| | Suggested Solution |
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| 1i | $\left(2x+3y+6z\right)^2 \le \left(2^2+3^2+6^2\right)\left(x^2+y^2+z^2\right)=49$<br>$2x+3y+6z\le 7$ |
| 1ii | $\begin{cases}2x+3y+6z=7\\x^2+y^2+z^2=1\end{cases}$<br>Since $2^2+3^2+6^2=7^2$, by observation, $x=\frac{2}{7}, y=\frac{3}{7}, z=\frac{6}{7}$. |
| 1iii | Suppose $\sum_{i=1}^{n} x_i^2 = 1$.<br>$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} 1\cdot x_i\right)^2 \le n\sum_{i=1}^{n} x_i^2 = n \Rightarrow \sum_{i=1}^{n} x_i \le \sqrt{n}$<br>Since if we let $x_i = \frac{1}{\sqrt{n}}$ for all $1\le i\le n$, we yield $\sum_{i=1}^{n} x_i^2 = n\left(\frac{1}{\sqrt{n}}\right)^2 = 1$ and $\sum_{i=1}^{n} x_i = n\left(\frac{1}{\sqrt{n}}\right)=\sqrt{n}$, the maximum possible value of $\sum_{i=1}^{n} x_i$ is $\sqrt{n}$. |
| 1iv | Suppose there are $n$ squares of lengths $x_i$ contained in the unit square.<br>Their total area is $\sum_{i=1}^{n} x_i^2 \le 1$, and their total perimeter is $18=\sum_{i=1}^{n} 4x_i$.<br>By part (iii), $\frac{18}{4}=\sum_{i=1}^{n} x_i \le \sqrt{n} \Rightarrow n\ge 20.25$.<br>Hence, there must be more than 20 such squares. |
| 2ia | Number of ways $=6\times 5^7 = 468750$ |
| 2ib | Label the 6 designs using 1, 2, 3, 4, 5 and 6. Let $A_i$ be the set of ways that the design $i$ is not used.<br>$\left\|A_1\cup A_2\cup A_3\cup A_4\cup A_5\cup A_6\right\|$<br>$=\sum\left\|A_i\right\|-\sum\left\|A_i\cap A_j\right\|+\sum\left\|A_i\cap A_j\cap A_k\right\|-\sum\left\|A_i\cap A_j\cap A_k\cap A_l\right\|+\sum\left\|A_i\cap A_j\cap A_k\cap A_l\cap A_m\right\|$<br>$-\left\|A_1\cap A_2\cap A_3\cap A_4\cap A_5\cap A_6\right\|$<br>$=\binom{6}{1}5^8-\binom{6}{2}4^8+\binom{6}{3}3^8-\binom{6}{4}2^8+\binom{6}{5}1^8-0$<br>$=1488096$<br>Required number<br>$=\left\|S\right\|-\left\|A_1\cup A_2\cup A_3\cup A_4\cup A_5\cup A_6\right\|$, where $S$ is the set of ways without restriction<br>$=6^8-1488096=191520$<br>**Alternative solution**<br>Considering that $8=3+1+1+1+1+1=2+2+1+1+1+1$, we can either have 3 of the same designs, or 2 pairs of 2 different designs.<br>Number of ways $= {}^6C_1\times\frac{8!}{3!}+{}^6C_2\times\frac{8!}{2!2!}=191520$ |
| 2iia | Number of ways $=\binom{20+5}{5}=\binom{25}{5}=53130$. |

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| 2iib | Number of ways with at least one of each design $=\binom{20-6+5}{5}=\binom{19}{5}=11628.$<br>Number of ways with at least one of each design and more than 6 for the particular design<br>$=\binom{20-12+5}{5}=\binom{13}{5}=1287.$<br>Number of ways with at least one of each design and at most 6 for the particular design<br>$=11628-1287=10341.$<br><br>**Alternative solution**<br>By considering the different number of wristbands of the particular design, i.e. 1, 2, …, 6, we will have 19, 18, …, 14 wristbands of other designs.<br>Number of ways $=\binom{18}{4}+\binom{17}{4}+\binom{16}{4}+\binom{15}{4}+\binom{14}{4}+\binom{13}{4}=10341$. |
| 3ai | Since all the sequences cannot start or end with a black bead, $a_2=9$ |
| 3aii | Case 1: 2nd bead is not black<br>There are $a_{n-1}$ ways to arrange beads of length $n-1$ from the 2nd bead to the last bead and there are 3 ways (red, blue or green) to insert the 1st bead.<br>Number of ways $=3a_{n-1}$.<br><br>Case 2: 2nd bead is black<br>There are $a_{n-2}$ ways to arrange beads of length $n-2$ from the 3rd bead to the last bead and there are 2 ways to insert the 1st bead as its colour must be different from that of the 3rd bead.<br>Number of ways $=2a_{n-2}$.<br>Total no of sequences $a_n=3a_{n-1}+2a_{n-2}$ |
| 3aiii | $a_n=3a_{n-1}+2a_{n-2}$<br>$=3(3a_{n-2}+2a_{n-3})+2a_{n-2}$<br>$=11a_{n-2}+6a_{n-3}$ ,<br>$\vdots$<br>$=xa_2+ya_1$ for some positive integers $x$ and $y$.<br>Since $a_1$ and $a_2$ are divisible by 3, $a_n$ is divisible by 3 for all positive integers $n$. |
| 3aiv | Let $P_n$ be the statement that $a_{2n}=9m$, where $m\in\mathbb{Z}$, for all positive integers $n$.<br>Since $a_2=9$ is divisible by 9.<br>$\therefore P_1$ is true.<br>Assume that $P_k$ is true for some positive integer $k$, i.e. $a_{2k}=9p$, where $p\in\mathbb{Z}$.<br>Check $P_{k+1}$<br>$a_{2k+2}=3a_{2k+1}+2a_{2k}$<br>$=3(3q)+2(9p)$ [From (iii), $a_{2k+1}=3q$, where $q\in\mathbb{Z}$ ]<br>$=9(q+2p)$<br>$a_{2k+2}$ is divisible by 9 since $q+2p\in\mathbb{Z}$.<br>$\therefore P_{k+1}$ is true.<br>Since $P_1$ is true and $P_k$ is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, $a_{2n}$ is divisible by 9 for all positive integers $n$. |
| 3b | There are $2^{10}-1=1023$ non-empty possible subsets (pigeons) of the set of 10 positive integers.<br>Since the smallest possible sum is 10 and the largest possible sum is 90 + 91 + 92 + … + 99 = 945, there are 936 possible sums (pigeonholes) in total for subsets of 10 integers.<br>Hence, by the pigeonhole principle, there are at least 2 non-empty subsets with the same sum. Call these two subsets $X$ and $Y$. |

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| | If $X$ and $Y$ are disjoint, we are done.<br>If not, let $W = X \cap Y$. Then $X \setminus W$ and $Y \setminus W$ are the two required sets. |
| 4ai | Since $p > 1$, $\frac{1}{(n+1)^p} < \frac{1}{n^p}$ for all positive integers $n$.<br>$\sum_{n=2^{N-1}}^{2^N-1} \frac{1}{n^p} = \frac{1}{(2^{N-1})^p} + \frac{1}{(2^{N-1}+1)^p} + \ldots + \frac{1}{(2^N-1)^p}$<br>$< (2^N - 1 - 2^{N-1} + 1) \times \frac{1}{(2^{N-1})^p}$<br>$= \frac{(2^N - 2^{N-1})}{(2^{N-1})^p} = \frac{2^{N-1}(2-1)}{(2^{N-1})^p} = \frac{1}{(2^{N-1})^{p-1}}$ |
| 4aii | $\sum_{n=1}^{2^N-1} \frac{1}{n^p} = 1 + \sum_{n=2^1}^{2^2-1} \frac{1}{n^p} + \sum_{n=2^2}^{2^3-1} \frac{1}{n^p} + \ldots + \sum_{n=2^{N-1}}^{2^N-1} \frac{1}{n^p}$<br>$< 1 + \frac{1}{(2^1)^{p-1}} + \frac{1}{(2^2)^{p-1}} + \ldots + \frac{1}{(2^{N-1})^{p-1}}$<br>$= 1 + \frac{1}{(2^{p-1})^1} + \frac{1}{(2^{p-1})^2} + \ldots + \frac{1}{(2^{p-1})^{N-1}}$<br>$< \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{1}{1 - 2^{1-p}}$<br>Since the partial sum is increasing and bounded above, it must converge by the monotone convergence theorem. |
| 4aiii | Since $\{a_n\}_{n \ge 1}$ converges, it must be bounded.<br>Let $M > 0$ be such that $\lvert a_n \rvert = \lvert n^p b_n \rvert < M$ for all $n \ge 1$.<br>Then $\lvert b_n \rvert < \frac{M}{n^p}$ for all $n \ge 1$.<br>From **(ii)**, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, so $\sum_{n=1}^{\infty} \frac{M}{n^p} = M \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.<br>By the comparison test, $\sum_{n=1}^{\infty} \lvert b_n \rvert$ converges. |
| 4(b) | Observe that the terms are always positive and hence it is an increasing sequence since $x_{i+1} - x_i = \frac{1}{i\,x_i} > 0$.<br>Assume that the sequence is bounded above by $M$, i.e. for all $i \ge 1$, $x_i \le M$ which gives $\frac{1}{x_i} \ge \frac{1}{M}$.<br>By the method of difference,<br>$x_{k+1} - x_1 = \sum_{i=1}^{k} (x_{i+1} - x_i) = \sum_{i=1}^{k} \frac{1}{i\,x_i} \ge \sum_{i=1}^{k} \frac{1}{iM} = \frac{1}{M} \sum_{i=1}^{k} \frac{1}{i}$.<br>Hence, $x_{k+1} \ge 1 + \frac{1}{M} \sum_{i=1}^{k} \frac{1}{i} > \frac{1}{M} \sum_{i=1}^{k} \frac{1}{i}$.<br>Since $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges, the sequence $\{x_k\}_{k \ge 1}$ is unbounded above – this contradicts our initial assumption.<br>Hence, the sequence is unbounded and the sequence diverges. |
| 5ai | If $n$ is even, then |

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| ii | $u_n = \frac{u_{n-2}}{n} = \frac{\frac{u_{n-4}}{n-2}}{n}$ <br> $= \frac{u_{n-4}}{n(n-2)} = \frac{\frac{u_{n-6}}{n-4}}{n(n-2)}$ <br> $= \frac{u_{n-6}}{n(n-2)(n-4)}$ <br> $= \ldots$ <br> $= \frac{u_0}{n(n-2)(n-4)\ldots(4)(2)}$ <br> $= \frac{1}{n(n-2)(n-4)\ldots(4)(2)}$ <br><br> Since, $n$ is even, then $n = 2k$ for some integer $k$. <br> $u_n = u_{2k} = \frac{1}{2k(2k-2)(2k-4)\ldots(4)(2)}$ <br> $= \frac{1}{2^k k(k-1)(k-2)\ldots(2)(1)}$ <br> $= \frac{1}{2^k k!}$ <br> $= \frac{1}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!}$ <br><br> If $n$ is odd, then <br> $u_n = \frac{u_{n-2}}{n} = \frac{\frac{u_{n-4}}{n-2}}{n}$ <br> $= \frac{u_{n-4}}{n(n-2)} = \frac{\frac{u_{n-6}}{n-4}}{n(n-2)}$ <br> $= \frac{u_{n-6}}{n(n-2)(n-4)}$ <br> $= \ldots$ <br> $= \frac{u_1}{n(n-2)(n-4)\ldots(3)(1)}$ <br> $= 0$ <br><br> $\therefore u_n = \begin{cases} 0 & \text{, if } n \text{ is odd} \\ \frac{1}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!} & \text{, if } n \text{ is even} \end{cases}$ |
| ii | $f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$ and $f'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$ . Hence |

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| | $f''(x) - xf'(x) - f(x) = 0$<br>$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$<br>$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$<br>$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=0}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$<br>$\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2} - nc_n - c_n\right]x^n = 0$<br>Therefore, for all $n \in \mathbb{Z}, n \geq 0$, we have<br>$(n+2)(n+1)c_{n+2} - nc_n - c_n = 0$<br>$(n+2)(n+1)c_{n+2} = (n+1)c_n$<br>$c_{n+2} = \frac{c_n}{n+2}$ |
| iii | Since $f(0) = 1$ and $f'(0) = 0$, we have $c_0 = 1$ and $c_1 = 0$. From (i), we have $c_{2k} = \frac{1}{2^k k!}$ and $c_{2k+1} = 0$.<br>$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} c_{2k} x^{2k} = \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}$<br>$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^2}{2}\right)^k$<br>$= e^{\frac{x^2}{2}}$ |
| 5(b) | The total number of squares is $mn$. We want the number of shaded squares to be $\frac{1}{2}mn$, but we also know that the number of shaded squares is $2m + 2n - 4$. Hence,<br>$2m + 2n - 4 = \frac{1}{2}mn$<br>$mn - 4m - 4n + 8 = 0$<br>$m(n-4) - 4(n-4) - 8 = 0$<br>$(m-4)(n-4) = 8$<br>Hence, both $(m-4)$ and $(n-4)$ must be positive integer divisors of 8. This means that these are the possible values of *m* and *n*:<br>$n - 4 = 1,2,4,8$ and $m - 4 = 1,2,4,8 \Rightarrow n = 5,6,8,12$ and $m = 5,6,8,12$<br>$\therefore (n,m) = (5,12),(6,8),(8,6)$ or $(12,5)$ |
| 6i | Let $x_1, x_2, ..., x_{2n}$ be the positions of the $(+1)$'s and $(-1)$'s. With this arrangement, we are able to find a consecutive pair of $+1$ and $-1$ in the clockwise direction (i.e. in the clockwise direction, the $+1$ precedes the $-1$). Remove this pair of numbers and there will be $2n-2$ numbers left arranged in the circle.<br>We will repeat this procedure by removing consecutive pairs of $+1$ and $-1$ in the clockwise direction until there is a final pair of $+1$ and $-1$ left. Let $x_k$ be the position of this final $+1$.<br>We claim that $x_k$ is the starting position for which $T_i$ is never negative.<br><br>Since the pairs of $+1$'s and $-1$'s removed were consecutive, and within each pair, the $+1$ preceded the $-1$, there will always be an increase of the partial sum $T_k$ prior to a decrease. Hence, $T_i$ is never negative for all $1 \leq i \leq 2n$. |

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| | **Alternative Solution**<br>Let $x_1, x_2, ..., x_{2n}$ be the positions of the $(+1)$'s and $(-1)$'s, and let $x_1$ be the starting position. As we evaluate $T_i$ for $1 \le i \le 2n$, there exists a $k$ such that $T_k$ is minimum. We then claim that $x_{k+1}$ is a starting position for which $T_i$ is never negative for all $1 \le i \le 2n$.<br><br>Relabel $x_{k+1}, ..., x_{2n}, x_1, ..., x_k$ as $y_1, y_2, ..., y_{2n}$ respectively, so $y_1 = x_{k+1}$ is the starting position. To avoid confusion, we shall let $S_i$ be the new partial sum from position $y_1$ to $y_i$.<br>Since $T_k$ is a minimum, $S_i = T_{i+k} - T_k \ge 0$ for $1 \le i \le 2n-k$.<br>With equal number of $(+1)$'s and $(-1)$'s, $T_{2n} = 0$. Hence, $S_{2n-k} = -T_k$.<br>Furthermore, $T_j \ge T_k \Rightarrow -T_k + T_j \ge 0$ for all $1 \le j \le k$, we have that $S_i = S_{2n-k} + T_{i-2n+k} = -T_k + T_{i-2n+k} \ge 0$ for $2n-k+1 \le i \le 2n$. |
| 6ii | Note that $T_i \equiv i \pmod 2$ due to the following. Enumerate $T_i$ starting with the index $i=1$. We have $T_1 \equiv 1 \pmod 2$ regardless of the first value, $+1$ or $-1$. Subsequently, as the index *i* increases each time by 1, we add $+1$ or $-1$ to the value of $T_i$, changing its parity. Hence, $n + \sum_{i=1}^{2n} T_i \equiv n + \sum_{i=1}^{2n} i \equiv n + n = 2n \equiv 0 \ (\text{mod}2)$; i.e. $n + \sum_{i=1}^{2n} T_i$ is even. |
| 7i | $a > c\cos\theta + d\sin\theta$ and $b > c\sin\theta + d\cos\theta$ |
| 7ii | "$\Rightarrow$" (Proof by contraposition.) Suppose on the contrary that $d \ge b$, then $a > c \ge d \ge b$ and<br>$d\cos\theta + c\sin\theta \ge b(\cos\theta + \sin\theta) = b\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) > b$ since $0 < \theta < \frac{\pi}{2} \Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) > \frac{1}{\sqrt{2}}$.<br>"$\Leftarrow$" Choose $\theta$ small enough such that $c\sin\theta < \varepsilon := \min(a-c, b-d)$.<br>Then $c\cos\theta + d\sin\theta \le c\cos\theta + c\sin\theta < c + \varepsilon \le a$ and $d\cos\theta + c\sin\theta < d + \varepsilon \le b$. |
| 7iii | Let $\theta_0$ be the angle for which the $c \times d$ rectangle is strictly contained in the $a \times b$ rectangle. By (i), we must have $c\cos\theta_0 + d\sin\theta_0 < a$ and $c\sin\theta_0 + d\cos\theta_0 < b \Rightarrow c\cos\left(\frac{\pi}{2} - \theta_0\right) + d\sin\left(\frac{\pi}{2} - \theta_0\right) < b \le a$.<br><br>Let $\text{f}(\theta) = c\cos\theta + d\sin\theta$, $\theta \in \left[0, \frac{\pi}{2}\right]$, $\theta_1 = \min\left(\theta_0, \frac{\pi}{2} - \theta_0\right)$ and $\theta_2 = \max\left(\theta_0, \frac{\pi}{2} - \theta_0\right)$. Then we must have $\text{f}(\theta_1) < a$, $\text{f}(\theta_2) < a$ and $\frac{\pi}{4} \in [\theta_1, \theta_2]$.<br>If we can prove that f is decreasing on $[\theta_1, \theta_2]$, then $a > \text{f}\left(\frac{\pi}{4}\right) = \frac{c+d}{\sqrt{2}}$ and we are done.<br><br>Since $\text{f}(\theta) > 0$ and $\text{f}''(\theta) = -c\cos\theta - d\sin\theta < 0$ for $\theta \in \left[0, \frac{\pi}{2}\right]$, f has a maximum at $\theta_{\max}$, where<br>$\text{f}'(\theta_{\max}) = -c\sin\theta_{\max} + d\cos\theta_{\max} = 0 \Rightarrow \theta_{\max} = \tan^{-1}\frac{d}{c} \le \frac{\pi}{4}$.<br><br>Since $\text{f}(0) = c \ge a$ and f is increasing on $[0, \theta_{\max}]$, we must have $\theta_{\max} < \theta_1$. Since f is decreasing on $\left[\theta_{\max}, \frac{\pi}{2}\right]$, it is also decreasing on $[\theta_1, \theta_2]$ and we are done. |
| 7iv | A $c \times d$ rectangle (with $c \ge d$) can be strictly contained in an $a \times a$ square if and only if $a > c$ or $a\sqrt{2} > c + d$.<br><br>Proof<br>$(\Rightarrow)$: Suppose that a $c \times d$ rectangle (with $c \ge d$) can be strictly contained in an $a \times a$ square. If $a > c$, we are done. Otherwise if $a \le c$, then by (iii), we have $a\sqrt{2} > c + d$. |

$(\Leftarrow)$: Suppose $a > c$ or $a\sqrt{2} > c + d$.

If $a > c$, then by (ii), a $c \times d$ rectangle can be strictly contained in an $a \times a$ square iff $a > d$. But since $a > c \ge d$, the rectangle can always be contained in the square.

If $a\sqrt{2} > c + d$, substituting $\theta = \frac{\pi}{4}$ into the inequalities in (i) yields $c\cos\frac{\pi}{4} + d\sin\frac{\pi}{4} = \frac{c+d}{\sqrt{2}} < a$ and $c\sin\frac{\pi}{4} + d\cos\frac{\pi}{4} = \frac{c+d}{\sqrt{2}} < a$. This shows that the rectangle can be contained in the $a \times a$ square.

**8i**

| $n$ (mod 12) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n^2$ (mod 12) | 0 | 1 | 4 | 9 | 4 | 1 | 0 | 1 | 4 | 9 | 4 | 1 |

By exhaustion, the elements of S(12) are all square numbers.

**8ii**

Let $N = 9$. Since $4^2 = 16 \equiv 7 \pmod 9$, $7 \in \mathrm{S}(9)$ and 7 is not a square number.

**8iii**

Since $(N + r)^2 \equiv r^2 \pmod N$ for all $r \in \mathbb{Z}$ with $0 \le r < \sqrt{N}$, there are at least $\sqrt{N}$ unique $r$'s.

Hence, $\left\{0,1,4,\ldots,\left(\lfloor \sqrt{N-1} \rfloor\right)^2\right\} \subseteq \mathrm{S}(N)$ and $\mathrm{S}(N)$ has at least $\sqrt{N}$ elements.

NB *We use* $\lfloor \sqrt{N-1} \rfloor$ *instead of* $\lfloor \sqrt{N} \rfloor$ *to be precise, since if N is a square number, then* $\lfloor \sqrt{N} \rfloor^2 = N$ *is not an element of* $S(N)$.

**8iv**

Suppose $\exists x, n, \lambda \in \mathbb{Z}$ such that $x^2 = 17 + 2^n\lambda$, with $n \ge 5$.

If $\lambda$ is even, then $x^2 = 17 + 2^{n+1}\left(\frac{\lambda}{2}\right) \Rightarrow x^2 \equiv 17 \pmod{2^{n+1}}$, i.e. $17 \in \mathrm{S}\left(2^{n+1}\right)$.

If $\lambda$ is odd, let $\mu = \dfrac{\lambda + x + 2^{n-2}}{2}$. Note that $\mu \in \mathbb{Z}$ since $\lambda$, $x$ are odd and $n \ge 5$.

$$\begin{aligned} \text{Then } 2^{n+1}\mu + 17 &= 2^n\left(\lambda + x + 2^{n-2}\right) + 17 \\ &= x^2 + 2^n x + 2^{2n-2} \\ &= \left(x + 2^{n-1}\right)^2. \end{aligned}$$

Hence, $\left(x + 2^{n-1}\right)^2 \equiv 17 \pmod{2^{n+1}}$, i.e. $17 \in \mathrm{S}\left(2^{n+1}\right)$.

**8v**

Let $\mathrm{P}_n$ be the statement: $17 \in \mathrm{S}\left(2^n\right), n \ge 5$.

For $n = 5$, $7^2 = 49 \equiv 17 \pmod{2^5}$. Hence, $\mathrm{P}_5$ is true.

By (iv), we have shown that $17 \in \mathrm{S}\left(2^n\right) \Rightarrow 17 \in \mathrm{S}\left(2^{n+1}\right)$, i.e., $\mathrm{P}_n \Rightarrow \mathrm{P}_{n+1}$ for $n \ge 5$.

By induction, $\mathrm{P}_n$ is true for all $n \ge 5$.

From (iii), we have shown that there are at least $\sqrt{N}$ numbers in $\mathrm{S}(N)$, where all these elements are square numbers. Since 17 is not a square number, and is also in $\mathrm{S}\left(2^n\right)$ for $n \ge 5$, then we must have at least $1 + \sqrt{2^n}$ elements in $\mathrm{S}\left(2^n\right)$.

<u>**Alternative Solution**</u>

Case 1: $n$ is even

$$\left(2^{\frac{n}{2}} + 1\right)^2 = 2^n + 2^{\frac{n}{2}+1} + 1 \equiv 2^{\frac{n}{2}+1} + 1 \pmod{2^n}$$

Then $\left\{1^2, 2^2, \ldots, 0 \equiv \left(2^{\frac{n}{2}}\right)^2, 2^{\frac{n}{2}+1}+1\right\} \subseteq \mathrm{S}\left(2^n\right)$, and this shows that $\mathrm{S}\left(2^n\right)$ has at least $1+\sqrt{2^n}$ elements.

Case 2: $n$ is odd

$$\left(2^{\frac{n+1}{2}}+1\right)^2 = 2^{n+1}+2^{\frac{n+3}{2}}+1 \equiv 2^{\frac{n+3}{2}}+1 \text{ (mod } 2^n\text{)}$$

Then $\left\{0^2, 1^2, 2^2, \ldots, \left(\left\lfloor 2^{\frac{n}{2}} \right\rfloor\right)^2, 2^{\frac{n+3}{2}}+1\right\} \subseteq \mathrm{S}\left(2^n\right)$, and this shows that $\mathrm{S}\left(2^n\right)$ has at least $1+\sqrt{2^n}$ elements.

Note that in both cases above, $2^{\frac{n}{2}+1}+1$ and $2^{\frac{n+3}{2}}+1$ are not square numbers.
Suppose that $p^2 = 2^q+1$ for some $p, q \in \mathbb{Z}^+$, then $(p+1)(p-1) = 2^q$.
This implies that $p+1 = 2^k$ and $p-1 = 2^l$ for some $k, l \in \mathbb{Z}^+$ with $k+l = q$.
Hence, $2^k - 1 = p = 2^l + 1 \Rightarrow 2\left(2^{l-1}+1\right) = 2^k \Rightarrow l = 1, k = 2 \Rightarrow q = 3$.

But $n \geq 5 \Rightarrow \frac{n}{2}+1, \frac{n+3}{2} \geq 4$, which says that $2^{\frac{n}{2}+1}+1$ and $2^{\frac{n+3}{2}}+1$ are not square numbers.