Name:__________________________________ Class:____________

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2020

MATHEMATICS
Higher 3

Paper 1

9820/01
25 September 2020
3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 4 printed pages.

1 Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \, dx$.

(i) Sketch the graph of $y = \cot x$ for $0 < x < \pi$. [2]

(ii) For $n \geq 2$, prove that $I_n + I_{n-2} = \dfrac{1}{n-1}$. [4]

(iii) By considering $I_n - I_{n+1}$ or otherwise, show that $I_n > I_{n+1}$. Hence prove that I_n is convergent and state the limit of I_n as n tends to infinity. [4]

(iv) Find the sum of the infinite series $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$. [4]

2 **(i)** For all positive real numbers a_1, a_2, b_1 and b_2, use the AM-GM inequality to show that

$$\frac{a_1 b_1}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}} \leq \frac{1}{2}\left(\frac{a_1^2}{a_1^2 + a_2^2} + \frac{b_1^2}{b_1^2 + b_2^2}\right).$$ [2]

Write down a similar inequality for $\dfrac{a_2 b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}$ and show that

$$\frac{a_1 b_1}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}} + \frac{a_2 b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}} \leq 1.$$ [3]

(ii) By considering $\sum_{i=1}^{n} \dfrac{a_i b_i}{AB}$ where $A = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$ and $B = \sqrt{b_1^2 + b_2^2 + \ldots + b_n^2}$, use the answer to part **(i)** to prove Cauchy-Schwarz inequality. [4]

(iii) Hence or otherwise find the maximum value of $2x + 3y + 4z$ given that

$$x^2 + y^2 + z^2 = 1.$$ [2]

3 **(a)** Prove that any subset of 55 numbers chosen from the set $\{1, 2, 3, 4, \ldots, 100\}$ must contain two numbers differing by 6. [6]

(b) Let n, a, b, x be positive integers.

(i) Show that $n^a - 1$ is a factor of $n^{ax} - 1$. [2]

(ii) Prove that $\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1$. [6]

4 **(i)** Prove that for each positive integer n, $15^n - 7^n$ is divisible by 8. [5]

(ii) Show that a linear polynomial with integer coefficients $\mathrm{f}(x)$ such that $\mathrm{f}(7) = 5$ and $\mathrm{f}(15) = 9$ doesn't exist. [3]

(iii) Hence or otherwise prove that there is no polynomial with integer coefficients $\mathrm{P}(x)$ such that $\mathrm{P}(7) = 5$ and $\mathrm{P}(15) = 9$. [4]

5 **(i)** Show that the differential equation

$$\frac{du}{dx} = 3\mathrm{f}(x) - \frac{2u}{x}$$

can be transformed into the equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3x^2 \mathrm{f}(x)$$

by the substitution $z = x^2 u$. [3]

(ii) A solution curve C of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\mathrm{e}^{2y}}{x^3} + \frac{1}{x}$$

passes through the point (1, 0).

Using the substitution $u = -\dfrac{1}{2\mathrm{e}^{2y}}$, find y in terms of x. [8]

(iii) Hence sketch C, stating the equations of all the asymptotes and the coordinates of the points where the curve crosses the x-axis. [2]

6 For all real numbers x and y, use the Triangle inequality to prove that

(a) $\left|\dfrac{x^3 + y^3}{x^2 + y^2}\right| \le |x| + |y|$.

Hence find the limit $\left|\dfrac{x^3 + y^3}{x^2 + y^2}\right|$ as x and y tends to 0. [5]

(b) $\big||x| - |y|\big| \le |x - y|$.

Hence is the inequality $\left||x|^2 - |y|^2\right| \le (x-y)^2$ true? Justify your answer. [6]

7 Belinda has a huge supply of pens that are red, blue, green and black.

(i) She decides to give 7 of the pens to her best friend, Catherine. Find the number of ways if she has to give at least one pen of every colour. [2]

(ii) On another occasion, she decides to give one pen each to 7 of her friends. Find the number of ways if she has to give at least one pen of every colour. [4]

(iii) Belinda puts 17 red pens, 23 blue pens, 6 green pens and 9 black pens into a case. What is the least number of pens that she must choose to ensure that there are 7 pens of the same colour? [2]

(iv) On another day, she decides to place the pens in a row, such that any green pen is in between 2 non-green pens of different colours. Let x_n be the number of such arrangements of the pens of length n.

(a) Find x_1 and x_2. [2]

(b) Show that for $n \ge 3$, $x_n = Ax_{n-1} + Bx_{n-2}$, where A and B are constants to be determined. [4]

(c) Find x_6. [2]

8 Given non-negative integers r and n where $r \geq n$, the *Stirling number of the second kind*, $S(r,n)$, is defined as the number of ways of distributing r distinct objects into n identical boxes such that no box is empty.

(i) By using a combinatorial argument, prove that

(a) $S(n,\ 2) = 2^{n-1} - 1$, [2]

(b) $S(r,\ n) = S(r-1,\ n-1) + nS(r-1,\ n)$. [2]

Using the above results, derive a formula for $S(n+1,\ 3)$, for $n \geq 3$. [2]

(ii) Find the number of ways 5 distinct beads are placed in 3 identical strings where some of the strings may be empty. [3]