EUNOIA JUNIOR COLLEGE
JC2 Preliminary Examination 2020
General Certificate of Education Advanced Level
Higher 3

MATHEMATICS 9820/01

21 September 2020

Paper 1 [100 marks]

3 hours

Additional Materials: Answer Booklet
List of Formulae (MF26)

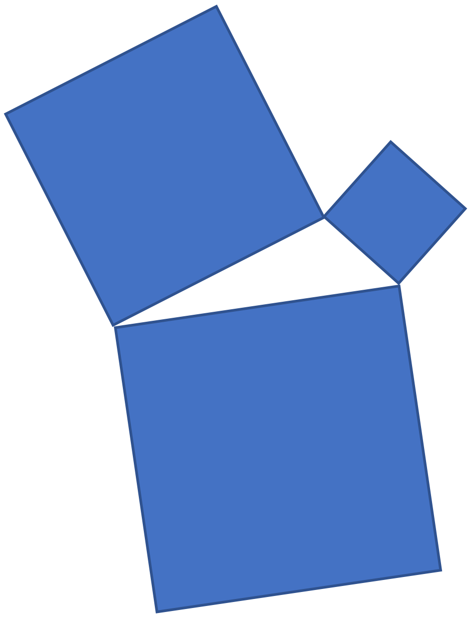
READ THESE INSTRUCTIONS FIRST
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, all work must be handed in.
If you have used any additional booklets, please insert them inside the 12-page Answer Booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 a, b and c are the lengths of the three sides of a triangle with perimeter 2. Squares are constructed on each of the three sides of the triangle, as shown in the diagram.



Let S be the sum of the areas of the three squares.

(i) Using the Cauchy-Schwarz inequality, prove that $\frac{4}{3} \le S$, and find the condition that must be satisfied by a, b and c in order to have $\frac{4}{3} = S$. [5]

(ii) **(a)** Explain why the length of any side of the triangle, i.e. a, b and c, are all less than 1. [1]

(b) Hence or otherwise, show that $1+(b+c-1)^2 > b^2+c^2$. [3]

(iii) Hence explain why b^2+c^2 is always less than $1+(1-a)^2$. [1]

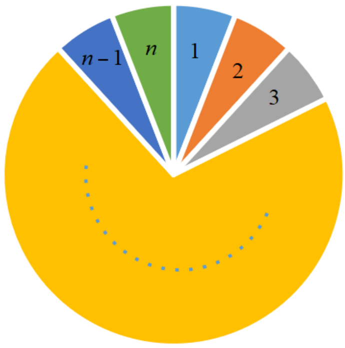
(iv) Deduce that $S < 2$. [2]

(v) Give an example of a set of side lengths $\{a, b, c\}$ such that $S = 1.36$. [1]

2 **(a)** A subphrase is a sequence of letters that appears together in the exact order within the word given. For example, "JUST" and "IF" is a subphrase in the word REJUSTIFY, but not "REST".

Find the number of rearrangements possible for the 9 letters of the word REJUSTIFY in which the following phrases are not found as a subphrase of the rearrangement: "JUST', "RYE", "IF". [4]

(b) Seven distinct colours are used to colour the sectors of a circle shown on the right.



Let a_n denote the number of ways to colour a circle with n sectors such that each sector is coloured by one colour and any two adjacent sectors must be coloured by different colours.

(i) Find the values of a_1 and a_2. [2]

(ii) Explain why $a_n + a_{n-1} = 7\left(6^{n-1}\right)$ for $n \geq 3,\ n \in \mathbb{Z}^+$. [3]

(iii) Use Mathematical Induction to show that

$$a_n = 6(-1)^n + 6^n \text{ for all } n \geq 3,\ n \in \mathbb{Z}^+. \quad [4]$$

3 **(a)** A shop sells doughnuts which comes in four flavours – vanilla, strawberry, chocolate and banana. The shop has a large number of doughnuts of each flavour.

(i) Mr. Lim wishes to buy a set of 12 doughnuts such that there is at least one doughnut of each flavour. In how many ways can he do this? [3]

(ii) Mrs. Tan wishes to buy 12 doughnuts for her 4 children, where each child can get any number of doughnuts of any flavour, including none (subject to a total of 12 doughnuts). In how many ways can she do this? [3]

(iii) Mr. Soh wishes to buy 2 doughnuts for each of 6 boys. In how many ways can he do this? [3]

(b) Let $x_1, x_2, \ldots, x_n$ be any sequence of integers. By using the pigeonhole principle, or otherwise, prove that there is a consecutive subsequence with a sum that is divisible by n. [5]

4 **(a)** Given a positive integer c, show that if c^3 is odd, then c is odd. [2]

(b) Given positive integers a, b, c, and d, if

- $\gcd(a+b, c) = 2d$
- $\gcd(a, b) = 28$
- $14 \mid c$

show that $7 \mid d$. [3]

(c) Show that if a and b are odd integers with $a > b > 1$, then $a^2 - b^2$ is divisible by 8. [4]

(d) It is known that $\gcd(p,\ q) = \gcd(p-q,\ q) = \gcd(p+q,\ q)$ for any positive integers p and q.

(i) Using the property above, show that if $\gcd(a+b,\ a-b) = 1$ for positive integers a and b with $a > b > 1$, then $\gcd(a, b) = 1$. [3]

(ii) Provide a counterexample to show why the converse does not necessarily hold. [1]

5 **(a)** The *Bernoulli equation* has the general form

$$\frac{dy}{dx} + f(x)y = g(x)y^a, \qquad a \neq 0,\ 1.$$

(i) Show that the *Bernoulli equation* can be transformed into

$$\frac{dz}{dx} + (1-a)f(x)z = (1-a)g(x), \qquad a \neq 0,\ 1.$$

by the substitution $z = y^{1-a}$. [3]

(ii) By considering $\frac{d}{dx}\left(ze^{(1-a)\int f(x)\,dx}\right)$, show that the general solution for the first order linear equation in **(i)** is $z = e^{(a-1)\int f(x)\,dx}\int (1-a)e^{(1-a)\int f(x)\,dx} g(x)\,dx$. [3]

(iii) A solution curve of the differential equation $\frac{dy}{dx} = -y^2 \sin x + y$ passes through the point $(0,1)$. Find the equation of the curve. [6]

6 **(i)** Let x be an integer. By considering $x \equiv k \pmod{8}$, where k is an integer satisfying $0 \le k \le 7$, show that $x^2 \equiv r$, where $r \in \{0, 1, 4\}$. [3]

(ii) Hence, prove that if n is a positive integer of the form $8m+7$, then there do not exist integers a, b, c such that $n = a^2 + b^2 + c^2$. [3]

(iii) Hence, or otherwise, using mathematical induction, prove that no integer of the form $4^k(8m+7)$, where $k \in \mathbb{N}$, can be written as the sum of square of three integers. Hint: Use **(ii)** to prove the case where $k = 1$. [6]

7 Let f be a periodic differentiable function with period 2π. It is known that such a function can be represented by a linear combination of sines and cosines.

Define $S_n(x)$ as the n^{th} partial sum of the Fourier series of the function f described as follows:

For non-negative integers n,

$$S_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx),\ x \in \mathbb{R},$$

where for $k = 1, 2, \ldots,$

the coefficients of the cosines and sines can be computed as

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{f}(t)\,\mathrm{d}t, \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{f}(t)\cos(kt)\,\mathrm{d}t, \qquad \text{and} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{f}(t)\sin(kt)\,\mathrm{d}t.$$

(a) Let $\mathrm{f}(x) = \cos(mx)$, where $m = 1, 2, \ldots.$ For $k = 0, 1, \ldots,$ show that

(i) $b_k = 0$,

(ii) $a_k = 0,\ k \ne m$,

(iii) $a_m = 1$. [5]

Define $D_n(z) = \dfrac{1}{2\pi}\left[1 + 2\sum_{k=1}^{n}\cos(kz)\right],\ z \in \mathbb{R}.$

(b) Using the definitions of a_k and b_k, and interchanging integrals and summations, show that $S_n(x) = \int_{-\pi}^{\pi} D_n(x-t)\mathrm{f}(t)\,\mathrm{d}t$. [3]

(c) By considering $\sin\left(\dfrac{z}{2}\right)D_n(z)$ and using a method of differences, or otherwise, show that $D_n(z) = \dfrac{\sin\left(n+\dfrac{1}{2}\right)z}{2\pi\sin\left(\dfrac{z}{2}\right)}$. [2]

8 Let $\{a_n\}$ and $\{b_n\}$ be two infinite sequences with $n = 0,1,2,...$ and all terms non-negative. The Comparison Test states that if $b_n \geq a_n$ for all non-negative integers and $\sum_{i=0}^{\infty} b_i$ converges, then $\sum_{i=0}^{\infty} a_i$ converges.

(a) Let the set A denote the set of positive integers n containing the digit 9 and the set B denote the set of positive integers n not containing the digit 9.

Also, let

$$c_0 = 1 + \frac{1}{2} + ... + \frac{1}{8}$$

$$c_1 = \frac{1}{10} + \frac{1}{11} + ... + \frac{1}{18} + \frac{1}{20} + ... + \frac{1}{88}$$

$$\vdots$$

$$c_k = \frac{1}{10^k} + \frac{1}{10^k + 1} + ... + \frac{1}{\underbrace{8...8}_{k+1 \text{ digits}}}$$ and so on, where $k \in \mathbb{Z}^+$.

(i) Show that $c_k \leq 8\left(\frac{9}{10}\right)^{k-1}$. Hence or otherwise, show that

$$\sum_{n \in B} \frac{1}{n} = 1 + \frac{1}{2} + ... + \frac{1}{8} + \frac{1}{10} + ... + \frac{1}{18} + \frac{1}{20} + ... + \frac{1}{88} + \frac{1}{100} + ...$$

is convergent. [3]

(ii) Determine if $\sum_{n \in A} \frac{1}{n}$ is convergent or divergent. [1]

(b) **(i)** Show that $\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} < \frac{1}{2}$ for all positive integers k.

Hence, use the Limit Comparison Test to show that

$$\sum_{k=0}^{\infty} \left(\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right)$$

converges. [3]

(ii) Given that

- $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^{r-1}}{1-x^8} \, dx = \frac{1}{2^{\frac{r}{2}}} \sum_{i=0}^{\infty} \frac{1}{16^i (8i+r)}$,
- $y^5 + y^4 + 2y^3 - 4 \equiv (y-1)(y^2+2)(y^2+2y+2)$ and
- $y^4 + 4 \equiv (y^2+2y+2)(y^2-2y+2)$,

find the exact value of $\sum_{k=0}^{\infty} \left(\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right)$ with the use of the substitution $y = x\sqrt{2}$ when performing the integration. [6]

End of Paper