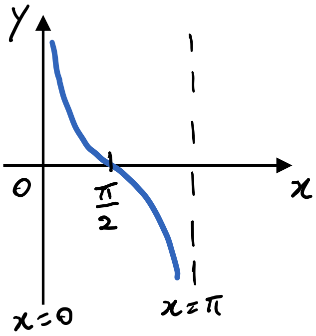
2020 JPJC H3 Math Solutions

Qn	Solution		
(i)			
(ii)	$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \, dx$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \cot^2 x \, dx$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \left(\text{cosec}^2 x - 1 \right) dx$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^{n-2} x)(\text{cosec}^2 x) - \cot^{n-2} \, dx$ $= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (-\text{cosec}^2 x)(\cot^{n-2} x) \, dx - I_{n-2}$ $I_n + I_{n-2} = -\left[\frac{\cot^{n-1} x}{n-1} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $= -\frac{1}{n-1}\left(\cot^{n-1}\frac{\pi}{2} - \cot^{n-1}\frac{\pi}{4} \right)$ $I_n + I_{n-2} = \frac{1}{n-1}$ (shown)		

Qn	Solution		
1 **(iii)**	Consider $I_n - I_{n+1} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x - \cot^{n+1} x \, \mathrm{d}x$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^n x)(1-\cot x) \, \mathrm{d}x$ Since $0 < \cot x < 1$ for $\frac{\pi}{4} < x < \frac{\pi}{2}$ then $1-\cot x > 0$ therefore $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^n x)(1-\cot x) \, \mathrm{d}x$ is positive $I_n - I_{n+1} > 0 \;\Rightarrow\; I_n > I_{n+1}$ (shown) Since I_n is strictly decreasing and bounded below by 0 ($I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \, \mathrm{d}x \geq 0$) by Monotone Sequence Theorem, I_n is convergent with limit 0.		
(iv)	$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = (\cancel{I_2} + I_0) - (\cancel{I_4} + \cancel{I_2}) + (\cancel{I_6} + \cancel{I_4}) - (\cancel{I_8} + \cancel{I_6}) + \ldots$ $= I_0$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 \, \mathrm{d}x$ $= \frac{\pi}{2} - \frac{\pi}{4}$ $= \frac{\pi}{4}$		

Qn	Solution		
2 **(i)**	Let $X=\dfrac{a_1^2}{a_1^2+a_2^2}$ and $Y=\dfrac{b_1^2}{b_1^2+b_2^2}$ AM-GM Inequality: $$\sqrt{X\,Y}\le\frac{X+Y}{2}$$ $$\sqrt{\frac{a_1^2}{a_1^2+a_2^2}}\sqrt{\frac{b_1^2}{b_1^2+b_2^2}}\le\frac{1}{2}\left(\frac{a_1^2}{a_1^2+a_2^2}+\frac{b_1^2}{b_1^2+b_2^2}\right)$$ $$\frac{a_1b_1}{\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}}\le\frac{1}{2}\left(\frac{a_1^2}{a_1^2+a_2^2}+\frac{b_1^2}{b_1^2+b_2^2}\right)$$ Thus, $$\frac{a_2b_2}{\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}}\le\frac{1}{2}\left(\frac{a_2^2}{a_1^2+a_2^2}+\frac{b_2^2}{b_1^2+b_2^2}\right)$$ $$\frac{a_1b_1}{\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}}+\frac{a_2b_2}{\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}}$$ $$\le\frac{1}{2}\left(\frac{a_1^2}{a_1^2+a_2^2}+\frac{b_1^2}{b_1^2+b_2^2}\right)+\frac{1}{2}\left(\frac{a_2^2}{a_1^2+a_2^2}+\frac{b_2^2}{b_1^2+b_2^2}\right)$$ $$=\frac{1}{2}\left(\frac{a_1^2+a_2^2}{a_1^2+a_2^2}+\frac{b_1^2+b_2^2}{b_1^2+b_2^2}\right)$$ $=1$ (shown)		

2 **(ii)**	$$\frac{a_1b_1}{AB} \le \frac{1}{2}\left(\frac{a_1{}^2}{A^2}+\frac{b_1{}^2}{B^2}\right)$$ $$\frac{a_2b_2}{AB} \le \frac{1}{2}\left(\frac{a_2{}^2}{A^2}+\frac{b_2{}^2}{B^2}\right)$$ $$\vdots$$ $$\frac{a_nb_n}{AB} \le \frac{1}{2}\left(\frac{a_n{}^2}{A^2}+\frac{b_n{}^2}{B^2}\right)$$ $$\sum_{i=1}^{n}\frac{a_ib_i}{AB} \le \frac{1}{2}\left[\left(\frac{a_1{}^2}{A^2}+\frac{a_2{}^2}{A^2}+\ldots+\frac{a_n{}^2}{A^2}\right)+\left(\frac{b_1{}^2}{B^2}+\frac{b_2{}^2}{B^2}+\ldots\frac{b_n{}^2}{B^2}\right)\right]$$ $$=\frac{1}{2}\left(\frac{a_1{}^2+a_2{}^2+\ldots+a_n{}^2}{A^2}+\frac{b_1{}^2+b_2{}^2+\ldots+b_n{}^2}{B^2}\right)$$ $$=\frac{1}{2}(1+1)$$ $$=1$$ $$\sum_{i=1}^{n}\frac{a_ib_i}{AB} \le 1$$ $$\sum_{i=1}^{n}a_ib_i \le AB = \sqrt{a_1{}^2+a_2{}^2+\ldots+a_n{}^2}\sqrt{b_1{}^2+b_2{}^2+\ldots+b_n{}^2}$$ $$\left(\sum_{i=1}^{n}a_ib_i\right)^2 \le \left(\sum_{i=1}^{n}a_i{}^2\right)\left(\sum_{i=1}^{n}b_i{}^2\right)$$ Which is the Cauchy-Schwarz Inequality		
(iii)	$$\left(2x+3y+4z\right)^2 \le \left(2^2+3^2+4^2\right)\left(x^2+y^2+z^2\right)$$ $$=29$$ $$2x+3y+4z \le \sqrt{29}$$ Maximum value is $\sqrt{29}$		

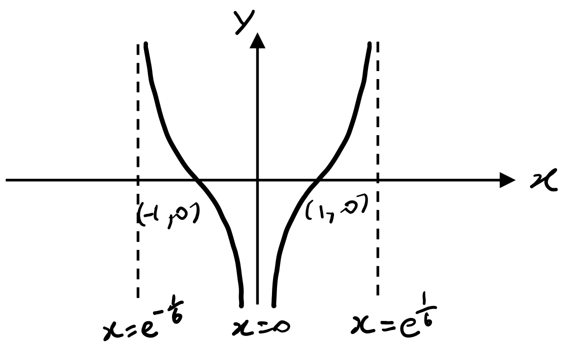
Qn	Solution		
3 **(a)**	Any of the number in the subset of 55 numbers must fall into one of the six groups that are congruence to 0, 1, 2, 3, 4 and 5 modulo 6. By Pigeonhole Principle, there must be a group with more than nine numbers since if each group have only nine then the total is 6 × 9 = 54 < 55. Let $a_1, a_2, ..., a_{10}$ be these numbers such that $a_1 < a_2 < ... < a_{10}$. Since $a_{i+1} \equiv a_i \pmod 6$ which implies that $a_{i+1} - a_i \equiv 0 \pmod 6$. Thus $a_{i+1} - a_i \in \{6, 12, ...\}$. We claim that $a_{i+1} - a_i = 6$ for some *i*. For if not then for all *i*, $a_{i+1} - a_i \geq 12$ then $a_{10} - a_1 \geq 9 \times 12 = 108$ which is impossible since a_1 and $a_{10} \in \{1, 2, ..., 100\} \Rightarrow a_{10} - a_1 < 100$. Thus $a_{i+1} - a_i = 6$ for some *i*. There exist two number in the subset of 55 numbers that differ by 6.		

3 **(b)** **(i)**	Consider $n^{ax}-1=0$ $n^{ax}=1$ $n^{a}=(1)^{\frac{1}{x}}$ $n^{a}=1$ Hence $n^{a}-1$ is a factor of $n^{ax}-1$.		
(ii)	Because $\gcd(a,b)$ divides both *a* and *b*. Therefore $a=(\gcd(a,b))(x)$ and $b=(\gcd(a,b))(y)$ for integer *x* and *y*. By (i), $n^{\gcd(a,b)}-1$ is a factor of $n^{(\gcd(a,b))(x)}-1$ $\Rightarrow n^{\gcd(a,b)}-1$ divides $n^{(\gcd(a,b))(x)}-1=n^{a}-1$ Similarly, $n^{\gcd(a,b)}-1$ is a factor of $n^{(\gcd(a,b))(y)}-1$ $\Rightarrow n^{\gcd(a,b)}-1$ divides $n^{(\gcd(a,b))(y)}-1=n^{b}-1$ Thus, $\underline{n^{\gcd(a,b)}-1 \le \gcd(n^{a}-1, n^{b}-1)}$ On the other hand, we can find positive integers *x* and *y* such that $ax_0-by_0=\gcd(a,b)$ and by part (i) $n^{a}-1$ divides $n^{ax_0}-1$ $n^{b}-1$ divides $n^{by_0}-1$ Consider $(n^{ax_0}-1)-(n^{by_0}-1)=n^{ax_0}-n^{by_0}$ $=n^{by_0}(n^{ax_0-by_0}-1)$ $\gcd(n^{a}-1, n^{b}-1)$ divides LHS of this equation and has no common factor with n^{by_0}. Therefore it must divides $n^{ax_0-by_0}-1$. Thus, $\underline{n^{\gcd(a,b)}-1 \ge \gcd(n^{a}-1, n^{b}-1)}$ Hence $\underline{n^{\gcd(a,b)}-1=\gcd(n^{a}-1, n^{b}-1)}$		

Qn	Solution		
4 **(i)**	Let P_n be the statement $15^n - 7^n$ is divisible by 8 for all positive integer *n*. When *n* = 1, $15^1 - 7^1 = 8$ which is divisible by 8. Thus, P_1 is true. Assume that P_k is true for some *k*, *k* = 1, 2, 3, … $15^k - 7^k$ is divisible by 8 To prove that P_{k+1} is true, i.e. $15^{k+1} - 7^{k+1}$ is divisible by 8 For P_k, $15^k - 7^k$ is divisible by 8 $\Rightarrow 15^k - 7^k = 8m$ for some integer *m*. $\text{LHS} = 15^{k+1} - 7^{k+1}$ $\equiv 15(15^k) - 7(7^k)$ $\equiv 15(8m + 7^k) - 7(7^k)$ from P_k $= 8(15m) + 7^k(15) - 7^k(7)$ $= 8[15m + 7^k]$ which is divisble by 8 Thus, P_{k+1} is true. Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, then by mathematical induction, P_n is true for all positive integer *n*.		

Qn	Solution		
4 (ii)	Assume that such linear polynomial exist $\mathrm{f}(x) = ax + b$ where a and b are integers. $\mathrm{f}(7) = 7a + b = 5 \quad ---(1)$ $\mathrm{f}(15) = 15a + b = 9 \quad ---(2)$ $(2)-(1): 8a = 4$ $a = \frac{1}{2}$ Which contradicts that a is an integer. Thus such linear polynomial doesn't exist.		
4 (iii)	Assume that such a polynomial $\mathrm{P}(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ does exist. $\mathrm{P}(7) = a_n(7^n) + a_{n-1}(7^{n-1}) + \ldots + a_0 = 5 \quad ---(1)$ $\mathrm{P}(15) = a_n(15^n) + a_{n-1}(15^{n-1}) + \ldots + a_0 = 9 \quad ---(2)$ $(2)-(1): a_n(15^n - 7^n) + a_{n-1}(15^{n-1} - 7^{n-1}) + \ldots + a_1(15-7) = 4$ By part (i), $15^n - 7^n$ is divisible by 8 for all positive integer n which will imply that 4 is divisible by 8 which is a contradiction thus such polynomial doesn't exist.		

Qn	Solution		
5 (i)	$z = x^2 u$ $\frac{\mathrm{dz}}{\mathrm{d}x} = 2xu + x^2 \frac{\mathrm{d}u}{\mathrm{d}x}$ $= 2xu + x^2 \left(3\mathrm{f}(x) - \frac{2u}{x}\right)$ $= 2xu + 3x^2 \mathrm{f}(x) - 2xu$ $\frac{\mathrm{dz}}{\mathrm{d}x} = 3x^2 \mathrm{f}(x)$ [shown]		
(ii)	$u = -\frac{1}{2\mathrm{e}^{2y}} = -\frac{1}{2}\mathrm{e}^{-2y}$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^{-2y} \frac{\mathrm{d}y}{\mathrm{d}x}$ $= \mathrm{e}^{-2y}\left(\frac{3\mathrm{e}^{2y}}{x^3} + \frac{1}{x}\right)$ $= \frac{3}{x^3} + \frac{\mathrm{e}^{-2y}}{x}$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{3}{x^3} - \frac{2u}{x}$ using part (i) with $\mathrm{f}(x) = \frac{1}{x^3}$, therefore, $\frac{\mathrm{dz}}{\mathrm{d}x} = 3x^2\left(\frac{1}{x^3}\right)$ $\frac{\mathrm{dz}}{\mathrm{d}x} = \frac{3}{x}$ $z = \int \frac{3}{x}\,\mathrm{d}x$ $-\frac{1}{2}x^2\mathrm{e}^{-2y} = 3\ln\lvert x\rvert + c$ sub in (1, 0), $c = -\frac{1}{2}$		

	$-\frac{1}{2}x^2\mathrm{e}^{-2y}=3\ln\lvert x\rvert-\frac{1}{2}$ $x^2\mathrm{e}^{-2y}=1-6\ln\lvert x\rvert$ $\mathrm{e}^{-2y}=\frac{1-6\ln\lvert x\rvert}{x^2}$ $-2y=\ln\left(\frac{1-6\ln\lvert x\rvert}{x^2}\right)$ $y=-\frac{1}{2}\ln\left(\frac{1-6\ln\lvert x\rvert}{x^2}\right)$		
Qn	**Solution**		
5 **(iii)**	$\frac{1-6\ln\lvert x\rvert}{x^2}=0$ $\ln\lvert x\rvert=\frac{1}{6}$ $x=\pm\mathrm{e}^{\frac{1}{6}}$ 		

Qn	Solution		
6 **(a)**	Using Triangle Inequality, $\left\|\dfrac{x^3+y^3}{x^2+y^2}\right\| \le \left\|\dfrac{x^3}{x^2+y^2}\right\| + \left\|\dfrac{y^3}{x^2+y^2}\right\|$ $\le \left\|\dfrac{x^3}{x^2}\right\| + \left\|\dfrac{y^3}{x^2}\right\|$ $= \|x\| + \|y\|$ As $x \to 0$ and $y \to 0$, $\|x\| + \|y\| \to 0$ and since $\left\|\dfrac{x^3+y^3}{x^2+y^2}\right\| \ge 0$, by Squeeze Theorem, $\left\|\dfrac{x^3+y^3}{x^2+y^2}\right\| \to 0$. Therefore limit of $\left\|\dfrac{x^3+y^3}{x^2+y^2}\right\|$ is 0.		

Qn	Solution		
6 (b)	Consider $\lvert x-y+y\rvert$ and $\lvert y-x+x\rvert$ By Triangle inequality, $\lvert x-y+y\rvert \le \lvert x-y\rvert+\lvert y\rvert$ $\lvert x\rvert \le \lvert x-y\rvert+\lvert y\rvert$ $\lvert x\rvert-\lvert y\rvert \le \lvert x-y\rvert$ $\lvert y-x+x\rvert \le \lvert y-x\rvert+\lvert x\rvert$ $\lvert y\rvert \le \lvert y-x\rvert+\lvert x\rvert$ $\lvert y\rvert-\lvert x\rvert \le \lvert y-x\rvert$ $\lvert x\rvert-\lvert y\rvert \ge -\lvert y-x\rvert$ Therefore, $-\lvert x-y\rvert \le \lvert x\rvert-\lvert y\rvert \le \lvert x-y\rvert$ $\big\lvert \lvert x\rvert-\lvert y\rvert \big\rvert \le \lvert x-y\rvert$ (shown) $\left\lvert \lvert x\rvert^2-\lvert y\rvert^2 \right\rvert \le \left\lvert (\lvert x\rvert-\lvert y\rvert)(\lvert x\rvert+\lvert y\rvert) \right\rvert$ $= \big\lvert \lvert x\rvert-\lvert y\rvert \big\rvert \times \big\lvert \lvert x\rvert+\lvert y\rvert \big\rvert$ $\le \lvert x-y\rvert \times \big\lvert \lvert x\rvert+\lvert y\rvert \big\rvert$ Thus for $\lvert x-y\rvert \times \big\lvert \lvert x\rvert+\lvert y\rvert \big\rvert \le (x-y)^2$ we need $\big\lvert \lvert x\rvert+\lvert y\rvert \big\rvert \le \lvert x-y\rvert$ which is not true. Counter-example: $x = 5$ and $y = 3$.		

Qn	Solution		
7 (i)	Let the pens be identical balls and the different coloured pens be 4 distinct boxes. Number of ways to distribute 7 balls into 4 distinct boxes such that none is empty = Number of ways to distribute 3 balls into 4 distinct boxes $= \binom{3+4-1}{3} = \binom{6}{3}$		
	$= 20$		
7 (ii)	No. of ways to select the pens (no restriction) = 4^7 = 16 384		
	Let A_1 be the case where she does not give a red pen. Let A_2 be the case where she does not give a blue pen. Let A_3 be the case where she does not give a green pen. Let A_4 be the case where she does not give a black pen. By Principle of Inclusion and Exclusion Number of ways = $4^7 - \binom{4}{1}3^7 + \binom{4}{2}2^7 - \binom{4}{3}1^7 + \binom{4}{4}0^7$		
	= 8 400		
7 (iii)	Only red, blue and black pens can have 7 pens. In order to have 7 pens, by PP, we must have at least 6 × 3 + 1 = 19 beads.		
	But we need to include the green pens as they have the chance of being selected. So total we have 19 + 6 = 25 pens.		

Qn	Solution		
7 (iv) (a)	Note that all the sequences cannot start or end with a green pen, $x_1 = 3$		
	$x_2 = 9$		
7 (iv) (b)	There are 2 cases to be considered. <u>Case 1</u>: 2nd pen is not green Consider the last (n – 1) pens, with the first of these sequences, not green in colour. No. of choices for the remaining (n – 1) pens = x_{n-1}		
	No. of choices for the 1st pen = 3 Total number of ways = $3x_{n-1}$ i.e. A = 3		
	<u>Case 2</u>: 2nd pen is green Consider the last (n – 2) pens, with the first of these sequences, not green in colour. No. of choices for the remaining (n – 2) pens = x_{n-2}		
	No. of choices for the 2nd pen (green in colour) = 1 No. of choices for the 1st pen = 2 Total number of ways = $2x_{n-2}$ i.e. B = 2 By AP, total no of ways $= 3a_{n-1} + 2a_{n-2}$		
7 (iv) (c)	$x_3 = 3x_2 + 2x_1 = 3(9) + 2(3) = 33$ $x_4 = 3x_3 + 2x_2 = 3(33) + 2(9) = 117$ $x_5 = 3x_4 + 2x_3 = 3(117) + 2(33) = 417$		
	$x_6 = 3x_5 + 2x_4 = 3(417) + 2(117) = 1485$		

Qn	Solution		
8 (i) (a)	Assume that the containers are different, there are 2 choices for each of the n distinct objects, hence 2^n ways to distribute the objects.		
	However, there are 2 ways in which one of the containers will be empty, hence $2^n - 2$ ways to arrange the objects such that no container is empty. Since the containers are identical, there are $\dfrac{2^n - 2}{2} = 2^{n-1} - 1$ ways. $\therefore S(n,\ 2) = 2^{n-1} - 1$		
8 (i) (b)	Let the r distinct objects be A, B, C, …, R. **<u>Case 1</u>: *A* is alone in a box** There remains $(r - 1)$ distinct objects to be distributed into $(n - 1)$ identical boxes with no empty boxes. $\therefore$ no. of ways = $S(r - 1, n - 1)$		
	<u>Case 2</u>: *A* is not alone in a box We first distribute the other $(r - 1)$ distinct objects into n identical boxes such that no box is empty. By definition, this can be done in $S(r - 1, n)$ ways. Then, we place object A into one box. There are n boxes so there are n ways. By the Multiplication Principle, number of ways = $n\,S(r - 1, n)$ $\therefore$ By the Addition Principle, total no. of ways: $S(r, n) = S(r - 1, n - 1) + n\,S(r - 1, n)$.		
	$\therefore\ S(n+1,\ 3) = S(n,\ 2) + 3S(n,\ 3)$ $= 2^{n-1} - 1 + 3S(n,\ 3)$		
8 (ii)	Required no. of ways $= S(5,\ 3) + S(5,\ 2) + S(5,\ 1)$		
	$= 2^3 - 1 + 3S(4,\ 3) + 2^4 - 1 + 1$ $= 23 + 3[S(3,\ 2) + 3S(3,\ 3)]$ $= 23 + 3[2^2 - 1 + 3]$		
	= 41		