Answer **all** the questions.

1 Let $I_r = \dfrac{1}{r!}\displaystyle\int_0^p e^{-x}x^r\,\mathrm{d}x$ for integer $r \geq 0$, where p is a positive constant.

(i) Show that $I_r = \mathrm{I}_{r-1} - \dfrac{1}{r!}e^{-P}p^r$ for $r \geq 1$. **[2]**

(ii) Show that for all non-negative integers n,

$$I_n = 1 - e^{-p}\sum_{r=0}^{n}\frac{p^r}{r!}.$$ **[4]**

(iii) Deduce the limit of I_n as $n \to \infty$. **[1]**

Given that $p = 1$.

(iv) Show that $0 < I_n < \dfrac{1}{(n+1)!}$. **[2]**

Hence show that that $\displaystyle\sum_{r=0}^{n}\frac{1}{r!}$ differs from e by less than $\dfrac{\mathrm{e}}{(n+1)!}$. **[2]**

2 **(a)** Show that for all positive x, y, z,

$$x+y+z \leq 2\left\{\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\right\}.$$ **[3]**

(b) Let a, b, c, and d be positive real numbers.

(i) Show that $\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2 \geq a^{\frac{2}{3}}(a^2+8bc).$ **[4]**

(ii) Using the result in **part (i)**, prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1.$$ **[3]**

3 A differential equation is of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}\mathrm{f}\ \ xy\ \ .$$

(i) By using the substitution $z = xy$, show that

$$\ln|x| = \int \frac{1}{z\ \mathrm{f}(z)+1}\,\mathrm{d}z.$$ **[4]**

(ii) Solve the following differential equation, expressing y in terms of x:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y \ln x + \ln y \ . \qquad \textbf{[5]}$$

(iii) A solution curve, $y = \mathrm{g}(x)$, of the differential equation in part **(ii)** passes through the point $(1, k)$. Given also that the graph of $y = x^2 \mathrm{e}^{-x} \mathrm{g}(x)$ is a straight line, find the value of k. **[2]**

4(a) **(i)** Let p be a prime number. Prove that for positive integer r, $1 \le r < p$,

$$\binom{p}{r} \text{ is divisible by } p. \qquad \textbf{[3]}$$

(ii) Prove by induction the following statement:
For any positive integer n, $n^p - n$ is divisible by p. **[4]**

(iii) Hence prove Fermat's Little Theorem:
If p is prime and n is a positive integer such that $\gcd(n, p) = 1$, then

$$n^{p-1} \equiv 1 \pmod{p}. \qquad \textbf{[2]}$$

(b) Let Q be the set of all primes of the form $4k + 1$, for some integer k.
So $Q = \{5, 13, 17, 29, \ldots\}$

(i) Let $N = 2q_1q_2...q_n$, where $\{q_i \mid 1 \le i \le n\}$ is a finite subset of Q. Show that, for a prime p such that $p \mid N^2 + 1$, the smallest exponent m where $N^m \equiv 1 \pmod{p}$ is 4. **[3]**

(ii) Hence, using Fermat's Little Theorem, prove that there are infinitely many primes in Q. **[5]**

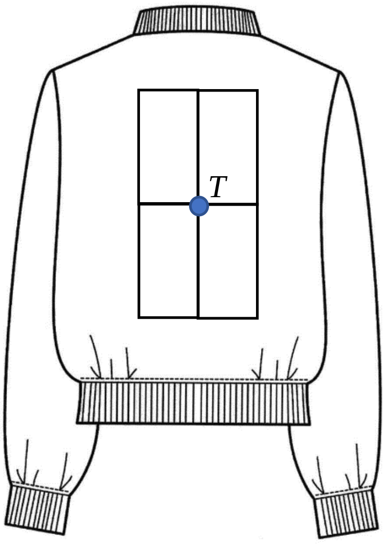
5 **(i)** Using the Well-ordering principle, prove that for a, b integers, the greatest common divisor of a and b is 1 if and only if there exists m, n, integers such that

$$ma + nb = 1. \qquad \textbf{[7]}$$

(ii) Prove that, for integers a, b such that $\gcd(a, b) = 1$, if a and b are both factors of an integer k, then ab is a factor of k. **[3]**

(iii) Find an integer x such that $x \equiv 11 \pmod{14}$ and $x \equiv 20 \pmod{25}$. **[3]**

6



The diagram shows the front of a jacket with a pattern formed from four congruent rectangles, the longer sides of each rectangle being vertical. Each of the four vertical rectangles is coloured with one of the c different colours. The pattern of the rectangles has a rotational symmetry about the point T if this pattern remains unchanged after a rotation about T by 180°.

(i) How many different patterns of rectangles are there

- **(a)** in total, **[1]**
- **(b)** with reflection symmetry in a vertical line **[2]**
- **(c)** with no rotational symmetry and no reflection symmetry? **[4]**

(ii) A man decides to give a jacket to each of 10 friends. He chooses from a range of jackets with 7 different designs. Use the principle of inclusion and exclusion to show that the number of ways that he can do this, buying at least one of each of the 7 designs, is

$$7^{10}-\binom{7}{1}6^{10}+\binom{7}{2}5^{10}-\binom{7}{3}4^{10}+\binom{7}{4}3^{10}-\binom{7}{5}2^{10}+\binom{7}{6}1^{10}.$$ **[5]**

7 In this question, assume that all coins of the same type are indistinguishable.

(i) Explain why the number of choices for n coins from a collection of coins of k types is $\begin{pmatrix} n+k-1 \\ n \end{pmatrix}$. **[2]**

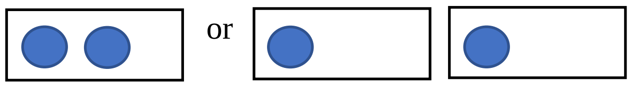
(ii) State the number of choices of n coins from a collection of coins of two types in each of the cases

(a) $n = 3$, **[1]**

(b) $n = 5$ and the 5 coins are not all of the same type. **[2]**

Let x_n be the number of ways of distributing n coins of one type into piggy banks placed in a line in such a way that each piggy bank contains 1, 2 or 3 coins. Thus $x_1 = 1$ and $x_2 = 2$ (see diagram below).

Diagram for the case when n = 2:



(iii) Find the value of x_3. **[2]**

(iv) Explain why $x_n = x_{n-1} + x_{n-2} + x_{n-3}$, for $n \geq 4$. Hence find x_7. **[4]**

Let y_n be the number of ways of distributing n coins of two types into piggy banks placed in a line in such a way that each piggy bank contains 1, 2 or 3 coins.

(v) Find a recurrence relation for y_n, for $n \geq 4$. **[3]**

8 **(a)** Prove that if $n+1$ distinct integers are selected from the set $\{1, 2, 3, \ldots, 2n\}$, then at least 2 of the integers must be relatively prime. **[2]**

(b) At an event, n participants deposited their belongings with the organizer and received an electronic identification tag. Due to a programming bug, the data for the tags are randomly permuted. Show that the probability at least one participant gets back his belongings is $\sum_{r=1}^{n} \frac{(-1)^{r+1}}{r!}$. **[4]**

(c) A stall sells 10 types of bubble tea. A group of friends decides to order 25 cups of bubble tea as takeaway.

They are given 2 small plastic bags and 2 large plastic bags, all different colours. The small bags can hold up to 5 cups each and the large bags can hold up to 10 cups each. How many ways can the friends carry their order in the bags assuming no bag is empty, and it does not matter what types of bubble tea are in each bag. **[6]**

End of Paper