Qn	Solutions
1	Let P(n) be the proposition "$\left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\ldots\left(1-\frac{1}{\sqrt{n}}\right)<\frac{2}{n^2}$" for integers $n \geq 2$. When $n=2$, $\text{LHS}=1-\frac{1}{\sqrt{2}}=0.293<\frac{1}{2}=$ RHS. Therefore, P(2) is true. Assume P(k) is true for some arbitrary positive integer $k \geq 2$, i.e. $\left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\ldots\left(1-\frac{1}{\sqrt{k}}\right)<\frac{2}{k^2}$. When $n=k+1$, $\text{LHS}=\left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\ldots\left(1-\frac{1}{\sqrt{k}}\right)\left(1-\frac{1}{\sqrt{k+1}}\right)$ $<\frac{2}{k^2}\left(1-\frac{1}{\sqrt{k+1}}\right)$ $=\frac{2}{k^2}\left(\frac{\sqrt{k+1}-1}{\sqrt{k+1}}\right)\left(\frac{\sqrt{k+1}+1}{\sqrt{k+1}+1}\right)$ $=\frac{2}{k^2}\left(\frac{k}{k+1+\sqrt{k+1}}\right)$ $=\frac{2}{k}\left(\frac{1}{k+1+\sqrt{k+1}}\right)$ $=\frac{2}{k(k+1)+k\sqrt{k+1}}$ $<\frac{2}{k(k+1)+\sqrt{k+1}\sqrt{k+1}}$ (*) $=\frac{2}{(k+1)^2}=\text{RHS}$ (*) holds because: $k^2-2k+1=(k-1)^2 \geq 0$ $\Rightarrow k^2 \geq 2k-1=k+k-1 \geq k+2-1=k+1$ as $k \geq 2$. Therefore $k>\sqrt{k+1}$. Since P(2) is true, and if P(k) is true then P(k +1) is true, by Mathematical Induction, P(n) holds for all integers $n \geq 2$.

2	Let $a = \gcd(a_1,\ a_2, \ldots,\ a_n)$ and $b = \gcd\left(\frac{a_1 + a_2}{2},\ \frac{a_2 + a_3}{2}, \ldots,\ \frac{a_n + a_1}{2}\right)$. We shall show that $a \mid b$ and $b \mid a$. To show $a \mid b$: There exists integers α_k such that $a_k = a\alpha_k$ for $k = 1, 2, \ldots, n.$ Therefore, $$\frac{a_k + a_{k+1}}{2} = \frac{\alpha_k + \alpha_{k+1}}{2} a,$$ where $k = 1, 2, \ldots, n$ and we take $a_{n+1} = a_1$ and $\alpha_{n+1} = \alpha_1$. Since $a_1,\ a_2, \ldots,\ a_n$ are odd integers, it follows that $\alpha_1,\ \alpha_2, \ldots,\ \alpha_n$ are also odd, and $\frac{\alpha_k + \alpha_{k+1}}{2} \in \mathbb{Z}$. Hence a divides each of $\frac{a_1 + a_2}{2},\ \frac{a_2 + a_3}{2}, \ldots,\ \frac{a_n + a_1}{2}$ and $a \mid b.$ To show $b \mid a$: There exists integers β_k such that $\frac{a_k + a_{k+1}}{2} = b\beta_k$ for $k = 1, 2, \ldots, n$ and $a_{n+1} = a_1$. From $a_k + a_{k+1} = 2b\beta_k$, we see that b divides $a_k + a_{k+1}$. --- (1) Taking the sum from $k = 1$ to $k = n,$ $\frac{a_1 + a_2}{2} + \frac{a_2 + a_3}{2} + \ldots + \frac{a_n + a_1}{2} = b(\beta_1 + \beta_2 + \ldots + \beta_n)$ i.e. $a_1 + a_2 + \ldots + a_n = b(\beta_1 + \beta_2 + \ldots + \beta_n)$ Hence, b divides $a_1 + a_2 + \ldots + a_n$ --- (2) Taking the sum for $k = 1, 3, 5, \ldots, n-2,$ (note that $n-2$ is odd since n is odd) $\frac{a_1 + a_2}{2} + \frac{a_3 + a_4}{2} + \ldots + \frac{a_{n-2} + a_{n-1}}{2} = b(\beta_1 + \beta_3 + \ldots + \beta_{n-2})$ $a_1 + a_2 + \ldots + a_{n-1} = 2b(\beta_1 + \beta_3 + \ldots + \beta_{n-2})$ Hence, b divides $a_1 + a_2 + \ldots + a_{n-1}$ ---(3) From (2) and (3), we have b divides a_n. Using (1) repeatedly, we can conclude that b divides a_k for $k = 1, 2, \ldots, n$. Hence $b \mid a.$

3 (i)	$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}\,x = \int_{1/m}^{m} (x-1) + \frac{1}{x+1}\,\mathrm{d}\,x$ $= \left[\frac{x^2}{2} - x + \ln(x+1)\right]_{1/\mathrm{m}}^{m}$ $= \frac{m^4 - 1 - 2m(m^2-1)}{2m^2} + \ln\left(\frac{m+1}{\frac{1}{m}+1}\right)$ $= \frac{\left(m^2-1\right)\left(m^2-2m+1\right)}{2m^2} + \ln\left(\frac{m^2+m}{m+1}\right)$ $= \frac{(m-1)^3(m+1)}{2m^2} + \ln m$
3 (ii)	$x = \frac{1}{u}, \quad \frac{\mathrm{d}\,x}{\mathrm{d}\,u} = -\frac{1}{u^2}$ $\int_b^a \frac{1}{x^n(x+1)}\,\mathrm{d}\,x = \int_{1/b}^{1/a} \frac{1}{\left(\frac{1}{u}\right)^n\left(\frac{1}{u}+1\right)}\left(-\frac{1}{u^2}\right)\mathrm{d}\,u$ $= -\int_{1/b}^{1/a} \frac{u^{n-2}}{\left(\frac{1}{u}+1\right)}\,\mathrm{d}\,u = \int_{1/a}^{1/b} \frac{u^{n-2}}{\left(\frac{1}{u}+1\right)}\,\mathrm{d}\,u$ $= \int_{1/a}^{1/b} \frac{u^{n-1}}{(u+1)}\,\mathrm{d}\,u$
3 (iii)	$\int_1^2 \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}\,x = \int_1^2 \frac{x^5}{x^3(x+1)}\,\mathrm{d}\,x + \int_1^2 \frac{x^3}{x^3(x+1)}\,\mathrm{d}\,x + \int_1^2 \frac{1}{x^3(x+1)}\,\mathrm{d}\,x$ $= \int_1^2 \frac{x^2}{(x+1)}\,\mathrm{d}\,x + \int_1^2 \frac{1}{(x+1)}\,\mathrm{d}\,x + \int_1^2 \frac{1}{x^3(x+1)}\,\mathrm{d}\,x$ $= \int_1^2 \frac{x^2}{(x+1)}\,\mathrm{d}\,x + \left[\ln(1+x)\right]_1^2 + \int_1^2 \frac{1}{x^3(x+1)}\,\mathrm{d}\,x$ $= \int_1^2 \frac{x^2}{(x+1)}\,\mathrm{d}\,x + \ln\frac{3}{2} + \int_{1/2}^{1} \frac{u^2}{(u+1)}\,\mathrm{d}\,u$ $= \int_{1/2}^{2} \frac{x^2}{(x+1)}\,\mathrm{d}\,x + \ln\frac{3}{2}$ $= \frac{(2-1)^3(2+1)}{2\times 2^2} + \ln 2 + \ln\frac{3}{2}$ $= \frac{3}{8} + \ln 3$

4 (a)	Since x and y are odd, there exists integers m and n such that $x=2m+1$ and $y=2n+1$. Then $x^2+y^2=(2m+1)^2+(2n+1)^2$ $\quad=4\left(m^2+m+n^2+n\right)+2$ $\quad\equiv 2 \pmod 4$ Note that x^2+y^2 is even. For even perfect squares, they take the form $(2k)^2=4k^2\equiv 0 \pmod 4$. Hence x^2+y^2 cannot be a perfect square.
4 (b)	Since n is composite, there exists numbers p and q such that $n=pq$. Note that $1<p,q<n$. Case 1: $p\neq q$ Since p and q are distinct and both are less than n, both show up as different factors in $(n-1)!$. Hence $n=pq\mid(n-1)!$. Case 2: $p=q$ i.e. $n=p^2$ If $p=2$, $n=4$ and the result does not hold as 4 does not divide $3!=6$. If $p>2$, then $1<2p<p^2$. As $2p\leq p^2-1=n-1$, $2p$ is one of the factors in $(n-1)!$. Note that p is another factor in $(n-1)!$. This implies that $2p^2\mid(n-1)!$. Since $n=p^2\mid 2p^2$, we have $n\mid(n-1)!$.

5 (i)	r_{n+1} is the number of ways the first and last post are painted red, which means that the nth post must be either white or green. Hence if we remove the $(n+1)$ th post, then we will have the number of ways that the first post is red and the last (nth) post is either white or green. $r_{n+1}+r_n=s_n+r_n=2^{n-1}$, since this will be the number of ways the first post out of n posts is red.

5 (ii)	Let P_n be the proposition that $r_n = \dfrac{2^{n-1}+2(-1)^{n-1}}{3}$ is true for all $n \in \mathbb{Z}^+$ Consider P_1: $r_1 = 1$ $\dfrac{2^{1-1}+2(-1)^{1-1}}{3} = \dfrac{1+2}{3} = 1 = r_1$ Hence P_1 is true Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $r_k = \dfrac{2^{k-1}+2(-1)^{k-1}}{3}$ Consider P_{k+1}: $r_{k+1} + r_k = 2^{k-1}$ $\Rightarrow r_{k+1} = 2^{k-1} - \dfrac{2^{k-1}+2(-1)^{k-1}}{3}$ $= \dfrac{3\left(2^{k-1}\right) - 2^{k-1} - 2(-1)^{k-1}}{3}$ $= \dfrac{2\left(2^{k-1}\right) + 2(-1)(-1)^{k-1}}{3}$ $= \dfrac{2^k + 2(-1)^k}{3}$ Therefore P_k is true $\Rightarrow$ P_{k+1} is true for some $k \in \mathbb{Z}^+$ Since P_1 is also true, by MI, P_n is true for all $n \in \mathbb{Z}^+$
5 (iii)	$r_{n+1} = \dfrac{2^n + 2(-1)^n}{3}$, and we can then 'merge' the first and last red post together to form a circle of *n* posts with no adjacent posts of the same colours. However, as the first post (take the widest fence post) can be of 3 colours, no of ways = $3 \times \dfrac{2^n + 2(-1)^n}{3} = 2^n + 2(-1)^n$

<table>
<tr><td>6</td><td>This is the equivalent of finding how many ways to partition the number ‘8’:

<u>Case I (with max value of 1)</u>

1, 1, 1, 1, 1, 1, 1, 1

<u>Case II (with max value of 2)</u>

2, 1, 1, 1, 1, 1, 1
2, 2, 1, 1, 1, 1,
2, 2, 2, 1, 1
2, 2, 2, 2

<u>Case III (with max value of 3)</u>

3, 1, 1, 1, 1, 1
3, 2, 1, 1, 1
3, 2, 2, 1
3, 3, 1, 1
3, 3, 2

<u>Case IV (with max value of 4)</u>

4, 1, 1, 1, 1
4, 2, 1, 1
4, 2, 2,
4, 3, 1
4, 4

<u>Case V (with max value of 5)</u>

5, 1, 1, 1
5, 2, 1
5, 3
<u>Case VI (with max value of 6)</u>

6, 1, 1
6, 2

<u>Case VII (with max value of 7 or 8)</u>

7, 1
8

Total number of ways = 22</td></tr>
</table>

6 (i)	No of ways to distribute remaining 8 books to 3 shelves such that no shelves have 8 books $= {}^{8+3-1}\mathrm{C}_{3-1} - 3 = 42$ Therefore no of ways = $42 \times 4 = 168$
6 (ii)	Total number of ways with no restrictions $= {}^{16+4-1}\mathrm{C}_{4-1} = 969$ Number of ways with one shelf having at least 9 books $= 4\left({}^{7+4-1}\mathrm{C}_{4-1}\right) = 480$ Hence no of ways with all shelves at most 8 books $= 969 - 480 = 489$
6 (iii)	Let $2a$ be the number of books on first shelf Let $2b$ be the number of books on second shelf Let $2c-1$ be the number of books on third shelf Let $2d-1$ be the number of books on fourth shelf Since all shelves are not empty, a,b,c,d are all bigger than or equal to 1. We have $2a+2b+2c-1+2d-1=16$ $\Rightarrow a+b+c+d=9$ So we let a,b,c,d all have minimum 1, and this will mean we need to distribute the remaining '5' to 4 shelves ${}^{5+4-1}\mathrm{C}_{4-1} = 56$ Therefore total number of ways = ${}^{4}\mathrm{C}_2 \times 56 = 336$ **Alternative solution**: We choose 2 shelves out of 4 to contain even number of books: ${}^{4}\mathrm{C}_2$ We place one book to each shelf to have odd number of books, and two books to each shelf to have even number of books. For the remaining 10 books to be considered as 5 pairs, the number of ways to distribute to 4 shelves is ${}^{5+4-1}\mathrm{C}_{4-1} = {}^{8}\mathrm{C}_3 = 56$. Therefore total number of ways = ${}^{4}\mathrm{C}_2 \times 56 = 336$ **Alternative solution**: No. of ways for no shelf to be empty = ${}^{12+4-1}\mathrm{C}_{4-1} = 455$ No. of ways for all shelves to contain odd number of books = ${}^{6+4-1}\mathrm{C}_{4-1} = 84$ (after placing 1 book in each shelf, we pair the remaining 12 books as 6 pairs) No. of ways for all shelves to contain even number of books = ${}^{4+4-1}\mathrm{C}_{4-1} = 35$ (after placing 2 books in each shelf, we pair the remaining 8 books as 4 pairs) Therefore total number of ways = 455-84-35= 336

7 (a)

$$\frac{(a+1)^3}{b}+\frac{(b+1)^3}{c}+\frac{(c+1)^3}{a} \geq 3\sqrt[3]{\frac{(a+1)^3(b+1)^3(c+1)^3}{abc}}$$

$$\frac{(a+1)^3}{b}+\frac{(b+1)^3}{c}+\frac{(c+1)^3}{a} \geq 3\frac{(a+1)(b+1)(c+1)}{\sqrt[3]{abc}}$$

Since $3\frac{(a+1)(b+1)(c+1)}{\sqrt[3]{abc}} = 3\frac{\left(a+\frac{1}{2}+\frac{1}{2}\right)\left(b+\frac{1}{2}+\frac{1}{2}\right)\left(c+\frac{1}{2}+\frac{1}{2}\right)}{\sqrt[3]{abc}}$

$$\geq 3\frac{3\sqrt[3]{\frac{a}{4}}\bullet 3\sqrt[3]{\frac{b}{4}}\bullet 3\sqrt[3]{\frac{c}{4}}}{\sqrt[3]{abc}} = 81/4$$

7 (b)

Since $\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z} = 3-\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 1$

$$(x+y+z)\left(\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}\right) \geq \left(\sqrt{x}\sqrt{\frac{x-1}{x}}+\sqrt{y}\sqrt{\frac{y-1}{y}}+\sqrt{z}\sqrt{\frac{z-1}{z}}\right)^2$$

$$= \left(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\right)^2$$

$$\therefore \left(x+y+x\right)\bullet 1 \geq \left(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\right)^2$$

$$\therefore \sqrt{\left(x+y+x\right)} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$

7 (c)

Since $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)\left(2b^2+3c^2+6d^2\right) \geq \left(\sqrt{\frac{1}{2}\times 2b^2}+\sqrt{\frac{1}{3}\times 3c^2}+\sqrt{\frac{1}{6}\times 6d^2}\right)^2$

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)\left(2b^2+3c^2+6d^2\right) \geq \left(b+c+d\right)^2$$

$$5-a^2 \geq \left(3-a\right)^2$$

$a^2-3a+2 \leq 0$

$1 \leq a \leq 2$

7 (d)

Prove by contrapositive

To prove: if $a^2+b^2+c^2 < abc$, then $a+b+c < abc$.

Since $a, b, c > 0$ and $a^2+b^2+c^2 < abc$

$\therefore\ a^2 < abc \Rightarrow a < bc$

Similarly $b < ac$

$c < ab$

Therefore $a+b+c < ab+bc+ac$

Since $ab \leq \frac{a^2+b^2}{2}$, $bc \leq \frac{b^2+c^2}{2}$, $ac \leq \frac{a^2+c^2}{2}$

$\Rightarrow\ ab+bc+ac \leq a^2+b^2+c^2$

Therefore $a+b+c < ab+bc+ac \leq a^2+b^2+c^2 < abc$

Original statement is true.

8 (i)	$k(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)+R$
8 (ii) (a)	$P(x)=m(x-1)(x-2)\ldots(x-N)+1$ Therefore, $P(N+1)=m(N)(N-1)\ldots(1)+1=mN!+1$
8 (ii) (b)	$mN!+1=N+1$ Therefore, $m=\dfrac{1}{(N-1)!}$. $P(x)=\dfrac{(x-1)(x-2)\ldots(x-N)}{(N-1)!}+1$ $P(N+r)=\dfrac{(N+r-1)(N+r-2)\ldots(r)}{(N-1)!}+1$ $=\dfrac{(N+r-1)!}{(N-1)!(r-1)!}+1$
8 (iii) (a)	$S(x)=(x-a)(x-b)(x-c)(x-d)+2009$ Suppose there exists *n* such that $S(n)=2020.$ Then $11=2020-2009=S(n)-2009=(n-a)(n-b)(n-c)(n-d)$ The factors of 11 are $\{-11,\ -1,\ 1,\ 11\}$. Letting $\{n-a,\ n-b,\ n-c,\ n-d\}$ be the 4 pigeons and the factors of 11 be the pigeon holes, by PHP, either all 4 distinct factors of 11 are used or at least 1 of the factors of 11 is used more than once. The 4 factors of 11 cannot be used at the same time as $-11\times-1\times1\times11=121$. Since $\{n-a,\ n-b,\ n-c,\ n-d\}$ are distinct, no factors of 11 can be used more than once. Hence, no such *n* exists.
8 (iii) (b)	$S(0)=abcd+2009=2090$ $abcd=81=3^4$ (table below) Number of ways = 5

a	b	c	d
-3^3	-1	1	3^1
-3^2	-3^1	1	3^1
-3^2	-1	1	3^2
-3^1	-1	1	3^3
-3^1	-1	3^1	3^2

9 (i)	(a) $\sinh^2 x+1=\dfrac{\left(e^x-e^{-x}\right)^2}{4}+1$ $=\dfrac{\left(e^{2x}-2+e^{-2x}\right)+4}{4}$ $=\left(\dfrac{e^x+e^{-x}}{2}\right)^2$ $=\cosh^2 x$ (b) $\dfrac{\mathrm{d}\cosh x}{\mathrm{d}\,x}=\dfrac{\mathrm{d}\left(\dfrac{e^x+e^{-x}}{2}\right)}{\mathrm{d}\,x}$ $=\dfrac{e^x-e^{-x}}{2}=\sinh x$
9 (ii)	$u^2+2u\sinh x-1=0$ $u^2+u(e^x-e^{-x})-1=0$ $(u-e^{-x})(u+e^x)=0$ Therefore $u=e^{-x}$ or $u=-e^x$ For : $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2\dfrac{\mathrm{d}y}{\mathrm{d}x}\sinh x-1=0$ Let $\dfrac{\mathrm{d}y}{\mathrm{d}x}=u$ Then $\dfrac{\mathrm{d}y}{\mathrm{d}x}=e^{-x}$ or $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-e^x$ Since $\dfrac{\mathrm{d}y}{\mathrm{d}x}>0$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x}=e^{-x}$ only $y=-e^{-x}+c$ Sub $y=0,\ x=0$ Then $c=1$ $y=-e^{-x}+1$

9 (iii)	$\sinh y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2\frac{\mathrm{d}y}{\mathrm{d}x}-\sinh y=0$ Letting $\frac{\mathrm{d}y}{\mathrm{d}x}=u$, $\sinh y\left(u^2\right)+2u-\sinh y=0$. By applying general formula $u=\frac{-2\pm\sqrt{4+4\sinh^2 y}}{2\sinh y}$ Since the result of (a)(i), $\sinh^2 x+1=\cosh^2 x$ $u=\frac{-2\pm 2\cosh y}{2\sinh y}=\frac{-1\pm\cosh y}{\sinh y}$ $u=\frac{-2\pm 2\cosh y}{2\sinh y}=\frac{-1\pm\cosh y}{\sinh y}$ Therefore $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1\pm\cosh y}{\sinh y}$ $\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{\sinh y}{-1\pm\cosh y}$ $\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{\sinh y}{-1+\cosh y}$ or $\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{\sinh y}{-1-\cosh y}$ $x=\ln\lvert\cosh y-1\rvert+C$ or $x=-\ln\lvert\cosh y+1\rvert+D$
	$x=\ln\lvert\cosh y-1\rvert+c$ or $x=-\ln\lvert\cosh y+1\rvert+d$ $A_1e^{x}=\cosh y-1$ $\quad$ $A_2e^{-x}=\cosh y+1$ When $y=0$, $\cosh y=1$, then $A_1=0$ (rej.) and $A_2=2$. Hence, $\cosh y=2e^{-x}-1$.

10(i)	$\begin{aligned}&\frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}\mathrm{C}_r}-\frac{1}{{}^{n+r}\mathrm{C}_r}\right)\\ &=\frac{r+1}{r}\left(\frac{r!(n-1)!}{(n+r-1)!}-\frac{r!n!}{(n+r)!}\right)\\ &=\frac{r+1}{r}\left(\frac{r!(n-1)!(n+r)-r!n!}{(n+r)!}\right)\\ &=\frac{r+1}{r}\left(\frac{r!(n-1)!\left[(n+r)-n\right]}{(n+r)!}\right)\\ &=(r+1)\left(\frac{r!(n-1)!}{(n+r)!}\right)\\ &=\frac{(r+1)!(n-1)!}{(n+r)!}\\ &=\frac{1}{\left(\frac{(n+r)!}{(r+1)!(n+r-(r+1))!}\right)}\\ &=\frac{1}{{}^{n+r}\mathrm{C}_{r+1}}\end{aligned}$
	$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{{}^{n+r}\mathrm{C}_{r+1}}=\lim_{N\to\infty}\frac{r+1}{r}\sum_{n=1}^{N}\left(\frac{1}{{}^{n+r-1}\mathrm{C}_r}-\frac{1}{{}^{n+r}\mathrm{C}_r}\right)\\ &=\lim_{N\to\infty}\frac{r+1}{r}\left(\begin{array}{ccc}\frac{1}{{}^{r}\mathrm{C}_r} & - & \cancel{\frac{1}{{}^{r+1}\mathrm{C}_r}}\\ \cancel{\frac{1}{{}^{r+1}\mathrm{C}_r}} & - & \cancel{\frac{1}{{}^{r+2}\mathrm{C}_r}}\\ & \vdots & \\ \cancel{\frac{1}{{}^{N+r-2}\mathrm{C}_r}} & - & \cancel{\frac{1}{{}^{N+r-1}\mathrm{C}_r}}\\ \cancel{\frac{1}{{}^{N+r-1}\mathrm{C}_r}} & - & \frac{1}{{}^{N+r}\mathrm{C}_r}\end{array}\right)\\ &=\frac{r+1}{r}\lim_{N\to\infty}\left(\frac{1}{1}-\frac{1}{{}^{N+r}\mathrm{C}_r}\right)\\ &=\frac{r+1}{r}\end{aligned}$

	since as $N \to \infty$, ${}^{N+r}\mathrm{C}_r \to \infty$ with r being constant, and hence $\dfrac{1}{{}^{N+r}\mathrm{C}_r} \to 0$ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{{}^{n+2}\mathrm{C}_3} = \sum_{n=1}^{\infty} \frac{1}{{}^{n+2}\mathrm{C}_3} - \frac{1}{{}^{3}\mathrm{C}_3}$ $= \dfrac{2+1}{2} - 1 = \dfrac{1}{2}$
10 (ii)	$\dfrac{1}{{}^{n+1}\mathrm{C}_3} = \dfrac{3!(n-2)!}{(n+1)!} = \dfrac{3!}{(n+1)n(n-1)}$ $= \dfrac{3!}{n^3 - n} > \dfrac{3!}{n^3}$ since n is positive $\dfrac{20}{{}^{n+1}\mathrm{C}_3} - \dfrac{1}{{}^{n+2}\mathrm{C}_5}$ $= \dfrac{20 \times 3!(n-2)!}{(n+1)!} - \dfrac{5!(n-3)!}{(n+2)!}$ $= \dfrac{5!(n-2)!(n+2) - 5!(n-3)!}{(n+2)!}$ $= \dfrac{5!(n-3)!\left[(n-2)(n+2)-1\right]}{(n+2)!}$ $= \dfrac{5!(n^2-5)}{(n+2)(n+1)n(n-1)(n-2)}$ $= \dfrac{5!(n^2-5)}{n(n^2-1)(n^2-4)}$ $= \dfrac{5!(n^5-5n^3) \div n^3}{(n^5-5n^3+4n)}$ $< \dfrac{5!}{n^3}$ since $(n^5-5n^3) < (n^5-5n^3+4n)$ when $n \geq 3$

10 (iii)	Consider $\sum_{n=3}^{\infty}\frac{3!}{n^3}<\sum_{n=3}^{\infty}\frac{1}{{}^{n+1}\mathrm{C}_3}=\sum_{n=2}^{\infty}\frac{1}{{}^{n+2}\mathrm{C}_3}=\frac{1}{2}$ Therefore $\sum_{n=1}^{\infty}\frac{3!}{n^3}<\frac{1}{2}+\frac{3!}{1^3}+\frac{3!}{2^3}=\frac{29}{4}$ $\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^3}<\frac{29}{24}=\frac{116}{96}$ Consider $\sum_{n=3}^{\infty}\frac{5!}{n^3}$ $>\sum_{n=3}^{\infty}\frac{20}{{}^{n+1}\mathrm{C}_3}-\sum_{n=3}^{\infty}\frac{1}{{}^{n+2}\mathrm{C}_5}$ $=20\sum_{n=2}^{\infty}\frac{1}{{}^{n+2}\mathrm{C}_3}-\sum_{n=1}^{\infty}\frac{1}{{}^{n+4}\mathrm{C}_5}$ $=20\left(\frac{1}{2}\right)-\frac{4+1}{4}=\frac{35}{4}$ Therefore $\sum_{n=1}^{\infty}\frac{5!}{n^3}>\frac{35}{4}+\frac{5!}{1^3}+\frac{5!}{2^3}=\frac{575}{4}$ $\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^3}>\frac{575}{4(120)}=\frac{115}{96}$ Hence $\frac{115}{96}<\sum_{n=1}^{\infty}\frac{1}{n^3}<\frac{116}{96}$ (shown)