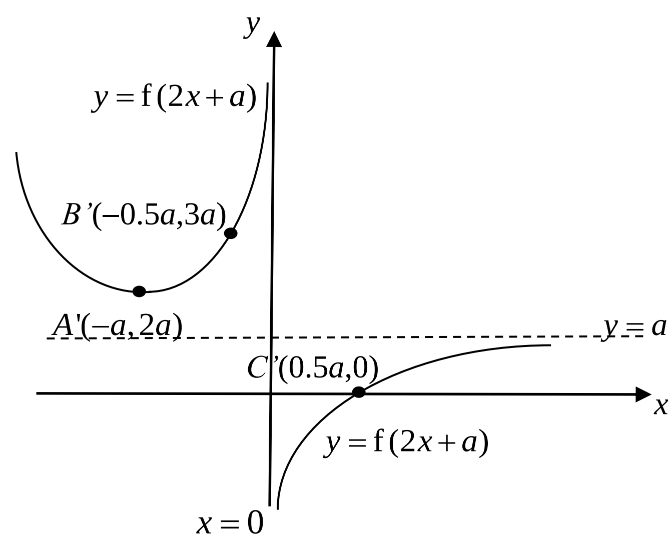
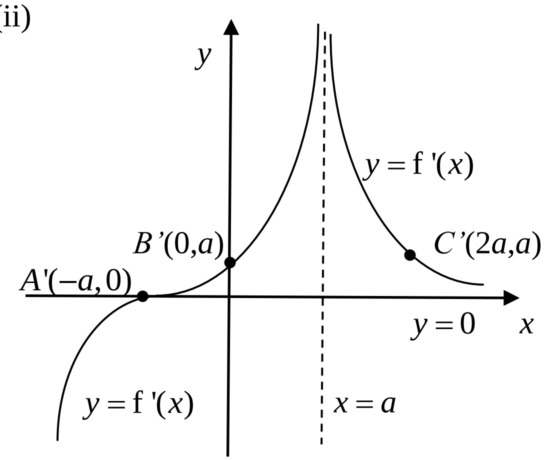
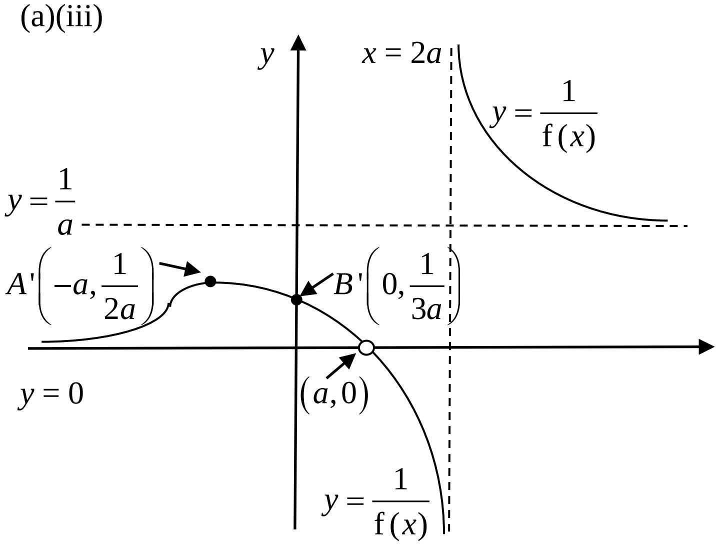
2019 RVHS H2 Maths Prelim P2 Solutions

1	Solution [6] System of linear Eqns	
	(i) $u_{n+1} = -3u_n + An + B$ When $n = 0$, $u_1 = -3u_0 + B$ ---- (*) $2 = -3(4) + B$ $B = 14$ When $n = 1$, $u_2 = -3u_1 + A + B$ ---- (**) $16 = -3(2) + A + 14$ $A = 8$	
	(ii) $u_0 = 4 \Rightarrow a \quad\quad + c = 4$ ---- (1) $u_1 = 2 \Rightarrow -3a + b + c = 2$ ---- (2) $u_2 = 16 \Rightarrow 9a + 2b + c = 16$ ---- (3) Using GC, $a = 1$, $b = 2$, $c = 3$	

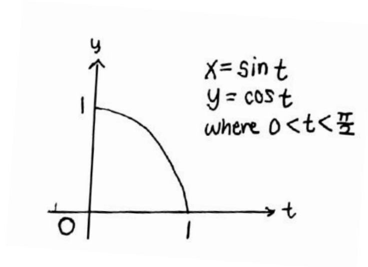
Question 2 [5] vectors		
(a)	Distance between p_1 and p_2 $=\dfrac{\left\lvert \overrightarrow{AB}\cdot(\mathbf{m}\times\mathbf{n})\right\rvert}{\lvert\mathbf{m}\times\mathbf{n}\rvert}$ $=\dfrac{\left\lvert \mathbf{b}\cdot(\mathbf{m}\times\mathbf{n})-\mathbf{a}\cdot(\mathbf{m}\times\mathbf{n})\right\rvert}{\lvert\mathbf{m}\times\mathbf{n}\rvert}$ $=\dfrac{\left\lvert \lvert\mathbf{b}\rvert\lvert(\mathbf{m}\times\mathbf{n})\rvert\cos 60^\circ-\lvert\mathbf{a}\rvert\lvert(\mathbf{m}\times\mathbf{n})\rvert\cos 45^\circ\right\rvert}{\lvert\mathbf{m}\times\mathbf{n}\rvert}$ $=\left\lvert\dfrac{\lvert\mathbf{b}\rvert}{2}-\dfrac{\lvert\mathbf{a}\rvert}{\sqrt{2}}\right\rvert$	
(b)	By ratio theorem, $\overrightarrow{OQ}=\dfrac{\mathbf{a}+2\mathbf{b}}{3}$. The locus of Q is a plane containing fixed point with position vector $\dfrac{\mathbf{a}+2\mathbf{b}}{3}$ and parallel to **m** and **n**. Equation of locus of Q: $\mathbf{r}=\dfrac{\mathbf{a}+2\mathbf{b}}{3}+\lambda\mathbf{m}+\mu\mathbf{n},\quad \lambda,\mu\in\mathbb{R}$	

3	Solution [9] Transformation	
	(a)(i) $\mathrm{f}(x) \to \mathrm{f}(x+a) \to \mathrm{f}(2x+a)$ 	
	(a)(ii) 	
	(a)(iii) 	

	(b) We first note the following transformation steps: $2y^2 - x^2 = 1$ $\downarrow$ Step 1: Replace $'x'$ by $'x-1'$ $2y^2 - (x-1)^2 = 1$ $\downarrow$ Step 2: Replace $'y'$ by $'\frac{y}{\sqrt{2}}'$ $2\left(\frac{y}{\sqrt{2}}\right)^2 - (x-1)^2 = 1$ $y^2 - (x-1)^2 = 1$ Thus, the sequence of transformations needed are as follow 1. A translation of 1 unit in the positive *x* axis direction; 2. A scaling parallel to the *y* axis of factor $\sqrt{2}$.	

4 Solution [10] Parametric Eqn + Integration Application (Area)

(i)



(ii)

Given $x=\sin t, \quad y=\cos t$ where $0<t<\dfrac{\pi}{2}$,

we have $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\dfrac{-\sin t}{\cos t}=-\tan t$

Thus, at the point where $t=p$, ie $(\sin p, \cos p)$, the equation of the tangent is

$$y-\cos p=-\tan p\left(x-\sin p\right) \text{ ---- } (*)$$

So,

$$y=-\left(\tan p\right)x+\frac{\sin^2 p}{\cos p}+\cos p$$

$$=-\left(\tan p\right)x+\frac{\sin^2 p+\cos^2 p}{\cos p}$$

$$=-\left(\tan p\right)x+\frac{1}{\cos p}$$

$$=-\left(\tan p\right)x+\sec p \text{ (shown)}$$

(iii)

The equation of the tangent is $y=-\left(\tan p\right)x+\sec p$,

Let $x=0$, $y=\sec p$. So, $Q=\left(0,\dfrac{1}{\cos p}\right)$.

Let $y=0$, $x=\dfrac{\sec p}{\tan p}=\dfrac{1}{\cos p}\times\dfrac{\cos p}{\sin p}=\dfrac{1}{\sin p}$.

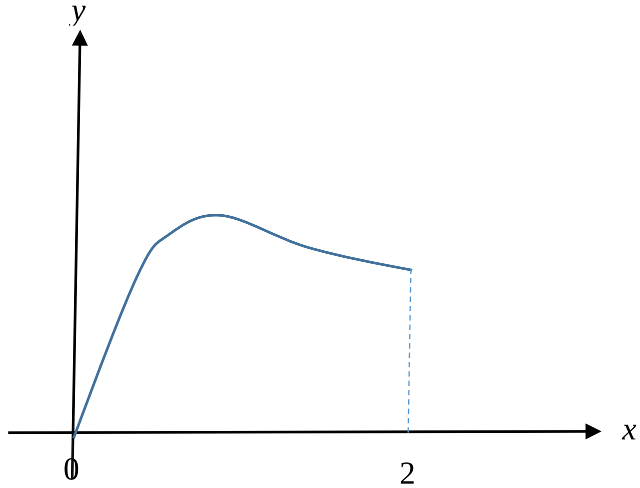
So, $P=\left(\dfrac{1}{\sin p},0\right)$

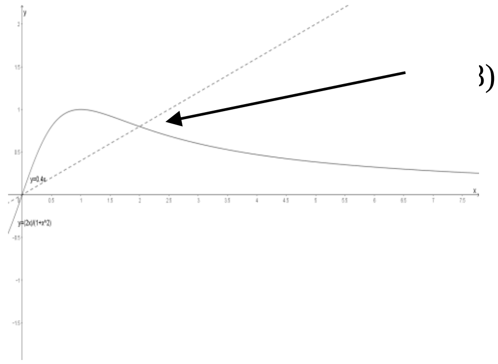
Hence mid point of PQ =

$$M=\left(\frac{1}{2}\left(\frac{1}{\sin p}+0\right),\frac{1}{2}\left(\frac{1}{\cos p}+0\right)\right)=\left(\frac{1}{2\sin p},\frac{1}{2\cos p}\right)$$

To find the Cartesian equation of the locus of the point *M*,

	We let $x=\dfrac{1}{2\sin p}$ and $y=\dfrac{1}{2\cos p}$ Then $\sin p=\dfrac{1}{2x}$ and $\cos p=\dfrac{1}{2y}$ And thus, $\left(\dfrac{1}{2x}\right)^2+\left(\dfrac{1}{2y}\right)^2=1$ Hence the Cartesian equation of the locus of M is $\dfrac{1}{x^2}+\dfrac{1}{y^2}=4$ or $x^2+y^2=4x^2y^2$	
	(iv) Exact area = $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} y\ \mathrm{d}x$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} y\ \dfrac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cos t)(\cos t)\ \mathrm{d}t$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 t\ \mathrm{d}t$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \dfrac{1+\cos 2t}{2}\ \mathrm{d}t$ $=\left[\dfrac{1}{2}t+\dfrac{\sin 2t}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$ $=\left(\dfrac{\pi}{8}+\dfrac{1}{4}\right)-\left(\dfrac{\pi}{12}+\dfrac{\sqrt{3}}{8}\right)$ $=\dfrac{\pi}{24}+\dfrac{1}{4}-\dfrac{\sqrt{3}}{8}$ units^2	

Question 5 [10]		
(i)		
(ii)	$x=\tan\theta \Rightarrow \dfrac{dx}{d\theta}=\sec^2\theta$ Volume $=\pi\int_0^2 \left(\dfrac{2x}{1+x^2}\right)^2 dx$ $=4\pi\int_0^2 \dfrac{x^2}{\left(1+x^2\right)^2} dx$ $=4\pi\int_0^{\tan^{-1}2} \dfrac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\, d\theta$ $=4\pi\int_0^{\tan^{-1}2} \dfrac{\tan^2\theta}{\sec^2\theta}\, d\theta$ $=4\pi\int_0^{\tan^{-1}2} \sin^2\theta\, d\theta$ $=4\pi\int_0^{\tan^{-1}2} \dfrac{1-\cos 2\theta}{2}\, d\theta$ $=2\pi\left[\theta-\dfrac{\sin 2\theta}{2}\right]_0^{\tan^{-1}2}$ $=2\pi\left[\theta-\sin\theta\cos\theta\right]_0^{\tan^{-1}2}$ $=2\pi\left[\tan^{-1}2-\frac{2}{\sqrt{5}}\frac{1}{\sqrt{5}}\right]$ $=2\pi\left[\tan^{-1}2-\frac{2}{5}\right]$ units^3	



Volume of plastic

$$= \frac{1}{3}\pi\left(\frac{4}{5}\right)^2(2) = \frac{32}{75}\pi \text{ units}^3$$

Cost of one paperweight

$$= \left(2\pi\left[\tan^{-1}2 - \tfrac{2}{5}\right] - \frac{32}{75}\pi\right)(3) + \left(\frac{32}{75}\pi\right)(1.2)$$

$= \$11$ (to the nearest dollar)

<table>
<tr><td colspan="3">Question 6 [6]</td></tr>
<tr><td>(i)</td><td>RRR BB W Y G
Step 1 : Arrange BB W Y G in $\frac{5!}{2!}$ ways
Step 2: Slot the RRR into the 6 appropriate slots _ to the left/ right of the arranged letters in Step:1 generally denoted by X

_X_X_X_X_X_

No. of ways $=\frac{5!}{2!}\times\binom{6}{3}=1200$

Note:
Number of ways in which RRR are separated
$\neq$ Total number of ways $-$ All R separated ----(*)

Why is this so?</td><td></td></tr>
<tr><td>(ii)</td><td>No. of ways for the letters to form a circle = $\frac{5!}{5}$
Since clockwise and anti-clockwise are indistinguishable in a ring,
No. of ways = $\frac{5!}{5}\div 2=12$

Note:
For a physical 3-dimensional ring made up of beads, what happen when you flip it the other side?</td><td></td></tr>
<tr><td>(iii)</td><td><u>Method 1</u>
III N TT A E
1. There is only 1 way to arrange A, E and N in alphabetical order.
2. Without restriction there are $(3!)$ ways to arrange A, E and N.

Without restriction, total number of ways to arrange all the letters $=\frac{8!}{3!2!}$ ---- (**)
To get number of ways to arrange all letters and A, E and N in alphabetical order, REMOVE all the UNWANTED arrangements in (**), using the ideas in (1) and (2).</td><td></td></tr>
</table>

	No. of ways to arrange all letters and A, E and N in alphabetical order $=\left[\dfrac{8!}{3!2!}\right]\div(3!)=\dfrac{8!}{3!3!2!}$ (Note: A, E and N need not be together.) Required prob = $\dfrac{8!}{3!3!2!}\Big/\dfrac{8!}{3!2!}=\dfrac{1}{3!}=\dfrac{1}{6}$ Method 2 Step 1: Other than A, E, N, the other letters are III TT, number of ways to arrange III TT $=\dfrac{5!}{3!2!}$ Step 2: After arranging III TT, we create slots _ to the left / right of each letter. _X_X_X_X_X_ Step3: We now slot in A, E, N into the slots so that A, E, N are in alphabetical order. We could group A, E, N as - 1 item: [AEN] - 2 items: [AE], [N] - 2 items: [A], [NE] - 3 items: [A], [E], [N] For example to slot 2 items: [AE], [N] into _X_X_X_X_X_, Number of ways $=\dfrac{5!}{3!2!}\times\dbinom{6}{2}$ Number of ways to arrange all letters and A, E and N in alphabetical order $=\dfrac{5!}{3!2!}\times\left[\dbinom{6}{1}+\dbinom{6}{2}+\dbinom{6}{2}+\dbinom{6}{3}\right]$	

7	Solution [6] DRV	
	(i) $\mathrm{P}(X=3)$ $=\frac{6-n}{6}\cdot\frac{5-n}{5}\cdot\frac{n}{4}$ $=\frac{(6-n)(5-n)n}{120}$	
	(ii) When $n=3$, $\mathrm{P}(W=1)$ $=\mathrm{P}(X\geq 4)$ = P(keys from first 3 trials can't unlock box) $=\left(\frac{3}{6}\right)\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)=\frac{1}{20}=0.05$ <table><tr><td>*w*</td><td>2</td><td>1</td></tr><tr><td>P(W = *w*)</td><td>0.95</td><td>0.05</td></tr></table> $\mathrm{E}(W)=2(0.95)+(0.05)=\$1.95$ Alternatively $\mathrm{P}(W=2)$ $=\mathrm{P}(X=1)+\mathrm{P}(X=2)+\mathrm{P}(X=3)$ $=\left(\frac{3}{6}\right)+\left(\frac{3}{6}\right)\left(\frac{3}{5}\right)+\left(\frac{3}{6}\right)\left(\frac{2}{5}\right)\left(\frac{3}{4}\right)$ $=\frac{19}{20}=0.95$	
	(iii) Let T denote the r.v the number of times that Alfred wins \$2 from a game, out of 5 games played. $T\sim\mathrm{B}(5,0.95)$ P(Alfred wins more at least \$9 out of 5 games) $=\mathrm{P}(T\geq 4)$ $=1-\mathrm{P}(T\leq 3)$ $=0.977$	

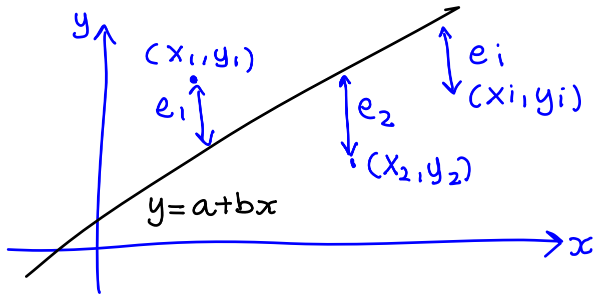
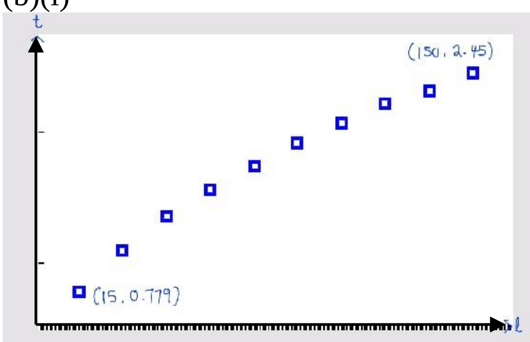
<table>
<tr><td colspan="3">Question 8 [7] Probability</td></tr>
<tr><td>(i)</td><td>P(Player draws 3 different numbers in the first round)
$= \dfrac{\binom{5}{1}\binom{2}{1}}{\binom{8}{3}} = \dfrac{5}{28}$</td><td></td></tr>
<tr><td>(ii)</td><td>Probability of sum of numbers from first round is 12 and sum of numbers from second round is 6)
$= \dfrac{\binom{5}{1}\binom{2}{2}}{\binom{8}{3}} \dfrac{\binom{5}{3}}{\binom{8}{3}}$
$= \dfrac{25}{1568}$

P(Sum of the numbers drawn adds up to 18)
= P(Sum of each of first and second round is 9)
+ P(Sum of first round is 12 and second round is 6)
$= \dfrac{\binom{5}{2}\binom{2}{1}}{\binom{8}{3}} \dfrac{\binom{5}{2}}{\binom{8}{3}} + \dfrac{25}{1568}$
$= \dfrac{125}{1568}$</td><td></td></tr>
<tr><td></td><td>Let X be the sum of the 2 numbers drawn.
<table><tr><td>x</td><td>4</td><td>7</td><td>12</td><td>15</td><td>20</td></tr><tr><td>P($X = x$)</td><td>$\dfrac{\binom{5}{2}}{\binom{8}{2}} = \dfrac{10}{28}$</td><td>$\dfrac{\binom{5}{1}}{\binom{8}{2}} = \dfrac{5}{28}$</td><td>$\dfrac{\binom{5}{1}\binom{2}{1}}{\binom{8}{2}} = \dfrac{10}{28}$</td><td>$\dfrac{\binom{2}{1}}{\binom{8}{2}} = \dfrac{2}{28}$</td><td>$\dfrac{\binom{2}{2}}{\binom{8}{2}} = \dfrac{1}{28}$</td></tr></table>
E(X)
$= 4\left(\dfrac{10}{28}\right) + 7\left(\dfrac{5}{28}\right) + 12\left(\dfrac{10}{28}\right) + 15\left(\dfrac{2}{28}\right) + 20\left(\dfrac{1}{28}\right)$
$= 8.75$</td><td></td></tr>
</table>

Question 9 [8] Binomial Dist		
(i)	Assumption: The candidature has no knowledge of which option is the correct answer and chooses one of the 5 options **randomly/by guessing**. This assumption is important to ensure that the probability of choosing a particular option is constant at 0.2.	
(ii)	Let *X* denotes the number of marks a candidate scores for the test, out of 12. X ~ B(12, 0.2) P(X=1) = 0.206 P(X=2) = 0.283 P(X=3) = 0.236 Mode of X is 2. Majority of the candidates will score 2 marks.	
(iii)	$P\left(X>4 \mid X\le 5\right)$ $=\dfrac{P(X=5)}{P(X\le 5)}$ $=0.0542\ \ (3\,\text{s.f})$	
(iv)	Assuming n is large, by CLT, $\overline{X}\sim N\left(2.4,\dfrac{1.92}{n}\right)$ approx. $P\left(\overline{X}\le 2.7\right)\ge 0.95$ $n=57, P\left(\overline{X}\le 2.7\right)=0.94893<0.95$ $n=58, P\left(\overline{X}\le 2.7\right)=0.95041>0.95$ Thus, least value of n is 58.	

Question 10 [9] Hypo Test		
(i)	$\bar{x}=-\frac{27}{70}+46=45.61428=45.6 \text{ (to 3 s.f.)}$ $s^2=\frac{1}{69}\left(30939-\frac{(-27)^2}{70}\right)=448.2403727=448 \text{ (to 3 s.f.)}$	
(ii)	Let μ_0 be the mean number of hours the engineer claimed. Test $H_0: \mu = \mu_0$ against $H_1: \mu > \mu_0$ at 5% significance level Test statistic: Under H_0, $Z=\frac{\bar{X}-\mu_0}{s/\sqrt{70}} \sim N(0,1)$ To reject H_0, $\frac{\bar{x}-\mu_0}{s/\sqrt{70}}>1.644853632$ $\mu_0<\bar{x}-1.644853632\frac{s}{\sqrt{70}}$ $\mu_0<41.45197671$ $\mu_0<41.5 \text{ (to 3 s.f.)}$ Thus, the greatest mean number of hours the engineers should claim is 41.5.	
(iii)	Based on n = 70, $\sum x=\left(-\frac{27}{70}+46\right)\times 70=3193$ Based on n = 75, $\bar{x}=\frac{3193+56+34+63+50+54}{75}=46$ $\sum(x-46)^2$ $=\sum(x-\bar{x})^2$ $=30939+(56-46)^2+(34-46)^2+(63-46)^2$ $+(50-46)^2+(54-46)^2$ $=31552$ $\therefore s^2=\frac{\sum(x-\bar{x})^2}{74}=426.3783783784=426.378 \text{ (3.s.f)}$	

	Test $H_0 : \mu = 41.5$ against $H_1 : \mu > 41.5$ at α% significance level Test statistic: Under H_0, $Z = \dfrac{\overline{X} - 41.5}{s\ /\sqrt{75}} \sim \mathrm{N}(0,1)$ By GC, p-value = 0.029559 Do not reject H_0, $p - value \geq \dfrac{\alpha}{100}$ $\Rightarrow \dfrac{\alpha}{100} \leq 0.029559$ $\Rightarrow \alpha \leq 2.9559$ The greatest integer value of α is 2. [If $\mu_0 = 41.4$, p-value = 0.026649, then the greatest integer value of α is 2. If $\mu_0 = 41.45197671$, p-value = 0.028230374, then the greatest integer value of α is 2.]	

11	Solution [12] Correlation & Regression	
	(a) Let the sample of bivariate data be (x_i, y_i) where $i = 1, 2, \ldots n$. Let $e_i = y_i - (a + bx_i)$ be the vertical deviation between the point (x_i, y_i) and the line $y = a + bx$.  The line $y = a + bx$ is the least square regression line for the sample of bivariate data if the sum of the squares of the vertical deviation $\sum_{i=1}^{n}(e_i)^2$ is the minimum.	
	(b)(i)  	
	(b)(ii) Model (B). As l increases, t increases at a decreasing rate, therefore model (B) is most appropriate. Note that for Model (A), the graph of $t = a + bl^2$ has a turning point located on the vertical axis, therefore Model A is not suitable.	
	(b)(iii) $t = a + b\ln l$ Using GC, $a = -1.370621901 \approx -1.37$ (3.s.f) $b = 0.7394875471 \approx 0.740$ (3 s.f) $r = 0.9871042012 \approx 0.987$ (3 s.f)	

	(b)(iv) $t = -1.370621901 + +0.7394875471\ln l$ ---- (1) To estimate *l* when t = 1.00, use the regression line of t on ln *l*, since *t* is the dependent variable. Sub $t = 1.00$ into (1), $l = 24.6743 \approx 24.7$ mm. The appropriate regression line is used since *t* is the dependent variable. Since $\lvert r \rvert \approx 0.987$ is close to 1, the model based on $t = a + b\ln l$, is a good fit for the data. $t = 1.00$ is within the input data range $0.779 \le t \le 2.45$, the estimate is an interpolation. Therefore, the estimate is reliable.	
	(b)(v) 1 millimeter = 0.03937 inch 1 inch = $\dfrac{1}{0.03937}$ millimeters *L* inches = $\dfrac{L}{0.03937}$ millimeters $(L = 0.03937l)$ $t = a + b\ln\dfrac{L}{0.03937}$ $t = a + b(\ln L - \ln 0.03937)$ $t = (a - b\ln 0.03937) + b\ln L$ $t = 1.02143631 + 0.7394875471\ln L$ $t = 1.02 + 0.739\ln L$ (3 s.f)	

12	Solution [12 marks] Normal Dist	
(i)	Let A be the mass of a randomly chosen apple. Suppose $A \sim N\left(70, 40^2\right)$ Then $\mathrm{P}\left(A<0\right)=0.0401\,(3\text{ s.f.})$, but it's impossible for apples to have negative mass, so the probability should be much closer to 0. Hence, the normal distribution is not a suitable distribution for the masses of apples.	
(ii)	Let Y be the mass of a pear. Let μ be the mean mass of the pears, and σ^2 be the variance in the mass of the pears. $Y \sim N\left(\mu, \sigma^2\right)$ Given, $\mathrm{P}\left(Y<148\right)=\mathrm{P}\left(Y>230\right)=\frac{1}{3}$ By symmetry, $\mu=\frac{148+230}{2}=189$ Let $Z=\frac{Y-189}{\sigma} \sim N\left(0,1\right)$ $\mathrm{P}\left(Y<148\right)=\frac{1}{3}$ $\mathrm{P}\left(Z<\frac{-41}{\sigma}\right)=\frac{1}{3}$ From GC, $\frac{-41}{\sigma}=-0.430727$ $\sigma=95.18783606$ $\sigma=95.2\,\text{g} \quad (3\text{ s.f.})$	
(iii)	Let T_A be the total mass of the 300 apples. Then $T_A=A_1+A_2+\ldots+A_{300}$ and $T_A \sim \mathrm{N}\left(300\times70, 300\times40^2\right)$ approx. by the Central Limit Theorem (CLT) as n = 300 is large.	
(iv)	Similarly, by letting T_P be the total mass of the 400 pears, we have $T_P \sim N\left(400\times189, 400\times9060.724134\right)$	

	So, we have $T_A \sim \mathrm{N}(21000, 480000)$ and $T_P \sim N(75600, 3624289.654)$ Thus, we have $R = 0.005T_A + 0.008T_P$ and $R \sim$ N(0.005$\times$21000+0.008$\times$75600, $0.005^2 \times 480000 + 0.008^2 \times 3624289.654$) or $R \sim$ N(709.8, 243.9545379) Since c is the running cost of the orchard, $\mathrm{P}(R > c) = 0.9$ From GC, c = 689.7833896 Therefore, the cost of running the orchard is \$689.78 (2 d.p) Note: In accounting terms, the amount of money collected from sales in a business, is referred to as the revenue.	