**HWA CHONG INSTITUTION**
**2020 JC2 Preliminary Examination**

# MATHEMATICS
## Higher 3

# 9820/01

Paper 1

**Wednesday** **23 September 2020** **3 hours**

Additional materials: 12-page Answer Booklet
List of Formula (MF26)
4-page Additional Answer Booklet (upon request)

**READ THESE INSTRUCTIONS FIRST**

Write your name and class on the 12-page Answer Booklet and any other additional 4-page Answer Booklets you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.
Do not write anything on the List of Formula (MF26).

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, slot any additional 4-page Answer Booklets used in your 12-page Answer Booklet and indicate on the 12-page Answer Booklet the number of additional 4-page Answer Booklets used (if any).

**This question paper consists of 5 printed pages and 1 blank page.**

**1.** Prove, by Mathematical Induction, for all integers $n \geq 2$,

$$\left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\ldots\left(1-\frac{1}{\sqrt{n}}\right)<\frac{2}{n^2}.$$ [5]

**2.** Given that $a_1, a_2, \ldots, a_n$ and $n$ are all odd integers, prove that the greatest common divisor of $a_1, a_2, \ldots, a_n$ is equal to the greatest common divisor of $\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}, \ldots, \frac{a_n+a_1}{2}$. [5]

**3.** **(i)** Show that, for $m > 0$,

$$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x = \frac{(m-1)^3(m+1)}{2m^2} + \ln m.$$ [3]

**(ii)** Show by means of substitution $x = \frac{1}{u}$, that

$$\int_b^a \frac{1}{x^n(x+1)}\,\mathrm{d}x = \int_{1/a}^{1/b} \frac{u^{n-1}}{u+1}\,\mathrm{d}u.$$ [2]

**(iii)** Hence or otherwise, without using calculator, evaluate

$$\int_1^2 \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x.$$ [3]

**4.** **(a)** If $x$ and $y$ are odd integers, prove that $x^2+y^2$ cannot be a perfect square. [3]

**(b)** Prove that if $n$ is a composite positive integer, then $n$ divides $(n-1)!$, except when $n = 4$. [6]

**5.** Casper has $n$ fence posts, each of different width, arranged in a straight line in front of his house. To surprise his wife, Ashley, Casper decides to paint the posts using three different colours red, white and green in such a way that no adjacent fence posts have the same colours. Let $r_n$ be the number of ways that Casper can paint the fence posts if he paints the first and last post red, and let $s_n$ be the number of ways that Casper can paint the posts if he paints the first post red but the last post with either of the other two colours.

**(i)** Explain why $r_{n+1} = s_n$. Hence determine the value of $r_{n+1} + r_n$ for all $n \in \mathbb{Z}^+$. [2]

**(ii)** Prove that for all $n \in \mathbb{Z}^+$,

$$r_n = \frac{2^{n-1} + 2(-1)^{n-1}}{3}.$$ [4]

**(iii)** Find the number of ways of painting the $n$ fence posts, given that $n \geq 3$, with the three different colours red, white and green in such a way that no adjacent fence posts have the same colours, if Casper wants to arrange the posts in a circle surrounding a newly built pond outside his house. Give your answer in terms of $n$. [3]

**6.** The school library is undergoing renovation, and Elijah the student librarian needs to pack the library books into boxes so that they can stored away from the library during the renovation.

Find the number of ways that Elijah can pack 8 identical library books on science into 10 identical boxes. [3]

When the renovation is completed, Elijah needs to arrange 16 identical library books on mathematics onto 4 distinct shelves. Find the number of ways to arrange these 16 identical mathematics book such that

**(i)** exactly one shelf have 8 books, [2]

**(ii)** each shelf can hold at most 8 books, [3]

**(iii)** exactly two shelves have an even number of books each, and none of the shelves are empty. [3]

**7.** **(a)** Given that $a, b, c \in \mathbb{R}^+$, by using AM-GM inequality, prove that

$$\frac{(a+1)^3}{b}+\frac{(b+1)^3}{c}+\frac{(c+1)^3}{a} \geq \frac{81}{4}.$$ [3]

**(b)** Given that $x \geq 1,\ y \geq 1,\ z \geq 1$, and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$, by using Cauchy-Schwarz Inequality, prove that

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$ [3]

**(c)** Given that $a+b+c+d=3$ and $a^2+2b^2+3c^2+6d^2=5$, prove that $1 \leq a \leq 2$. [3]

**(d)** Given that $a, b, c \in \mathbb{R}^+$, by proving the contrapositive or otherwise, if $a+b+c \geq abc$ then $a^2+b^2+c^2 \geq abc$. [4]

**8.** **(i)** A polynomial of degree $n$ leaves a remainder of $R$ when divided by any of $(x-\alpha_1)$, $(x-\alpha_2)$,..., and $(x-\alpha_n)$. Given that the coefficient of $x^n$ is $k$, where $k \neq 0$, write down the polynomial. [1]

**(ii)** The polynomial $P(x)$ has degree $N$, where $N \geq 1$, and satisfies $P(1)=P(2)=\ldots=P(N)=1$. The coefficient of $x^N$ is denoted by $m$.

**(a)** Find an expression for $P(N+1)$. [2]

**(b)** Suppose $P(N+1)=N+1$. Find $P(N+r)$, where $r$ is a positive integer. [3]

**(iii)** The polynomial $S(x)$ with integer coefficients, is of degree 4. The coefficient of $x^4$ is 1. It satisfies $S(a)=S(b)=S(c)=S(d)=2009$, where $a$, $b$, $c$ and $d$ are distinct integers (not necessary positive), such that $a<b<c<d$.

**(a)** By considering the factors of $S(n)-2009$, use pigeon hole principle to show that there is no integer $n$ such that $S(n)=2020$. [4]

**(b)** Find the number of ways the integers $a$, $b$, $c$ and $d$ can be chosen such that $S(0)=2090$. [3]

**9.** **(i)** Some hyperbolic functions are defined as $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Prove that

**(a)** $\sinh^2 x + 1 = \cosh^2 x$, and [1]

**(b)** $\dfrac{d\cosh x}{d\,x} = \sinh x.$ [1]

**(ii)** Solve the equation $u^2 + 2u\sinh x - 1 = 0$ giving $u$ in terms of $x$. [2]

Hence find the particular solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx}\sinh x - 1 = 0$$

that satisfies $y = 0$ and $\dfrac{dy}{dx} > 0$ at $x = 0$. [3]

**(iii)** Find the general solution of the differential equation

$$\sinh y\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - \sinh y = 0.$$

Given further that $y = 0$ at $x = 0$, expressing your particular solution in the form $\cosh y = f(x)$. [6]

**10.** **(i)** Prove that, for any positive integers $n$ and $r$,

$$\frac{1}{{}^{n+r}C_{r+1}} = \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right).$$ [3]

Hence determine $\displaystyle\sum_{n=1}^{\infty}\frac{1}{{}^{n+r}C_{r+1}}$ in terms of $r$, and use the result obtained to deduce that $\displaystyle\sum_{n=2}^{\infty}\frac{1}{{}^{n+2}C_3} = \frac{1}{2}$. [5]

**(ii)** Show that, for $n \geq 3$,

$$\frac{3!}{n^3} < \frac{1}{{}^{n+1}C_3} \quad \text{and} \quad \frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} < \frac{5!}{n^3}.$$ [3]

**(iii)** Using the results in **(i)** and **(ii)**, show that

$$\frac{115}{96} < \sum_{n=1}^{\infty}\frac{1}{n^3} < \frac{116}{96}.$$ [3]

**END**

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