{6}=\frac{5(n-1)\pi}{6}$ For $w^n w^*$ to be purely imaginary $\arg\left(w^n w^*\right)=\frac{\pi}{2},\ \frac{\pi}{2}\pm\pi,\ \frac{\pi}{2}\pm 2\pi,\ldots$ $\arg\left(w^n w^*\right)=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$ $\arg\left(w^n w^*\right)=\frac{(2k+1)\pi}{2},\ k\in\mathbb{Z}$ $\frac{5(n-1)\pi}{6}=\frac{(2k+1)\pi}{2}$ $n-1=\frac{2k+1}{2}\left(\frac{6}{5}\right)$ $n=1+\frac{3(2k+1)}{5},\ k\in\mathbb{Z}$ $n=4,\ 10$ (when $k=2$ and $k=7$)

Section B: Probability and Statistics

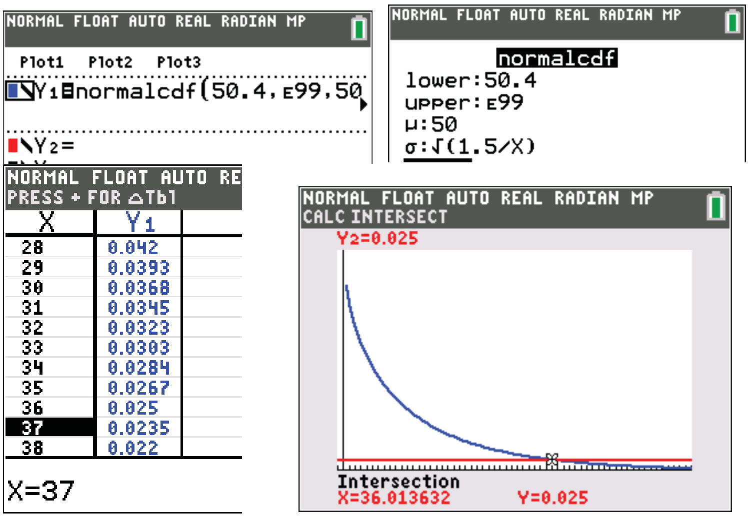
Qn	Solution
6(i)	Total no. of possible IN $=10^4 \times 10 = 100\ 000$ ways
6(ii)	Since the last letter of his IN is I, i.e. the remainder is 8, then the possible sum is 28 (since the sum must be at least 20) The possible sets of digits are $\{9,9,a,b\}$, where $a+b=10$. The possible values of $\{a,b\}$ are $\{2,8\}$, $\{3,7\}$, $\{4,6\}$ and $\{5,5\}$. Hence, no. of ways $=3\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!2!}\right)=$ 42 ways Hence, there are a total of 42 ways.
7(i)	 $\text{P}(\text{blue face shown and ball is red})$ $=\frac{7}{12}\left(\frac{1}{4}\right)$ $=\frac{7}{48}=0.146$ (3 sf)
7(ii)	P(mystery gift given) =P(all red) + P(all blue) $=\frac{1}{4}\left(\frac{3}{7}\right)+\left(\frac{7}{12}\right)\left(\frac{3}{8}\right)$ $=\frac{73}{224}$ $=0.32589=0.326$ (3 sf)

Qn	Solution
7(iii)	$\text{P}\left(\text{one red ball picked} \mid \text{did not win mystery gift}\right)$ $= \dfrac{\text{P}\left(\text{one red ball picked and did not win mystery gift}\right)}{1-\dfrac{73}{224}}$ $= \dfrac{7}{48} \div \left(1-\dfrac{73}{224}\right)$ $= \dfrac{98}{453}$ $= 0.216$ (3sf)
7 Last part	P(mystery gift given) = P(gift given in the first roll) + P(white face in 1st roll and gift given in the second roll) + P(white face in 1st 2 rolls and gift given in the third roll) +… $= \underbrace{\dfrac{73}{224}+\left(\dfrac{1}{6}\right)\left(\dfrac{73}{224}\right)+\left(\dfrac{1}{6}\right)^2\left(\dfrac{73}{224}\right)+\left(\dfrac{1}{6}\right)^3\left(\dfrac{73}{224}\right)+\ldots}_{\text{Sum of infinity with } a=\frac{73}{224} \text{ and } r=\frac{1}{6}}$ $= \dfrac{73}{224}\left[\dfrac{1}{1-\dfrac{1}{6}}\right]$ $= \dfrac{219}{560}$ $= 0.391$ (3 sf)
8 (a)	

Qn	Solution
	 
8 (b) (i)	<u>Using G.C.</u>  

Qn	Solution
	 A linear model may not be appropriate as the points do not lie close to a straight line. (Or the points do not follow a linear trend.)
8 (b)(ii)	**Using G.C.**  Note that from the scatter diagram suggests that as t increases, P increases at an increasing rate. Hence, Case (A) is most appropriate. Using G.C., $r = 0.995$ (3 s.f.)
8 (b) (iii)	Using G.C., $P = 3.6211 + 0.0022167t^2$ $P = 3.62 + 0.00222t^2$ (3 s.f.) In the year 2020, $t = 30$ Hence, when $t = 30$, $P = 3.6211 + 0.0022167(30)^2$ $= 5.6161$ Hence, the predicted population is approximately 5.62 million (3 s.f.)
8 (b) (iv)	The statistician's prediction is **not be reliable** as $t = 30$ is **not within data range**.

Qn	Solution
9(i)	A random sample means that each of the tablets were **selected independently** from each other *and* **each** tablet has an **equal chance** of being **selected**.
9(ii)	Let X be the amount in grams of active ingredient A in a particular type of health supplement tablets. Define μ : Let μ be the **population mean amount/mass of active ingredient *A*** in a particular type of health supplement tablets. Given that sample mean $\bar{x} = 50.6$, sample variance = 2.15 **Unbiased estimate** of population variance $= \frac{40}{39}(2.15) = 2.2051$ $H_0 : \mu = 50$ $H_1 : \mu \neq 50$ Under H_0, since ***n* = 40 is large, by Central Limit Theorem**, $\bar{X} \sim N\left(50,\ \frac{2.2051}{40}\right)$ approximately. Use *z*-test at $\alpha = 0.01$ Using GC, *p*-value = 0.010605 = 0.0106 (3 sf) Since *p*-value > α, do not reject H_0. There is **insufficient evidence** at 1% level of significance to conclude that the mean amount of active ingredient *A* has changed (or is not 50 mg).
9 last part	Let Y be the amount in grams of active ingredient A in the revised formula $H_0 : \mu = 50$ $H_1 : \mu > 50$ Under H_0, $\bar{Y} \sim N\left(50,\ \frac{1.5}{n}\right)$ Use *z*-test at $\alpha = 0.025$. __**Method 1**__ Test-statistic distribution: $Z = \frac{\bar{Y} - 50}{\sqrt{\frac{1.5}{n}}} \sim N(0,\ 1)$

Qn	Solution
	Corresponding test-statistic: $z=\dfrac{50.4-50}{\sqrt{\dfrac{1.5}{n}}}=\dfrac{0.4\sqrt{n}}{\sqrt{1.5}}$ Critical region: $z \geq 1.95996$ Since the chemist claim is valid, H0 is rejected. Test-statistic falls within the critical region. $\dfrac{0.4\sqrt{n}}{\sqrt{1.5}} \geq 1.95996$ $n \geq 36.0135$ $n \geq 37$ $\{n \in \mathbb{Z} : n \geq 37\}$ **<u>Method 2 (Using GC, Table/Graph method)</u>** For H0 to be rejected, p - value ≤ 0.025. p-value = $\mathrm{P}\left(\overline{Y} > 50.4\right) \leq 0.025$  $n \geq 37$ $\{n \in \mathbb{Z} : n \geq 37\}$
10 (i)	$\mathrm{P}(X=2)=\mathrm{P}(\text{point } A_2)$ $=\mathrm{P}(2 \text{ steps to the right and 3 steps downwards})$ $={}^5C_2\,p^2q^3$ $=10p^2q^3$ (Shown)
10 (ii)	Note that $\mathrm{P}(X=3)=\mathrm{P}(\text{point } A_3)$ $={}^5C_3\,p^3q^2=10p^3q^2$ Since the particle will end up at point A_2 most of the time, then mode occurs at $X=2$.

Qn	Solution
	$\therefore \mathrm{P}(X=3)<\mathrm{P}(X=2)$ $10p^3q^2<10p^2q^3$ $\dfrac{q}{p}>1 \Rightarrow 1-p>p$ $2p<1 \Rightarrow p<0.5$ In the same way, $\mathrm{P}(X=2)>\mathrm{P}(X=1)$ $10p^2q^3>5pq^4$ $2p>q$ $2p>1-p$ $3p>1 \Rightarrow p>\dfrac{1}{3}$ Combining both inequalities, $\dfrac{1}{3}<p<\dfrac{1}{2}$ (Shown)
10 (iii)	(table below)

Y	0	2	3	5
$\mathrm{P}(Y=y)$	0.07776	0.4224	0.299904	0.199936

$\mathrm{P}(Y=0)=\mathrm{P}(X=0)={}^5C_0(0.4)^0(0.6)^5=0.07776$

$\mathrm{P}(Y=2)=\mathrm{P}(X=i,\text{ where } i \text{ is even})$

$=\mathrm{P}(X=2)+\mathrm{P}(X=4)$

$=10(0.4)^2(0.6)^3+{}^5C_4(0.4)^4(0.6)^1=0.4224$

$\mathrm{P}(Y=3)=\mathrm{P}(X=i,\text{ where } i \text{ is odd and lose the game})$

$=0.6(\mathrm{P}(X=1)+\mathrm{P}(X=3)+\mathrm{P}(X=5))$

$=0.6\left[{}^5C_1(0.4)^1(0.6)^4+{}^5C_3(0.4)^3(0.6)^2+{}^5C_5(0.4)^5(0.6)^0\right]$

$=0.299904$

$\mathrm{P}(Y=5)=\mathrm{P}(X=i,\text{ where } i \text{ is odd and win the game})$

$=0.4(\mathrm{P}(X=1)+\mathrm{P}(X=3)+\mathrm{P}(X=5))$

$=0.4\left[{}^5C_1(0.4)^1(0.6)^4+{}^5C_3(0.4)^3(0.6)^2+{}^5C_5(0.4)^5(0.6)^0\right]$

$=0.199936$

Qn	Solution
10 (iv)	Using G.C. $\mathrm{E}(Y)=2.744192=2.74$ (3 sf) $\mathrm{Var}(Y)=1.3626^2=1.8567=1.86$ (3 sf)

Qn	Solution
11(i)	Let X be the mass, in g, of a randomly chosen dark truffle. $X \sim \mathrm{N}\left(17,\ 1.3^2\right)$ Let $T = X_1 + X_2 + X_3 + X_4$ $\mathrm{E}(T) = 4(17) = 68$ $\mathrm{Var}(T) = 4\left(1.3^2\right) = 6.76$ $T \sim \mathrm{N}(68,\ 6.76)$ $\mathrm{P}(T > 70) = 0.220878 = 0.221$ (3 sf)
11(ii)	Let Y be the **number of boxes** that <u>weigh more than 70 g</u>, out of 20 boxes. $Y \sim \mathrm{B}(20,\ 0.220878)$ $\mathrm{P}(Y > 3) = 1 - \mathrm{P}(Y \le 3)$ $= 0.67448$ $= 0.674$ (3 sf) **Assumption:** The mass of the empty box is negligible. (**Other possible answer:** The **event that a mass of a box of 4 dark truffles** has mass more than 70g is **independent** of other boxes.)
11(iii)	Let W be the mass, in g, of a randomly chosen salted caramel ganache. $W \sim \mathrm{N}\left(\mu,\ \sigma^2\right)$ Given $\mathrm{P}(W < 12) = \mathrm{P}(W > 15)$ and $\mathrm{P}(W \le 15) = 0.97$ <u>**Method 1**</u> By symmetry, $\mu = \dfrac{12+15}{2} = 13.5$ $\mathrm{P}(W \le 15) = 0.97$ $\mathrm{P}\left(Z \le \dfrac{15-13.5}{\sigma}\right) = 0.97$ $\dfrac{1.5}{\sigma} = 1.88079$ $\sigma = 0.797537$ $\sigma^2 = 0.636065$ $\sigma^2 = 0.636$ (3 sf)

Qn	Solution
	<u>Method 2</u> $\mathrm{P}(W < 12) = 0.03$ $\mathrm{P}\left(Z < \frac{12-\mu}{\sigma}\right) = 0.03$ $\frac{12-\mu}{\sigma} = -1.88079$ $\mu - 1.88079\sigma = 12$ ---(1) $\mathrm{P}(W \le 15) = 0.97$ $\mathrm{P}\left(Z < \frac{15-\mu}{\sigma}\right) = 0.03$ $\frac{15-\mu}{\sigma} = 1.88079$ $\mu + 1.88079\sigma = 15$ ---(2) Solving equation (1) and (2), $\mu = 13.5, \quad \sigma = 0.797537$ $\sigma^2 = 0.636065$ $\sigma^2 = 0.636$ (3 sf)
11(iv)	$W \sim \mathrm{N}(13.5,\ 0.636065)$ Find $\mathrm{P}\left(0.28(W_1 + W_2 + \ldots + W_6) < 0.34T\right)$. Let $S = 0.28(W_1 + W_2 + \ldots + W_6) - 0.34T$ $\mathrm{E}(S) = 0.28(6)(13.5) - 0.34(68) = -0.44$ $\mathrm{Var}(S) = 0.28^2(6)(0.636065) + 0.34^2(6.76) = 1.08065$ $S \sim \mathrm{N}(-0.44,\ 1.08065)$ $\mathrm{P}(S < 0) = 0.66394 = 0.664$ (3 sf)