Answer **all** the questions.

1 **(i)** The real numbers x, y and z are such that

$$x^2 + y^2 + z^2 = 1.$$

Use the Cauchy-Schwarz inequality to prove that

$$2x + 3y + 6z \leq 7.$$ [3]

(ii) Solve the simultaneous equations

$$2x + 3y + 6z = 7$$
$$x^2 + y^2 + z^2 = 1.$$ [2]

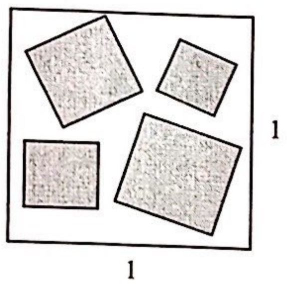
(iii) Let n be a positive integer. The real numbers x_i, where $1 \leq i \leq n$, are such that

$$\sum_{i=1}^{n} x_i^2 = 1.$$

Find, with proof, the maximum possible value of

$$\sum_{i=1}^{n} x_i.$$ [3]

(iv) A unit square contains a number of non-overlapping squares as shown in the diagram.



Given that the total perimeter of the non-overlapping squares is 18, prove that there are more than 20 such squares. [4]

2 A shop sells wristbands which are available in six different designs. The shopkeeper has a large number of wristbands of each design.

(i) The shopkeeper places eight wristbands in a line. In how many ways can she do this

(a) if no two wristbands of the same design are to be next to each other, [2]

(b) if each design must be used at least once? [4]

(ii) A customer wishes to buy twenty wristbands. In how many ways can he do this

(a) if there are no other restrictions, [2]

(b) if he buys at least one of each design and at most 6 of a particular design? [4]

3 **(a)** Amy has a huge supply of beads that are red, blue, green and black in colour. She proceeds to string them in a row, such that any black bead is in between 2 non-black beads of different colours. Let a_n be the number of such arrangements of the beads of length n. Thus $a_1 = 3$, as the colour of the bead is either red, blue or green.

(i) Find a_2. [1]

(ii) Show that for $n \geq 3$, $a_n = 3a_{n-1} + 2a_{n-2}$. [3]

(iii) Deduce that a_n is divisible by 3, for all positive integers n. [2]

(iv) Use mathematical induction to prove that a_{2n} is divisible by 9, for all positive integers n. [3]

(b) It is given that a set contains ten positive integers of two digits each. Show that there are two disjoint non-empty subsets with the same sum of the elements. [4]

4 [You may use the following result without proof in this question:
A monotone sequence u_n is one such that $u_n \leq u_{n+1}$ or $u_n \geq u_{n+1}$ for all $n \in \mathbb{Z}^+$. If u_n is a monotone sequence of real numbers, then the sequence has a finite limit if and only if the sequence is bounded.]

(a) Let $p > 1$.

(i) Show that

$$\sum_{n=2^{N-1}}^{2^N-1} \frac{1}{n^p} < \frac{1}{\left(2^{N-1}\right)^{p-1}}$$

for all integers $N \geq 2$. [2]

(ii) Hence prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. [3]

(iii) Let a_n and b_n be sequences such that $a_n = n^p b_n$ and a_n converges.

Prove that $\sum_{n=1}^{\infty} |b_n|$ converges. [3]

(b) A sequence is defined by

$$x_1 = 1 \text{ and } x_{i+1} = x_i + \frac{1}{i\,x_i} \text{ for } i \geq 1.$$

If the sequence converges, find its limit. Otherwise, prove that the sequence diverges. [4]

[Turn over

5 (a) (i) A sequence $u_0, u_1, u_2, \ldots$ is such that $u_0 = 1$, $u_1 = 0$ and

$$u_{n+2} = \frac{u_n}{n+2}.$$

Find u_n in terms of n, distinguishing the cases where n is odd and n is even. [4]

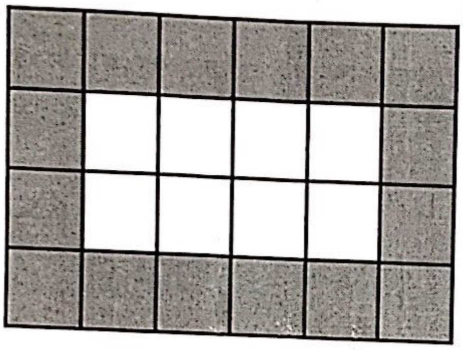
It is given that $\mathrm{f}(x) = \sum_{n=0}^{\infty} c_n x^n$, where c_n is a constant for each $n \in \mathbb{Z}$, $n \geq 0$.

(ii) Show that if $\mathrm{f}''(x) - x\mathrm{f}'(x) - \mathrm{f}(x) = 0$, then $c_{n+2} = \dfrac{c_n}{n+2}$. [5]

It is also known that $\mathrm{f}(0) = 1$ and $\mathrm{f}'(0) = 0$.

(iii) Hence, find a simplified expression for $\mathrm{f}(x)$. Your answer should not involve infinite sum(s). [2]

(b) A $n \times m$ rectangle is drawn on a grid and the border squares are shaded. The diagram below shows a 4×6 rectangle with the required shading of the border squares.



In this case, there are more shaded squares than non-shaded squares.
Find all values of n and m where there will be an equal number of shaded and non-shaded squares in the $n \times m$ rectangle. Justifying your answer. [4]

6 Let n be a positive integer and suppose that $2n$ numbers are arranged at different points around a circle, half of these numbers being $+1$ and half being -1. Moving clockwise around the circle from a given starting position, let T_i be the total of the first i numbers passed.

(i) Prove that there is a starting position on the circle for which no T_i is negative. [7]

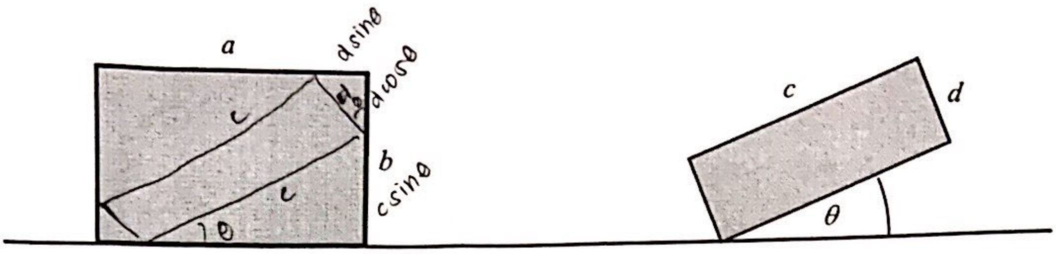
(ii) For any starting position, prove that

$$n + \sum_{i=1}^{2n} T_i$$

is even. [4]

7 Let a, b, c and d be positive numbers such that $a \geq b$ and $c \geq d$.

(i) Two rectangles are as shown, with θ the acute angle between the longest sides of the rectangles.



Given that, after a suitable translation, the $c \times d$ rectangle can be strictly contained in the $a \times b$ rectangle, write down two inequalities for a and b in terms c, d and θ. [2]

(ii) Given that $a > c$, prove that the $c \times d$ rectangle can be strictly contained in the $a \times b$ rectangle for some value of θ if and only if $b > d$. [3]

(iii) Given that $a \leq c$, prove that the $c \times d$ rectangle can be strictly contained in the $a \times b$ rectangle for some value of θ then $a > \dfrac{c+d}{\sqrt{2}}$. [4]

(iv) Obtain a necessary and sufficient condition for a $c \times d$ rectangle to be strictly contained in the $a \times a$ square. [2]

8 Let N be any positive integer and let $\mathrm{S}(N)$ be the set of remainders when the square numbers $0, 1, 4, 9, \ldots$ are divided by N.

(i) Prove that all the elements of $\mathrm{S}(12)$ are square numbers. [2]

(ii) Find an odd integer N and an element x of $\mathrm{S}(N)$ such that x is not a square number. [2]

(iii) For all positive integers N, prove that $\mathrm{S}(N)$ has at least $\sqrt{N}$ elements. [3]

(iv) It is given that there are integers x, λ and n, where $n \geq 5$, such that

$$x^2 = 17 + 2^n \lambda.$$

Prove that 17 is in $\mathrm{S}(2^{n+1})$. [3]

(v) For $n \geq 5$, prove that $\mathrm{S}(2^n)$ has at least $1 + \sqrt{2^n}$ elements. [4]