RAFFLES INSTITUTION
2020 YEAR 6 PRELIMINARY EXAMINATION
Higher 3

MATHEMATICS 9820

24 September 2020

3 hours

Additional materials: Answer Paper
Graph paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Let

$$S_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \cdots + (-1)^n \frac{1}{2n+1}.$$

(a) **(i)** Show that

$$\frac{1}{(4k+1)(4k+3)} < \frac{1}{4k(4k+4)}$$

for positive integers k. [1]

(ii) By using $S_{2m+1} = \sum_{k=0}^{m} \left(\frac{1}{4k+1} - \frac{1}{4k+3} \right)$, show that $S_{2m+1} < \frac{19}{24}$ for positive integers m. [3]

(b) **(i)** State the sum of $1 - x^2 + x^4 - \cdots + (-1)^n x^{2n}$. [1]

(ii) Hence show that

$$S_n = A + (-1)^n \int_0^1 \frac{x^{2n+2}}{1+x^2}\, dx,$$

where A is a constant to be determined. [3]

(iii) Deduce that $|S_n - A| < \frac{1}{2n+3}$, and hence determine the limit of the sequence S_n. [3]

2 **(a)** Let $I = \int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left[f(x) + f\left(\frac{1}{x}\right)\right]}\, dx$, where a is a positive constant and f is any function for which this integral exists.

Show using a suitable substitution that $I = \int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x}\right)}{x\left[f(x) + f\left(\frac{1}{x}\right)\right]}\, dx$, and evaluate I in terms of a. [3]

Hence evaluate exactly

(i) $\int_{\frac{1}{2}}^{2} \frac{\cos x}{x\left[\cos x + \cos\left(\frac{1}{x}\right)\right]}\, dx$, [1]

(ii) $I = \int_{\frac{1}{3}}^{3} \frac{\ln(1+x)}{x\left[2\ln(1+x) - \ln x\right]}\, dx.$ [2]

(b) Find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\, dx.$ [5]

3 A group of 10 people went to watch a concert, and were seated in a row.

(i) 6 identical bags of popcorn are to be distributed to the group. In how many ways can this be done if each person can have at most 1 bag of popcorn? [1]

(ii) 26 identical bottles of mineral water are to be distributed to the group. In how many ways can this be done if each person has a non-zero and even number of bottles of mineral water? [3]

(iii) 3 of them went to the toilet during the first intermission. In how many ways can this happen if no two of them were sitting next to each other? [3]

(iv) During the second intermission, all 10 of them left their seats before returning. Show that the number of arrangements if each person has a different person seated to his or her right after returning is given by $\sum_{k=0}^{9}(-1)^k\binom{9}{k}(10-k)!$. [4]

4 **(a)** Let $a_1, a_2, ..., a_n$ be real numbers.

(i) Show that $a_1^2 + a_2^2 + a_3^2 \geq a_1a_2 + a_2a_3 + a_1a_3$. [2]

(ii) Show that $a_1^2 + a_2^2 + a_3^2 + a_4^2 \geq \frac{2}{3}\left(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4\right)$.

[2]

(iii) Determine the largest constant C (in terms of n) such that

$$\sum_{i=1}^{n} a_i^2 \geq C \sum_{1\leq i<j\leq n} a_ia_j .$$

[3]

(b) Let x_1, x_2, x_3 be nonnegative real numbers satisfying $x_1 + x_2 + x_3 = 1$. Determine, with proof, the minimum value of

$$\left(x_1 + 3x_2 + 5x_3\right)\left(x_1 + \frac{x_2}{3} + \frac{x_3}{5}\right).$$

[4]

(c) Let $y_1, y_2, ..., y_n$ be real numbers satisfying $|y_i| < 1$ for all i, and

$$|y_1| + |y_2| + ... + |y_n| = 31 + |y_1 + y_2 + ... + y_n|.$$

Determine, with proof, the minimum value of n. [4]

5 Let $\mathrm{f}, \mathrm{g} : [0, \infty) \to [0, \infty)$ be two continuous functions. Suppose for $t \ge 0$,

$$\mathrm{f}'(t) \le \mathrm{f}(t)\mathrm{g}(t).$$

(i) Show that $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{f}(t)\mathrm{e}^{-\int_0^t \mathrm{g}(s)\,\mathrm{d}s}\right) \le 0$ and hence deduce that $\mathrm{f}(t) \le \mathrm{f}(0)\mathrm{e}^{\int_0^t \mathrm{g}(s)\,\mathrm{d}s}$. [4]

Let $y_1(t)$ and $y_2(t)$ be solutions to the differential equation

$$y'(t) = \mathrm{f}(y) \qquad (*)$$

with $y(0) = 1$ for some function f.

Assume that for all $u, v \ge 0$, there exists a positive constant L such that

$$\left|\mathrm{f}(u) - \mathrm{f}(v)\right| \le L\left|u - v\right|.$$

Define $z(t) = \left(y_1(t) - y_2(t)\right)^2$.

(ii) Show that $z'(t) \le 2Lz(t)$ and hence show that the solution to the differential equation (*) is unique. [5]

(iii) Consider the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 2\sqrt{|w|}, \quad w(0) = 0. \qquad (**)$$

Show that for every $a > 0$, the function

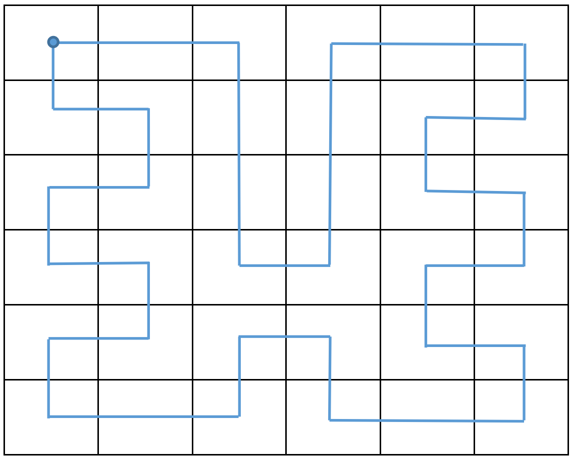
$$w_a(t) = \begin{cases} 0 & \text{if } 0 \le t < a \\ (t-a)^2 & \text{if } \quad t \ge a \end{cases}$$

is a solution to the differential equation (**). Explain why this does not contradict the result in **(ii)**. [4]

6 Given an $n \times n$ grid of squares, where $n > 1$, a *tour* is a path drawn within a grid such that

- along its way the path moves, horizontally or vertically, from the centre of one square to the centre an adjacent square;
- the path starts and finishes in the same square;
- the path visits the centre of every other square just once.

For example, below is a tour drawn in a 6×6 grid of squares which starts and finishes in the top-left square.



We first assume n is even for parts **(i)** to **(iv)**.

(i) With the aid of a diagram, show how a tour, which starts and finishes in the top-left square, can be drawn in any $n \times n$ grid. [2]

(ii) Is a tour still possible if the start/finish point is changed to the centre of a different square? Justify your answer. [1]

Suppose now that a robot is programmed to move along a tour of an $n \times n$ grid. The robot understands the two commands:

- command R which turns the robot clockwise through a right angle;
- command F which moves the robot forward to the centre of the next square.

The robot has a program; a list of commands, which it performs in the given order to complete a tour. In total, the command R appears r times and command F appears f times in the program.

(iii) Initially the robot is placed in the top-left square pointing to the right. Assuming the first command is an F, what is the value of f? Explain also why $r+1$ is a multiple of 4. [3]

(iv) Must the results of part **(iii)** still hold if the robot starts and finishes at the centre of a different square? Explain your reasoning. [2]

(v) Explain why a tour of an $n \times n$ grid is not possible when n is odd. [3]

7 Define by $B(n)$ the number of partitions of n into sums of powers of 2. For example, $B(6)=6$, since

$$\begin{aligned}6 &= 1+1+1+1+1+1\\ &= 1+1+1+1+2\\ &= 1+1+2+2\\ &= 1+1+4\\ &= 2+2+2\\ &= 4+2\end{aligned}$$

Prove that

(i) $B(2n+1)=B(2n)$ for all $n\geq 1$. [2]

(ii) $B(2n)=B(2n-1)+B(n)$ for all $n\geq 1$. [3]

(iii) $B(n)$ is even for all $n\geq 2$. [4]

We say that a partition is even if the number of parts is even, and a partition is odd otherwise. For example, $1+1+1+1+1+1$, $1+1+2+2$ and $4+2$ are even partitions of 6, whereas $1+1+1+1+2$, $1+1+4$ and $2+2+2$ are odd partitions of 6.

(iv) Show that there is an equal number of even and odd partitions of n for all $n\geq 2$. [3]

8 **(a)** Show that if an odd prime number is a sum of two squares, then it must be congruent to 1 modulo 4. [2]

(b) Suppose p is an odd prime of the form $4k+1$.

(i) Show that if n is a number in the set P, where $P=\{1, 2, ..., p-1\}$, then there exists a unique integer $q(n)$ in P such that $nq(n)\equiv 1 \pmod{p}$. [5]

(ii) Deduce that $(p-1)!\equiv -1 \pmod{p}$. [3]

(iii) Hence show that $\left[(2k)!\right]^2=-1$ (modulo p). [2]

(iv) By considering the set of integers $S=\left\{(2k)!x-y \mid 0\leq x,y<\sqrt{p}\right\}$, show that there are two numbers whose squares sum to p. [4]

[END OF PAPER]