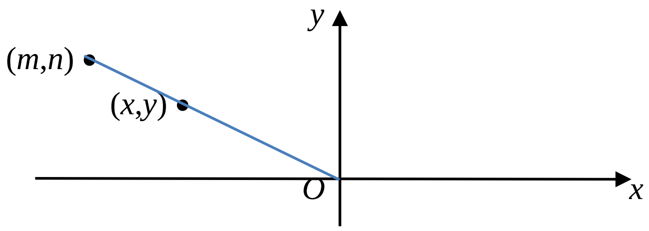
Solutions

1		Using the substitution $x=\frac{1}{t}$, $\int_1^\infty \frac{1}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x=\int_1^0 \frac{1}{\left(1+\frac{1}{t^2}\right)\left(1+\frac{1}{t^\alpha}\right)}\left(-\frac{1}{t^2}\right)\mathrm{d}t$ $=\int_0^1 \frac{t^\alpha}{(1+t^2)(1+t^\alpha)}\,\mathrm{d}t$ $=\int_0^1 \frac{x^\alpha}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x$
		$\int_0^\infty \frac{1}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x=\int_0^1 \frac{1}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x+\int_1^\infty \frac{1}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x$ $=\int_0^1 \frac{1}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x+\int_0^1 \frac{x^\alpha}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x$ $=\int_0^1 \frac{1+x^\alpha}{(1+x^2)(1+x^\alpha)}\,\mathrm{d}x$ $=\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$ $=\left[\tan^{-1}x\right]_0^1$ $=\frac{\pi}{4}$
		$\int_0^\infty \frac{3x^n-x^m+2}{(1+x^2)(1+x^m)(1+x^n)}\,\mathrm{d}x$ $=\int_0^\infty \frac{3(1+x^n)-(1+x^m)}{(1+x^2)(1+x^m)(1+x^n)}\,\mathrm{d}x$ $=3\int_0^\infty \frac{1}{(1+x^2)(1+x^m)}\,\mathrm{d}x-\int_0^\infty \frac{1}{(1+x^2)(1+x^n)}\,\mathrm{d}x$ $=3\left(\frac{\pi}{4}\right)-\frac{\pi}{4}$ $=\frac{\pi}{2}$

2		If we perform a transformation on S as follows: $$T:(x,y)\mapsto(x-a,y-b),$$ then $(a,b)\xrightarrow{T}(0,0)$ and $(c,d)\xrightarrow{T}(c-a,d-b)$. It is therefore clear that (a,b) and (c,d) are mutually visible if and only if $(0,0)$ and $(c-a,d-b)$ are.
		Let $m=c-a$ and $n=d-b$. We will prove that $(0,0)$ and (m,n) are mutually visible iff $\gcd(m,n)=1$. ($\Rightarrow$) Assume $(0,0)$ and (m,n) are mutually visible. If $m=0$, then $n=1$, for otherwise there will be an integer point between $(0,0)$ and (m,n), contradicting the assumption of mutual visibility. Similarly, if $n=0$, then $m=1$. Thus in both cases, $\gcd(m,n)=\gcd(0,1)=\gcd(1,0)=1$. So we assume $m,n\neq 0$. Suppose on the contrary, $\gcd(m,n)=d>1$. [Note: By definition, $d\geq 1$] Then there exist non-zero integers x and y (since $m,n\neq 0$) such that $$m=xd \text{ and } n=yd.$$ Then we have $\lvert m\rvert>\lvert x\rvert$ and $\lvert n\rvert>\lvert y\rvert$ and $\frac{n}{m}=\frac{y}{x}$. [Note: $\lvert m\rvert>\lvert x\rvert$ and $\lvert n\rvert>\lvert y\rvert$ implies that the integer point (x,y) lies strictly between the integer points $(0,0)$ and (m,n) since distance from $(0,0)$ to (m,n) $=\sqrt{m^2+n^2}$ $>\sqrt{x^2+y^2}$ since $\lvert m\rvert>\lvert x\rvert\Rightarrow m^2>x^2, \lvert n\rvert>\lvert y\rvert\Rightarrow n^2>y^2$ = distance from $(0,0)$ to (x,y)] This implies $(x,y)\in S$ lies on the line segment joining $(0,0)$ and (m,n), again contradicting the assumption of mutual visibility. Hence we must have $\gcd(m,n)=1$.
		($\Leftarrow$) Assume $\gcd(m,n)=1$.

		Suspose on the contrary, $0,0$ and m,n are not mutually visible, that is, there exists $x,y \in S$ on the line segment joining $0,0$ and m,n .  Then we have $\dfrac{n}{m}=\dfrac{y}{x} \Leftrightarrow \dfrac{n}{y}=\dfrac{m}{x}=r$ where $r > 1$ is a rational number in its lowest terms. [See above diagram] We write $r=\dfrac{p}{q}$ where $p,q \in \mathbb{Z}^{+}, p>q$ (since $r > 1$) and $\gcd\ p,q\ =1$ (since r is a fraction in its lowest terms) $\dfrac{n}{y}=r=\dfrac{p}{q} \Rightarrow nq=yp \Rightarrow p\|nq \Rightarrow p\|n$ since $\gcd\ p,q\ =1$. Similarly, $\dfrac{m}{x}=r=\dfrac{p}{q} \Rightarrow mq=xp \Rightarrow p\|mq \Rightarrow p\|m$ since $\gcd\ p,q\ =1$. Therefore p ($> q \geq 1$) is a common factor of m and n. Consequently, $\gcd\ m,n\ \geq p>1$, contradicting the assumption $\gcd\ m,n\ =1$. Hence $0,0$ and m,n are mutually visible.

3	**(i)**	Let $y=\left(1+\dfrac{x}{n}\right)^{n}$. Taking ln both sides, $\ln y=\ln\left(1+\dfrac{x}{n}\right)^{n}$ $\lim\limits_{n\to\infty}(\ln y)=\lim\limits_{n\to\infty}\left[\ln\left(1+\dfrac{x}{n}\right)^{n}\right]$ $=\lim\limits_{n\to\infty}\left[n\ln\left(1+\dfrac{x}{n}\right)\right]$ Since x is a fixed real number, $\dfrac{x}{n}\to 0$ as $n\to\infty$ so $\left\|\dfrac{x}{n}\right\|<1$ for a sufficiently large n. Thus for sufficiently large n,

		$\ln\left(1+\frac{x}{n}\right)=\frac{x}{n}-\frac{1}{2}\left(\frac{x}{n}\right)^2+\frac{1}{3}\left(\frac{x}{n}\right)^3-\ldots$ $\Rightarrow n\ln\left(1+\frac{x}{n}\right)=x-\frac{1}{2}\left(\frac{x^2}{n}\right)+\frac{1}{3}\left(\frac{x^3}{n^2}\right)-\ldots$ $\lim\limits_{n\to\infty}(\ln y)=\lim\limits_{n\to\infty}\left[n\ln\left(1+\frac{x}{n}\right)\right]$ $=\lim\limits_{n\to\infty}\left[x-\frac{1}{2}\left(\frac{x^2}{n}\right)+\frac{1}{3}\left(\frac{x^3}{n^2}\right)-\ldots\right]$ $=x$ By continuity of the ln function, $\ln\left(\lim\limits_{n\to\infty} y\right)=x$ $\Rightarrow \lim\limits_{n\to\infty} y=\mathrm{e}^x$ That is, $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n=\mathrm{e}^x$.
	(ii)	Define $b_m(j)=a_m(j)+1$. $a_m(j+1)=\left[a_m(j)\right]^2+2a_m(j)$ $\Rightarrow a_m(j+1)+1=\left[a_m(j)\right]^2+2a_m(j)+1$ $\Rightarrow a_m(j+1)+1=\left[a_m(j)+1\right]^2$ $\Rightarrow b_m(j+1)=\left[b_m(j)\right]^2$
	(iii)	Using the above equality and putting $j=m-1,m-2,\ldots,1$, we have $b_m(m)=\left[b_m(m-1)\right]^2$ $=\left[b_m(m-2)\right]^{2^2}$ $=\left[b_m(m-3)\right]^{2^3}$ $\vdots$ $=\left[b_m(m-m)\right]^{2^m}$ $=\left[b_m(0)\right]^{2^m}$ $=\left[a_m(0)+1\right]^{2^m}$ $=\left(1+\frac{x}{2^m}\right)^{2^m}$ Therefore, using the result in **(i)**, we have

		$a_m(m)+1=\left(1+\frac{x}{2^m}\right)^{2^m}$ $\lim_{m\to\infty} a_m(m)=\lim_{m\to\infty}\left(1+\frac{x}{2^m}\right)^{2^m}-1$ $=e^x-1$

4	**(i)**	$p_n = P\left(\begin{array}{l}\text{John scores } n-1 \text{ points}\\ \text{on subsequent tosses}\end{array}\middle\vert\,\text{first toss is a head}\right)P(\text{first toss is a head})$ $+P\left(\begin{array}{l}\text{John scores } n-2 \text{ points}\\ \text{on subsequent tosses}\end{array}\middle\vert\,\text{first toss is a tail}\right)P(\text{first toss is a tail})$ $=\frac{1}{3}p_{n-1}+\frac{2}{3}p_{n-2}$ $p_1 = P(\text{first toss is a head})=\frac{2}{3}$ $p_2 = P(\text{first toss is a tail})+P(\text{first 2 tosses are heads})=\frac{1}{3}+\left(\frac{2}{3}\right)^2=\frac{7}{9}$ $p_7=\frac{1591}{2187}=0.727$ (3 s.f.)
	(ii)	Consider the possible sequences of outcomes. Permissible sequences are such that John is not able to stop. Non-permissible sequences are struck-through. (table below) Note that permissible sequences are such that every 1st and 2nd, 3rd and 4th outcomes, and so on, are not the same. Define a map from the set of outcomes of n tosses of the coin to the set of binary sequences of length $\frac{n}{2}$ such that the $(2k-1)$-th and $(2k)$-th outcomes map to the set of binary digits $\{0,1\}$: $\text{TH}\mapsto 0$ and $\text{HT}\mapsto 1$. We have a bijection from the set of possible sequences of outcomes where John is not able to stop after n tosses and the set of binary sequences of length $\frac{n}{2}$.

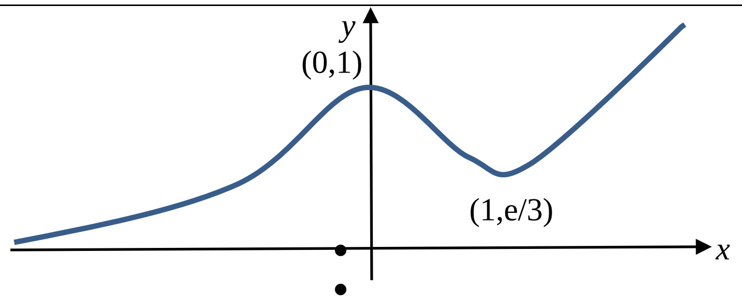
$n=1$	$n=2$	$n=3$	$n=4$
T	~~TT~~ TH	THT THH	~~THTT~~ THTH THHT ~~THHH~~
H	HT ~~HH~~	HTT HTH	~~HTTT~~ HTTH HTHT ~~HTHH~~

		The number of binary sequences of length $\frac{n}{2}$ is $2^{\frac{n}{2}}$. Thus the number of possible sequences of heads and tails for the n tosses is $2^{\frac{n}{2}}$.
	(iii)	Let x_i be the total amount John bet up to and including the i-th toss. Since John bet at least \$1 on each toss, $1 \le x_1 < x_2 < \ldots < x_{15} = 20$ and $10 \le x_1 + 9 < x_2 + 9 < \ldots < x_{15} + 9 = 29$. We have 30 positive integers (pigeons) $x_1,\ x_2,\ \ldots,\ x_{15},\ x_1+9,\ x_2+9,\ \ldots,\ x_{15}+9$ and 29 possible values for John's bets (pigeonholes). Hence $x_j = x_i + 9$ for some i and j. $$x_j - x_i = 9$$ That is, \$9 was bet from $(i+1)-th$ toss to the j-th toss.

5	**(a)**	Each of the 6 boxes contains one of the 3 toys: $\lvert S\rvert = 3^6$ outcomes. Let A_i denote the event that the 6 boxes do not contain toy i ($i = 1, 2, 3$) $\lvert A_i\rvert = 2^6$, $\lvert A_i \cap A_j\rvert = 1^6$, $\lvert A_1 \cap A_2 \cap A_3\rvert = 0$ Number of outcomes where all 3 different toys are obtained from the 6 boxes $= \lvert A_1' \cap A_2' \cap A_3'\rvert = \lvert S\rvert - \lvert A_1 \cup A_2 \cup A_3\rvert$ $$= 3^6 - \binom{3}{1}2^6 + \binom{3}{2}1^6 - 0$$ Probability $= 1 - \binom{3}{1}\left(\frac{2}{3}\right)^6 + \binom{3}{2}\left(\frac{1}{3}\right)^6$ $= \frac{20}{27}$ $= 0.741$ (3 s.f.)
	(b) (i)	$r_1 + r_2 + \ldots + r_n = r$ ____(1) Let $\left.\begin{aligned} r_1 &= x_n \\ r_2 &= x_{n-1} + x_n \\ r_3 &= x_{n-2} + x_{n-1} + x_n \\ &\vdots \\ r_n &= x_1 + x_2 + \ldots + x_n \end{aligned}\right\}$ (*) *Equivalently, define* $x_n = r_1$ $x_{n-1} = r_2 - r_1$ $x_{n-2} = r_3 - r_2$ $\vdots$ $x_1 = r_n - r_{n-1}$ Then (*) is equivalent to

		$x_1+2x_2+3x_3+\ldots+nx_n=r$ ___(2) and $r_1\le r_2\le r_3\le\ldots\le r_n$ $x_i\ge 0 \;\Leftrightarrow\; r_i\ge 0$ for $i=1,\ 2,\ \ldots,\ n$ The mapping $x_i\mapsto r_j$ given by (*) is bijective. Hence the number of non-negative integer solutions is equivalent to the number of non-negative integer solutions for (1), and this corresponds to the number of partitions of *r* into at most *n* parts.
	(ii)	The number of ways is given by the number of non-negative integer solutions of the equation $x_1+2x_2+3x_3=7$, and this corresponds to the number of partitions of 7 into at most 3 parts. $P(7,1)=1$ $P(7,2)=3$ $P(7,3)=4$ Hence total number of ways = 8
	(c) (i)	$4^6=4096$
	(ii)	$4\left(3^5\right)=972$
	(iii)	Number of ways of distributing 6 distinct objects (weeks) to 4 distinct boxes (modules) such that no box is empty $=S(6,4)4!$ $=\left[S(5,3)+4S(5,4)\right]4!$ $=\left[S(4,2)+3S(4,3)+4S(5,4)\right]4!$ $=\left[2^3-1+3\binom{4}{2}+4\binom{5}{2}\right]4!$ $=1560$ <u>Alternative method</u>: Number of onto mappings from $\{1,2,3,4,5,6\}$ to $\{1,2,3,4\}$ $=\underbrace{\binom{6}{3}4!}_{\text{3 weeks for 1 module \& 1 week each for the other 3 modules}}+\underbrace{\frac{\binom{6}{2}\binom{4}{2}}{2!}4!}_{\text{2 weeks each for 2 modules \& 1 week each for the other 2 modules}}$ $=1560$

6	**(i)**	$\left(x^2+x+1\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}-\left(x^2-x\right)y=0$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(\dfrac{x^2-x}{x^2+x+1}\right)y$ $\displaystyle\int\frac{1}{y}\,\mathrm{d}y=\int\left(\frac{x^2-x}{x^2+x+1}\right)\mathrm{d}x$ $\displaystyle\ln\lvert y\rvert=\int\left(\frac{x^2+x+1-(2x+1)}{x^2+x+1}\right)\mathrm{d}x$ $\displaystyle\quad=\int 1\,\mathrm{d}x-\int\left(\frac{2x+1}{x^2+x+1}\right)\mathrm{d}x$ $\quad=x-\ln\left(x^2+x+1\right)+C$ $\lvert y\rvert=\mathrm{e}^{x-\ln\left(x^2+x+1\right)+C}=\mathrm{e}^C\left(\dfrac{\mathrm{e}^x}{x^2+x+1}\right)$ $y=\underbrace{\pm\mathrm{e}^C}_{A}\left(\dfrac{\mathrm{e}^x}{x^2+x+1}\right)=\dfrac{A\mathrm{e}^x}{x^2+x+1}$
	(ii)	Put $x = 0$, $y = 1$ into GS gives $A=1$. Thus $y=\dfrac{\mathrm{e}^x}{x^2+x+1}$. Note that $y>0\ \forall x\in\mathbb{R}$.
	(iii)	Set $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ in the given DE gives $\left(x^2-x\right)y=0$ $\Rightarrow x^2-x=0$ since $y\neq 0$ $\Rightarrow x=0$ or 1 The stationary points are $(0,1)$ and $\left(1,\dfrac{1}{3}\mathrm{e}\right)$. We have $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(\dfrac{x^2-x}{x^2+x+1}\right)y$. Since $x^2+x+1, y>0$, we consider $x^2-x\begin{cases}>0, & x<0 \text{ or } x>1\\ =0, & x=0,1\\ <0, & 0<x<1\end{cases}$ Therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x}\begin{cases}>0, & x<0 \text{ or } x>1\\ =0, & x=0,1\\ <0, & 0<x<1\end{cases}$

		We infer that $(0,1)$ is a maximum and $\left(1,\frac{1}{3}\text{e}\right)$ a minimum point.
	(iv)	
	(v)	Differentiate $\left(x^2+x+1\right)\frac{\text{d}y}{\text{d}x}-\left(x^2-x\right)y=0$ w.r.t. x gives $\left(x^2+x+1\right)\frac{\text{d}^2y}{\text{d}x^2}+\left(2x+1\right)\frac{\text{d}y}{\text{d}x}-\left(x^2-x\right)\frac{\text{d}y}{\text{d}x}-\left(2x-1\right)y=0$ $\left(x^2+x+1\right)\frac{\text{d}^2y}{\text{d}x^2}-\left(x^2-3x-1\right)\frac{\text{d}y}{\text{d}x}-\left(2x-1\right)y=0$ $\left(x^2+x+1\right)\frac{\text{d}y}{\text{d}x}-\left(x^2-x\right)y=0\Rightarrow\frac{\text{d}y}{\text{d}x}=\frac{\left(x^2-x\right)y}{x^2+x+1}$ Substitute into above DE gives $\left(x^2+x+1\right)\frac{\text{d}^2y}{\text{d}x^2}-\left(x^2-3x-1\right)\left[\frac{\left(x^2-x\right)y}{x^2+x+1}\right]-\left(2x-1\right)y=0$ $\left(x^2+x+1\right)^2\frac{\text{d}^2y}{\text{d}x^2}-\left[\left(x^2-3x-1\right)\left(x^2-x\right)+\left(2x-1\right)\left(x^2+x+1\right)\right]y=0$ $\Rightarrow\left(x^2+x+1\right)^2\frac{\text{d}^2y}{\text{d}x^2}-\left(x^4-2x^3+3x^2+2x-1\right)y=0$ on simplifying.
	(vi)	Put $\frac{\text{d}^2y}{\text{d}x^2}=0$ into above DE gives $\left(x^4-2x^3+3x^2+2x-1\right)y=0\Rightarrow x^4-2x^3+3x^2+2x-1=0$ Define $\text{f}\left(x\right)=x^4-2x^3+3x^2+2x-1$. $\text{f}\left(0.3\right)=(0.3)^4-2(0.3)^3+3(0.3)^2+2(0.3)-1=-0.176<0$ $\text{f}\left(0.4\right)=(0.4)^4-2(0.4)^3+3(0.4)^2+2(0.4)-1=0.1776>0$ By the IVP, the equation $x^4-2x^3+3x^2+2x-1=0$ has a root in $0.3,0.4$. So *I* is near *P* and it is to the right of *P*. **Note:** The point *I* is called an **inflexion point** of *C*.

7	**(i)**	Firstly we assume $c_s \geq c_r$. Then $S - S' = a_r c_r - a_r c_s + a_s c_s - a_s c_r$ $= -a_r(c_s - c_r) + a_s(c_s - c_r)$ $= \underbrace{(a_s - a_r)}_{\geq 0}\underbrace{(c_s - c_r)}_{\geq 0}$ ≥ 0 since $a_s \geq a_r$ $(a_i\text{'s are in increasing order})$ and $c_s \geq c_r$. Hence $S \geq S'$. Next we assume $c_s \leq c_r$. Then similarly, we have $S - S' = \underbrace{(a_s - a_r)}_{\geq 0}\underbrace{(c_s - c_r)}_{\leq 0} \leq 0 \Rightarrow S \leq S'$.
	(ii)	Since $\sum_{i=1}^{n} a_i b_{n+1-i}$ is the sum of products of corresponding terms in the sequences $a_1, a_2, \ldots, a_n$ and $b_n, b_{n-1}, \ldots, b_1$ arranged in reverse order, it has the least value. Since $\sum_{i=1}^{n} a_i b_i$ is the sum of products of corresponding terms in the sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ arranged in the same increasing order, it has the largest value. By (i), we have $\sum_{i=1}^{n} a_i b_{n+1-i} \leq \sum_{i=1}^{n} a_i c_i \leq \sum_{i=1}^{n} a_i b_i$.
	(iii)	For the increasing sequence $a_1, a_2, \ldots, a_n$, we define the corresponding decreasing positive sequence $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$. By the Rearrangement Inequality, the sum $\sum_{i=1}^{n} \frac{a_i}{a_i} = n$ is least. Thus for any permutation $\frac{1}{b_1}, \frac{1}{b_2}, \ldots, \frac{1}{b_n}$ of $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$, $\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n$.
	(iv)	Define the sequence $b_i \;{}_{i=1,2,\ldots,n} = \left(\frac{1}{a_i}\right)_{i=1,2,\ldots,n}$. It is then clear that the sequences $a_i \;{}_{i=1,2,\ldots,n}$ and $b_i \;{}_{i=1,2,\ldots,n}$ are arranged in reverse order since the larger the a_i, the smaller the b_i and vice versa.

		By the rearrangement inequality, $a_1b_n + a_2b_1 + a_3b_2 + \ldots + a_nb_{n-1} \geq a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n$ $\frac{x_1}{c}\cdot\frac{c^n}{x_1x_2\cdots x_n} + \frac{x_1x_2}{c^2}\cdot\frac{c}{x_1} + \frac{x_1x_2x_3}{c^3}\cdot\frac{c^2}{x_1x_2} + \ldots + \frac{x_1x_2\cdots x_n}{c^n}\cdot\frac{c^{n-1}}{x_1x_2\cdots x_{n-1}}$ $\geq a_1\cdot\frac{1}{a_1} + a_2\cdot\frac{1}{a_2} + a_3\cdot\frac{1}{a_3} + \ldots + a_n\cdot\frac{1}{a_n}$ $\Rightarrow \frac{x_1}{c} + \frac{x_2}{c} + \frac{x_3}{c} + \ldots + \frac{x_n}{c} \geq n$ $\Rightarrow \frac{1}{n}\ x_1 + x_2 + \ldots + x_n\ \geq c$ $\Rightarrow \frac{1}{n}\ x_1 + x_2 + \ldots + x_n\ \geq \sqrt[n]{x_1x_2\cdots x_n}$ **Alternative Solution:** Define the sequence $b_1, b_2, \ldots, b_n$ Such that $b_1 = 1, b_2 = \frac{x_1}{c}, \ldots, b_n = \frac{x_1x_2\cdots x_{n-1}}{c^{n-1}}$. By (iii), $\sum_{i=1}^{n}\frac{a_i}{b_i} \geq n \Rightarrow \frac{x_1}{c} + \frac{\frac{x_1x_2}{c^2}}{\frac{x_1}{c}} + \ldots + \frac{\frac{x_1x_2\cdots x_n}{c^n}}{\frac{x_1x_2\cdots x_{n-1}}{c^{n-1}}} \geq n$ $\Rightarrow \frac{x_1}{c} + \frac{x_2}{c} + \ldots + \frac{x_n}{c} \geq n$ $\Rightarrow \frac{1}{n}\ x_1 + x_2 + \ldots + x_n\ \geq c$ $\Rightarrow \frac{1}{n}\ x_1 + x_2 + \ldots + x_n\ \geq \sqrt[n]{x_1x_2\cdots x_n}$ as before

8		Let n be an integer such that $n \geq 2$. Since 2 is prime, n = 2 has a trivial prime factor, that is, itself. Assume that the integers 2, 3, … , k each has a prime factor. (IH) Consider the integer $n = k + 1$. <u>Case 1:</u> If $k+1$ is prime, then it has a trivial prime factor, that is, itself and we are done. <u>Case 2:</u> If $k+1$ is not prime, then it is composite and so it has a proper factor $m \in \{2, 3, \ldots, k\}$. But by (IH), any integer in the set $\{2, 3, \ldots, k\}$ has a prime factor and so m has a prime factor which is automatically a prime factor of $k+1$.

		By strong induction, every integer $n \geq 2$ has a prime factor.
	(i)	Since $i < j \Rightarrow \mathrm{f}_i < \mathrm{f}_j$, f_n is strictly increasing without bound as $n \to \infty$. So f_n takes infinitely many values and so there are infinitely many Fermat numbers.
	(ii)	Let proposition $\mathrm{P}(n)$: $\mathrm{f}_0\mathrm{f}_1\cdots\mathrm{f}_{n-1} = \mathrm{f}_n - 2$ for all $n \in \mathbb{Z}^+$. Since $\mathrm{f}_0 = 3 = 5 - 2 = \mathrm{f}_1 - 2$, $\mathrm{P}(1)$ is true. Assume $\mathrm{P}(k)$ is true for some $k \in \mathbb{Z}^+$. That is, $\mathrm{f}_0\mathrm{f}_1\cdots\mathrm{f}_{k-1} = \mathrm{f}_k - 2$ ------ (IH) Multiply both sides of (IH) by f_k gives $\mathrm{f}_0\mathrm{f}_1\cdots\mathrm{f}_{k-1}\mathrm{f}_k = \mathrm{f}_k \;\; \mathrm{f}_k - 2$ $= \; 2^{2^k} + 1 \quad 2^{2^k} + 1 \; - 2$ $= \; 2^{2^k} + 1 \;\; 2^{2^k} - 1$ $= \; 2^{2^k} \; ^2 - 1$ $= 2^{2^{k+1}} - 1$ $= 2^{2^{k+1}} + 1 - 2$ $= \mathrm{f}_{k+1} - 2$ proving that $\mathrm{P}(k+1)$ is true. This proves that $\mathrm{P}(n)$ is true.
	(iii)	Let *m* and *n* be distinct non-negative integers. Without loss of generality, assume $m \underset{\neq}{<} n$ and so $\mathrm{f}_m \underset{\neq}{<} \mathrm{f}_n$. Let $d = \gcd(\mathrm{f}_m, \mathrm{f}_n)$. Then by definition, $d \mid \mathrm{f}_m$ and $d \mid \mathrm{f}_n$. Since the non-negative integers *m* and *n* are such that $m < n$, $m \leq n - 1$. Since $d \mid \mathrm{f}_m$, $d \mid \mathrm{f}_0\mathrm{f}_1\cdots\mathrm{f}_{n-1}$ as f_m is amongst the numbers $\mathrm{f}_0, \mathrm{f}_1 \ldots, \mathrm{f}_{n-1}$. Since $d \mid \mathrm{f}_n$, we conclude that $d \mid (\mathrm{f}_n - \mathrm{f}_0\mathrm{f}_1\cdots\mathrm{f}_{n-1}) \Rightarrow d \mid 2$ since $\mathrm{f}_0\mathrm{f}_1\cdots\mathrm{f}_{n-1} = \mathrm{f}_n - 2$ by Duncan's identity. Thus $d = 1$ or 2. If *d* = 2, then this implies both f_m and f_n are even which is a contradiction since they are both odd. So *d* = 1. Hence $\gcd(\mathrm{f}_m, \mathrm{f}_n) = d = 1$ and thus f_m and f_n are relatively prime.

	(iv)	Since every integer $n \geq 2$ has a prime factor, every Fermat number (which is ≥ 3) has a prime factor. By Polya's assertion, any pair of Fermat numbers are relatively prime, meaning they have no common prime factors. Since there are infinitely many Fermat numbers, there must be infinitely many prime numbers.