NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 3

MATHEMATICS **9820/01**

Paper 1 **21st September 2020**

3 Hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Use the substitution $x = \dfrac{1}{t}$ to show that

$$\int_1^{\infty} \frac{1}{\left(1+x^2\right)\left(1+x^{\alpha}\right)}\,\mathrm{d}x = \int_0^{1} \frac{x^{\alpha}}{\left(1+x^2\right)\left(1+x^{\alpha}\right)}\,\mathrm{d}x\,.$$

where α is a real number. [1]

Hence evaluate exactly

$$\int_0^{\infty} \frac{1}{\left(1+x^2\right)\left(1+x^{\alpha}\right)}\,\mathrm{d}x\,.$$ [4]

Deduce the exact value of

$$\int_0^{\infty} \frac{3x^n - x^m + 2}{\left(1+x^2\right)\left(1+x^m\right)\left(1+x^n\right)}\,\mathrm{d}x$$

where $m, n \in \mathbb{R}$. [3]

2 Let S be the set of integer points on the cartesian plane defined by

$$S = \{(i, j) : i, j \in \mathbb{Z}\}.$$

Any two distinct points (a,b) and (c,d) belonging to S are said to be **mutually visible** if there is no integer point on the line segment joining them.
The transformation T defined on S is given by

$$T : (x, y) \longmapsto (x-a, y-b).$$

Use the above transformation to explain why the distinct points (a,b) and (c,d) in S are mutually visible if and only if the points $(0,0)$ and $(c-a, d-b)$ in S are mutually visible. [2]

Hence prove that the distinct points (a,b) and (c,d) are mutually visible if and only if

$$\gcd(c-a, d-b) = 1.$$ [8]

3 **(i)** By considering $\ln\left(1+\frac{x}{n}\right)^n$, show that for a given $x \in \mathbb{R}$,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \mathrm{e}^x ,$$

justifying each step of your working. [4]

Let x be a given real number. For each m = 0, 1, 2, …, define a sequence $\{a_m(j)\}$, $j = 0,1,2,\ldots$ by the following conditions:

$$a_m(0) = \frac{x}{2^m} \text{ and } a_m(j+1) = \left[a_m(j)\right]^2 + 2a_m(j),\ j = 0,1,2,\ldots .$$

(ii) By writing $b_m(j) = a_m(j)+1$, show that

$$b_m(j+1) = \left[b_m(j)\right]^2 .$$ [2]

(iii) Hence evaluate $\lim_{n\to\infty} a_n(n)$ in terms of *x*. [4]

4 **(i)** John repeatedly tosses a biased coin. Each time he scores one point for a head and two points for a tail. The probability of getting a head is $\frac{2}{3}$. Let p_n denotes the probability that John's score is *n* points at some point. Show that $p_n = \frac{1}{3}p_{n-1} + \frac{2}{3}p_{n-2}$ for $n \geq 3$, and hence find p_7. [3]

(ii) John decides to stop when the number of heads and the number of tails he obtains differ by **two**. In a series of *n* successive tosses, he is still not able to stop. Supposed that *n* is even. By using a suitable bijection, show that the number of possible sequences of heads and tails for the *n* tosses is $2^{\frac{n}{2}}$. [4]

(iii) John then bet against himself on the results of his coin toss over a sequence of 15 tosses. For each toss, he bet at least one dollar and he can choose to bet an integer number of dollars. He ended up betting a total of 20 dollars. Show that there was a sequence of consecutive tosses where he bet exactly 9 dollars. [4]

5 **(a)** A retailer sells 3 different toys in mystery boxes, one to a box. Each toy is equally likely to be in any box. If a person buys 6 boxes, what is the probability of getting all 3 different toys? [3]

(b) **(i)** A partition of a positive integer r is a way of writing r as a sum of positive integers, where ordering of the integers is not considered. If $r = r_1 + r_2 + \ldots + r_n$ where $r_1 \geq r_2 \geq \ldots \geq r_n \geq 1$, we say that r is partitioned into n parts of sizes $r_1, r_2, \ldots, r_n$ respectively.

Prove that the number of non-negative integer solutions of

$$x_1 + 2x_2 + 3x_3 + \ldots + nx_n = r$$

is given by the number of partitions of r into parts at most equal to n. [4]

(ii) A boy has 13 identical marbles. He decides to give them away to three of his friends A, B and C such that A gets any number of marbles, B gets marbles in bundles of 2 and C gets marbles in bundles of 3 and none of the friends leave empty-handed. Find the number of ways the boy can do so. [3]

(c) A student takes 4 different modules for a particular subject and he intends to revise the content for the modules over 6 consecutive weeks. Determine the number of ways for the student to do his revision if he chooses to focus on exactly one module a week and

(i) he must revise at least one module, [1]

(ii) he does not spend consecutive weeks on each module, [2]

(iii) he must revise every module. [4]

6 **(i)** Find the general solution of the differential equation

$$\left(x^2+x+1\right)\frac{\mathrm{d}y}{\mathrm{d}x}-\left(x^2-x\right)y=0 \text{ where } y\neq 0,$$

expressing y explicitly as a function x. [4]

(ii) Hence find the equation of the solution curve C that passes through $(0,1)$. [2]

(iii) Show that C has two stationary points and determine their nature by considering the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for appropriate range of values of x. [4]

(iv) Sketch the curve C. [1]

(v) Show that

$$\left(x^2+x+1\right)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\left(x^4-2x^3+3x^2+2x-1\right)y=0.$$ [3]

(vi) P is a point on C at which $x=0.3$. Use the intermediate value property for continuous functions to show that there is a point I on C, near P, at which $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ and state whether I is to the left or right of P. [3]

7 Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two increasing positive sequences and let the sequence $c_1, c_2, \ldots, c_n$ be a permutation of the sequence $b_1, b_2, \ldots, b_n$.
Define the sums

$$S = a_1c_1 + a_2c_2 + \ldots + a_rc_r + \ldots + a_sc_s + \ldots + a_{n-1}c_{n-1} + a_nc_n$$
$$S' = a_1c_1 + a_2c_2 + \ldots + a_rc_s + \ldots + a_sc_r + \ldots + a_{n-1}c_{n-1} + a_nc_n$$

where the a_i's in S and S' are arranged in increasing order and S' is obtained from S by interchanging c_r and c_s while keeping the rest fixed.

(i) Prove that if $c_r \le c_s$, then $S \ge S'$ and if $c_r \ge c_s$, then $S \le S'$. [3]

(ii) Hence explain why the following **Rearrangement Inequalities** hold:

$$\sum_{i=1}^{n} a_i b_{n+1-i} \le \sum_{i=1}^{n} a_i c_i \le \sum_{i=1}^{n} a_i b_i \,.$$ [2]

(iii) Use the Rearrangement Inequality to prove that for the increasing positive sequence $a_1, a_2, \ldots, a_n$,

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \ge n$$

where the sequence $b_1, b_2, \ldots, b_n$ is a permutation of the sequence $a_1, a_2, \ldots, a_n$. [3]

(iv) Let $x_1, x_2, \ldots, x_n$ be a positive sequence and define

$$c = \sqrt[n]{x_1x_2\cdots x_n},\ a_1 = \frac{x_1}{c},\ a_2 = \frac{x_1x_2}{c^2},\ a_3 = \frac{x_1x_2x_3}{c^3}, \ldots,\ a_n = \frac{x_1x_2\cdots x_n}{c^n}.$$

By defining a suitable sequence $b_1, b_2, \ldots, b_n$, use the Rearrangement Inequality to prove the following AM-GM Inequality:

$$\frac{1}{n}\left(x_1 + x_2 + \ldots + x_n\right) \ge \sqrt[n]{x_1x_2\cdots x_n}\,.$$ [4]

8 Use strong induction to prove that every integer $n \geq 2$ has a prime factor. [4]

In 1640, the French mathematician Pierre de Fermat (1607 – 1665) studied what is now known as **Fermat numbers**, which are numbers of the form

$$\mathrm{f}_n = 2^{2^n} + 1, \quad n = 0, 1, 2, \ldots.$$

The first two Fermat numbers are

$$\mathrm{f}_0 = 2^{2^0} + 1 = 3, \ \mathrm{f}_1 = 2^{2^1} + 1 = 5 \text{ etc.}$$

(i) Explain why there are infinitely many Fermat numbers. [1]

In 1964, D.C. Duncan discovered the following identity, known as **Duncan's identity**:

$$\mathrm{f}_0 \mathrm{f}_1 \cdots \mathrm{f}_{n-1} = \mathrm{f}_n - 2$$

where $n \geq 1$.

(ii) Prove Duncan's identity by induction. [4]

In 1925 the Hungarian mathematician George Polya asserted that if m and n are distinct non-negative integers, then f_m and f_n are relatively prime.

(iii) Use Duncan's identity to prove Polya's assertion. [4]

(iv) Hence argue that there are infinitely many prime numbers. [2]