Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 7

Name: ________________________________ (____) Date: ________________

Class: Sec 4______

Revision 7: Binomial Theorem

1. **Binomial Expansion**

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \boxed{\binom{n}{r}a^{n-r}b^r} + \ldots + b^n$$

where n is a positive integer and the binomial coefficient $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$= \frac{n(n-1)\ldots(n-r+1)}{r(r-1)(r-2)\times\ldots\times 3\times 2\times 1}$$

2. **The T_{r+1} term**

The $(r+1)^{\text{th}}$ term, $T_{r+1} = \binom{n}{r}a^{n-r}b^r$

3. **Some Important Identities**

- $\binom{n}{1} = \binom{n}{n-1} = n$
- $\binom{n}{2} = \binom{n}{n-2} = \dfrac{n(n-1)}{2!}$
- $\binom{n}{3} = \binom{n}{n-3} = \dfrac{n(n-1)(n-2)}{3!}$

Example

1. Find the term independent of x in the expansion of $\left(2x^2 - \dfrac{4}{x^4}\right)^6$.

$$Tr+1 = \binom{6}{r}(2x^2)^{6-r}\left(-\frac{4}{x^4}\right)^r$$

$$= \binom{6}{r}(2)^{6-r}(-4)^r x^{12-6r}$$

When $12-6r=0$, $r=2$

Term independent of x $= \binom{6}{2}(2)^4(-4)^2$

$= 3840$

2. Write down the first three terms in the expansion, in ascending powers of x, of $\left(2-\frac{x}{4}\right)^n$, where n is a positive integer greater than 2.

The first two terms in the expansion, in ascending powers of x, of $(1+x)\left(2-\frac{x}{4}\right)^n$ are $a+bx^2$, where a and b are constant.

(ii) Find the value of n.

(iii) Hence, find the value of a and of b. N2009

(i) $\left(2-\frac{x}{4}\right)^n$

$$= 2^n + \binom{n}{1}2^{n-1}\left(\frac{x}{4}\right) + \binom{n}{2}2^{n-2}\left(-\frac{x}{4}\right)^2 + \cdots$$

$$= 2^n - 2^{n-1}n(2^{-2}x) + 2^{n-2}\left[\frac{n(n-1)}{2}(2^{-2}x)^2\right] + \cdots$$

$$= 2^n - 2^{n-3}nx + 2^{n-7}n(n-1)x^2 + \cdots$$

(ii) $(1+x)\left(2-\frac{x}{4}\right)^n$

$$= (1+x)[2^n - 2^{n-3}nx + 2^{n-7}n(n-1)x^2 + \cdots]$$

$$= 2^n - 2^{n-3}nx + 2^{n-7}n(n-1)x^2 + 2^n x - 2^{n-3}nx^2 + \cdots$$

$$= 2^n + (2^n - 2^{n-3}n)x + [2^{n-7}n(n-1) - 2^{n-3}n]x^2 + \cdots$$

$(1+x)\left(2-\frac{x}{4}\right)^n = a + bx^2 + \cdots$

- The expansion on the right has a term in x missing, indicating the coefficient of x is 0.

Comparing coefficient of x, $2^n - 2^{n-3}n = 0$

$2^n\left(1-\frac{n}{8}\right) = 0$

Since $2^n > 0$, $1-\frac{n}{8} = 0$

$n = 8$

(iii) Comparing constant term, $a = 2^8$

$= 256$

Comparing coefficient of x^2, $b = 2(8)(7) - 2^5(8)$

$= -144$

Exercise

1. (a) Write down and simplify the first four terms of $\left(1+\frac{x}{2}\right)^8$ in ascending powers of x. Hence, find the value of 1.002^8 correct to 4 decimal places.

$$\left(1+\frac{x}{2}\right)^8 = 1+\binom{8}{1}\left(\frac{x}{2}\right)+\binom{8}{2}\left(\frac{x}{2}\right)^2+\binom{8}{3}\left(\frac{x}{2}\right)^3+\cdots$$
$$= 1+4x+7x^2+7x^3+\cdots$$

$$1.002^8 = \left(1+\frac{0.004}{2}\right)^8$$
$$= 1+4(0.004)+7(0.004)^2+7(0.004)^3+\cdots$$
$$\approx 1.0161$$

(b) If the first three terms of the expansion of $(2-ax)^n$ in ascending powers of x are $32-240x+720x^2$, find n and a.

$$(2-ax)^n = 2^n+\binom{n}{1}2^{n-1}(-ax)+\binom{n}{2}2^{n-2}(-ax)^2+\cdots$$
$$= 2^n-2^{n-1}anx+2^{n-3}n(n-1)a^2x^2+\cdots$$

Comparing coefficient of terms,

$2^n = 32$

$n = 5$

$2^{n-1}an = 240$

$80a = 240$

$a = -3$

(c) Find the term independent of x in the expansion of $\left(2x-\frac{1}{x}\right)^8$.

$$T_{r+1} = \binom{8}{r}(2x)^{8-r}\left(-\frac{1}{x}\right)^r$$
$$= \binom{8}{r}(2^{8-r})(-1)^r x^{8-2r}$$

When $8-2r = 0$, $r = 4$

$$\text{Constant term} = \binom{8}{4}(2^4)(-1)^4$$
$$= 1120$$

2. (a) (i) Write in ascending powers of x, the first three terms in the expansion of $(1+ax)^8$.

(ii) Given that the first three terms in the expansion of $(b+2x)(1+ax)^8$ are $5-38x+cx^2$, find the values of a, b and c.

(i) $$\begin{aligned}(1+ax)^8 &= 1+\binom{8}{1}(ax)+\binom{8}{2}(ax)^2+\cdots \\ &= 1+8ax+28a^2x^2+\cdots\end{aligned}$$

(ii) $$\begin{aligned}&(b+2x)(1+ax)^8 \\ &= (b+2x)(1+8ax+28a^2x^2+\cdots) \\ &= b+8abx+28a^2bx^2+2x+16ax^2+\cdots \\ &= b+(8ab+2)x+(28a^2b+16a)x^2+\cdots\end{aligned}$$

Comparing coefficient of terms,
$b=5$

$$\begin{aligned}8ab+2 &= -38 \\ 40a &= -40 \\ a &= -1\end{aligned}$$

$$\begin{aligned}c &= 28a^2b+16a \\ &= 28(-1)^2(5)+16(-1) \\ &= 14\end{aligned}$$

(b) Find the term independent of x in the expansion of $\left(2x^3+\dfrac{1}{3x^2}\right)^{10}$.

$$\begin{aligned}T_{r+1} &= \binom{10}{r}(2x^3)^{10-r}\left(\frac{1}{3x^2}\right)^r \\ &= \binom{10}{r}(2)^{10-r}\left(\frac{1}{3}\right)^r x^{30-5r}\end{aligned}$$

When $30-5r=0$, $r=6$

$$\begin{aligned}\text{Constant term} &= \binom{10}{6}(2)^4\left(\frac{1}{3}\right)^6 \\ &= \frac{1120}{243}\end{aligned}$$

3. (a) Find the coefficient of x^7 in the expansion of $\left(2x^2+\frac{1}{x}\right)^{11}$.

$$T_{r+1}=\binom{11}{r}(2x^2)^{11-r}\left(\frac{1}{x}\right)^r$$
$$=\binom{11}{r}(2)^{11-r}x^{22-3r}$$

When $22-3r=7$, $r=5$

$$\text{Coefficient of } x^7=\binom{11}{5}(2)^6$$
$$=29568$$

(b) (i) Find the first three terms in the expansion of $(1-2x)^6$ in ascending powers of x, simplifying the coefficients.

(ii) Given that the expansion of $(1-2x)^6(ax+b)$ is $3 + cx + 204x^2 + \ldots$, find the values of a, b and c.

(ii) $(1-2x)^6=1+\binom{6}{1}(-2x)+\binom{6}{2}(-2x)^2+\cdots$
$$=1-12x+60x^2+\cdots$$

(iii) $(1-2x)^6(ax+b)$
$$=(ax+b)(1-12x+60x^2+\cdots)$$
$$=ax-12ax^2+b-12bx+60bx^2+\cdots$$
$$=b+(a-12b)x+(60b-12a)x^2+\cdots$$

Comparing coefficient of terms, $b=3$

$$60b-12a=204$$
$$12a=180-204$$
$$a=-2$$
$$c=a-12b$$
$$=-2-36$$
$$=-38$$

4. Find the coefficient of x in the expansion of $\left(x+\frac{2}{x^2}\right)^7$.

$$T_{r+1}=\binom{7}{r}x^{7-r}\left(\frac{2}{x^2}\right)^r$$
$$=\binom{7}{r}(2^r)x^{7-3r}$$

When $7-3r=1$, $r=2$

$$\text{coefficient of } x=\binom{7}{2}(2^2)$$
$$=84$$

5. (a) Evaluate the term which is independent of x in the expansion of $\left(x^2 - \frac{1}{2x^7}\right)^{18}$.

$$T_{r+1} = \binom{18}{r} x^{2(18-r)} \left(\frac{1}{2x^7}\right)^r$$
$$= \binom{18}{r}\left(\frac{1}{2}\right)^r x^{36-9r}$$

When $36 - 9r = 0$, $\quad r = 4$

$$\text{coefficient of } x = \binom{18}{4}\left(\frac{1}{2}\right)^4$$
$$= \frac{765}{4}$$

(b) (i) Expand $(1+px+x^2)^n$ in ascending powers of x up to the term in x^2, where n is a positive integer.

(ii) If the coefficients of x and x^2 in the expansion in (b) (i) are 2 and $\frac{59}{5}$ respectively, find the values of p and n.

(i) $$(1+px+x^2)^n = [1+(px+x^2)]^n$$
$$= 1 + \binom{n}{1}(px+x^2) + \binom{n}{2}(px+x^2)^2 + \cdots$$
$$= 1 + n(px+x^2) + \frac{n(n-1)(p^2x^2)}{2} + \cdots$$
$$= 1 + npx + \left[n + \frac{n(n-1)}{2}p^2\right]x^2 + \cdots$$

(ii) $np = 2 \Rightarrow p = \frac{2}{n}$ ……(1)

$n + \frac{n(n-1)}{2}p^2 = \frac{59}{5}$ ……(2)

Sub (1) into (2) $\quad n + \frac{n(n-1)}{2}\left(\frac{2}{n}\right)^2 = \frac{59}{5}$

$$n + \frac{2(n-1)}{n} = \frac{59}{5}$$
$$5n^2 + 10n - 10 = 59n$$
$$5n^2 - 49n - 10 = 0$$
$$(5n+1)(n-10) = 0$$
$$n = \frac{1}{5} \text{ (NA)},\ 10$$
$$p = \frac{1}{5}$$

6. In the expansion of $(x+p)(x+q)^7$, the coefficient of x^7 is $\frac{4}{3}$ and there is no term in x^6. Given that $q \neq 0$, find the values of the constants p and q.

Hence, show that the coefficient of x^5 is $-\frac{28}{27}$.

$$(x+p)(x+q)^7$$
$$= (x+p)\left[x^7 + \binom{7}{1}qx^6 + \binom{7}{2}q^2x^5 + \binom{7}{3}q^3x^4 + \cdots\right]$$
$$= (x+p)(x^7 + 7qx^6 + 21q^2x^5 + 35q^3x^4 + \cdots)$$
$$= x^8 + 7qx^7 + 21q^2x^6 + px^7 + 7pqx^6 + 21pq^2x^5 + 35q^3x^5 + \cdots$$

Coefficient of $x^4 = \frac{4}{3}$

$$7q + p = \frac{4}{3}$$
$$p = \frac{4-21q}{3} \qquad \ldots\ldots(1)$$
$$21q^2 + 7pq = 0 \qquad \ldots\ldots(2)$$

Sub (1) into (2)

$$21q^2 + 7q\left(\frac{4-21q}{3}\right) = 0$$
$$81q^2 + 36q - 189q^2 = 0$$
$$108q^2 - 36q = 0$$
$$36q(3q-1) = 0$$
$$q = 0 \text{ (NA)}, \ \frac{1}{3}$$
$$p = -1$$

Coefficient of x^5 $= 21pq^2 + 35q^3$
$$= 21(-1)\left(\frac{1}{3}\right)^2 + 35\left(\frac{1}{3}\right)^3$$
$$= -\frac{28}{27}$$

7. (a) (i) Find the first four terms of the expansion, in ascending powers of x, of $(3+x)^5$.

Hence obtain
(ii) the coefficient of x^3 in the expansion of $(3+x)^5(2x^2+x-1)$,
(iii) the coefficient of y^3 in the expansion of $(3-y+y^2)^5$.

(i) $(3+x)^5 = 3^5 + \binom{5}{1}(3^4)x + \binom{5}{2}(3^3)x^2 + \binom{5}{3}(3^2)x^3 + \cdots$
$= 243 + 405x + 270x^2 + 90x^3 + \cdots$

(ii) $(3+x)^5(2x^2+x-1)$
$= (2x^2+x-1)(243+405x+270x^2+90x^3+\cdots)$
Coefficient of $x^3 = 2(405) + 270 - 90$
$= 990$

(iii) $(3-y+y^2)^5$
$= [3+(-y+y^2)]^5$
$= 243 + 405(-y+y^2) + 270(-y+y^2)^2 + 90(-y+y^2)^3 + \cdots$
$= 243 + 405(y+y^2) + 270(y^2-2y^3) - 90y^3 + \cdots$
Coefficient of $x^3 = -2(270) - 90$
$= -630$

(b) Determine if the term independent of x exists in the expansion of $(x^2-2x)^{100}$.

$T_{r+1} = \binom{100}{r} x^{2(100-r)}(-2x)^r$
$= \binom{100}{r}(-2)^r x^{200-r}$
When $200-r=0,\quad r=200$
Yes, the term independent of x does not exist.

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8. Find the value of k for which the coefficient of x^2 in the expansion of $(4+kx)(2-x)^6$ is zero. Hence, evaluate the coefficient of x in the expansion.

$(4+kx)(2-x)^6$
$= (4+kx)\left[2^6 + \binom{6}{1}(2^5)(-x) + \binom{6}{2}(2^4)(-x)^2 + \cdots\right]$
$= (4+kx)(64 - 192x + 240x^2 + \cdots)$

Coefficient of $x^2 = 0$
$4(240) - 192k = 0$
$k = 5$

Coefficien t of $x = 4(-192) + 64k$
$= -448$

9. Given that the coefficient of $\dfrac{1}{x^3}$ is 512 in the expansion $\left(\dfrac{2}{x}+ax^2\right)^9$, where $a < 0$.

(i) Find the value of a.

(ii) Hence, using the value of a found in (i), show that the term in $\dfrac{1}{x^4}$ does not exist in the expansion $\left(\dfrac{2}{x}+ax^2\right)^9\left(\dfrac{1}{8x}+\dfrac{x^2}{12}\right)$.

(i) $$T_{r+1} = \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(ax^2)^r$$
$$= \binom{9}{r}(2)^{9-r}a^r x^{-(9-r)+2r}$$
$$= \binom{9}{r}(2)^{9-r}a^r x^{3r-9}$$

When $3r-9=-3$, $3r=6$

$r=2$

coefficien t of $\dfrac{1}{x^3}=512$

$$\binom{9}{2}(2)^7a^2=512$$

$$4608a^2=512$$

$$a^2=\frac{1}{9}$$

$$a=-\frac{1}{3}\ (a<0)$$

(ii) When $3r-9=-6$, $3r=3$

$r=1$

$$\left(\frac{2}{x}-\frac{x^2}{3}\right)^9\left(\frac{1}{8x}+\frac{x^2}{12}\right)$$
$$=\left(\frac{1}{8x}+\frac{x^2}{12}\right)\left[\cdots+\binom{9}{1}(2)^8\left(-\frac{1}{3x^6}\right)+\frac{512}{x^3}+\cdots\right].$$
$$=\left(\frac{1}{8x}+\frac{x^2}{12}\right)\left[\cdots-\frac{768}{x^6}+\frac{512}{x^3}\right]$$

Coefficient of $\dfrac{1}{x^4}$ $=\dfrac{1}{8}(512)+\left(\dfrac{1}{12}\right)(-768)$

$=0$

Term in $\dfrac{1}{x^4}$ does not exist.

10. The first two non-zero terms in the expansion of $(1+ax)(1+bx)^9$ in ascending powers of x are 1 and $-5x^2$. Given that $a > 0$ and $b < 0$, find the values of a and b.

$$(1+ax)(1+bx)^9 = (1+ax)\left[1+\binom{9}{1}(bx)+\binom{9}{2}(bx)^2+\cdots\right]$$
$$= (1+ax)(1+9bx+36b^2x^2+\cdots)$$
$$= 1+9bx+36b^2x^2+ax+9abx^2+\cdots$$
$$= 1+(a+9b)x+(9ab+36b^2)x^2+\cdots$$

$a+9b=0$
$a=-9b$ ……(1)

$9ab+36b^2=-5$ ……(2)

Sub (1) into (2) $\quad 9(-9b)b+36b^2=-5$

$45b^2=5$

$b^2=\dfrac{1}{9}$

$b=-\dfrac{1}{3} \quad (b<0)$

$a=3$

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11. Find the value of m for which the coefficient of x^3 in the expansion of $\left(2-\frac{1}{2}x\right)^6-(1+mx)^7$ is $-\frac{125}{8}$.

Coefficient of $x^3 = -\dfrac{125}{8}$

$$\binom{6}{3}(2)^3\left(-\frac{1}{2}\right)^3-\binom{7}{3}m^3=-\frac{125}{8}$$

$$-20-35m^3=-\frac{125}{8}$$

$$35m^3=-20+\frac{125}{8}$$

$$=-\frac{35}{8}$$

$$m^3=-\frac{1}{8}$$

$$m=-\frac{1}{2}$$

12. In the expansion of $(1-2x)^n(x+p)$, the coefficient of x and x^2 are 37 and -306 respectively. Given that $n>2$ and $p \neq 0$, find the values of the constants n and p.

$$
\begin{aligned}
&(1-2x)^n(x+p) \\
&= (x+p)\left[1+\binom{n}{1}(-2x)+\binom{n}{2}(-2x)^2+\cdots\right] \\
&= (x+p)[1-2nx+2n(n-1)x^2+\cdots] \\
&= x-2nx^2+p-2npx+2n(n-1)px^2+\cdots \\
&= p+x-2npx-2nx^2+2n(n-1)px^2+\cdots \\
&= p+(1-2np)x+[2n(n-1)p-2n]x^2+\cdots
\end{aligned}
$$

$$
\begin{aligned}
&1-2np=37 \\
&2np=-36 \\
&p=-\frac{18}{n} \quad \ldots\ldots(1)
\end{aligned}
$$

$$2n(n-1)p-2n=-306 \ldots\ldots(2)$$

Sub (1) into (2)
$$
\begin{aligned}
&2n(n-1)\left(-\frac{18}{n}\right)-2n=-306 \\
&-18n+18-n=-153 \\
&-19n=-171 \\
&n=9 \\
&p=-2
\end{aligned}
$$

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13. Given that the expansion of $(a+x)(1-3x)^n$ in ascending powers of x is $5-134x+bx^2+\ldots$ find the values of a, n and b.

$$
\begin{aligned}
&(a+x)(1-3x)^n \\
&= (a+x)\left[1+\binom{n}{1}(-3x)+\binom{n}{2}(-3x)^2+\cdots\right] \\
&= (a+x)\left[1-3nx+\frac{9n(n-1)}{2}x^2+\cdots\right] \\
&= a-3nax+\frac{9n(n-1)a}{2}x^2+x-3nx^2+\cdots \\
&= a+x-3nax+\frac{9n(n-1)a}{2}x^2-3nx^2+\cdots \\
&= a+(1-3na)x+\left[\frac{9n(n-1)a}{2}-3n\right]x^2+\cdots
\end{aligned}
$$

Comparing coefficient of terms,

$$
\begin{aligned}
&a=5 \\
&1-3na=-134 \\
&-15n=-135 \\
&n=9
\end{aligned}
$$

$$
\begin{aligned}
n &= \frac{9(9)(8)(5)}{2}-27\cdots \\
&=1593
\end{aligned}
$$

14. Given that the first three terms in the expansion of $(2+px)^n$, where $n > 0$, are $32-40x+5rx^2$. Find the values of n, p and r.

$$(2+px)^n$$
$$= 2^n + \binom{n}{1}(2^{n-1})(px) + \binom{n}{2}(2^{n-2})(px)^2 + \cdots$$
$$= 2^n + 2^{n-1}npx + 2^{n-3}n(n-1)p^2x^2 + \cdots$$

Comparing coefficient of terms,

$$2^n = 32$$
$$n = 5$$

$$2^{n-1}np = -40$$
$$80p = -40$$
$$p = -\frac{1}{2}$$

$$5r = 2^{n-3}n(n-1)p^2$$
$$r = \frac{1}{5} \times 4(5)(4)\left(\frac{1}{2}\right)^2$$
$$= 4$$

15. (a) Find the coefficient of x in the expansion of $\left(2x+\frac{1}{4x^2}\right)^{16}$.

$$T_{r+1}=\binom{16}{r}(2x)^{16-r}\left(\frac{1}{4x^2}\right)^{r}$$
$$=\binom{16}{r}(2)^{16-3r}x^{16-3r}$$

When $16-3r=1,\quad r=5$

coefficient of $x=\binom{16}{5}(2)$

$=8736$

(b) In the expansion of $(1+4x)^6(1-3x)^n$, where n is a positive integer, the coefficient of x^2 is –84. Find the value of n.

$$(1+4x)^6(1-3x)^n$$
$$=\left[1+\binom{6}{1}(4x)+\binom{6}{2}(4x)^2+\cdots\right]\left[1+\binom{n}{1}(-3x)+\binom{n}{2}(-3x)^2+\cdots\right]$$
$$=(1+24x+240x^2+\cdots)\left[1-3nx+\frac{9n(n-1)x^2}{2}+\cdots\right]$$

Coefficient of x^2 = –84

$$-72n+240+\frac{9n(n-1)}{2}=-84$$
$$-144n+480+9n^2-9n=-168$$
$$9n^2-153n+648=0$$
$$n^2-17n+72=0$$
$$(n-8)(n-9)=0$$
$$n=8,\ 9$$

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 9

Name: ______________________________ (____)   Date: ________________

Class: Sec 4______

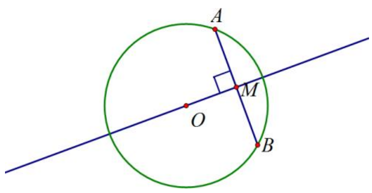
Revision 9: Coordinate Geometry of Circles

1. **Deriving the equation of a circle from the coordinates of its centre and its radius**
   The equation of a circle with centre $C(a, b)$ and radius r units is
   - $(x-a)^2+(y-b)^2=r^2$.

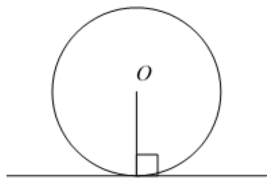
2. **Deriving the coordinates of the centre and radius of a circle from its equation**
   If the equation of a circle is $x^2+y^2+2gx+2fy+c=0$,
   - the coordinates of the centre are $C(-g,-f)$
   - the radius is $r=\sqrt{a^2+b^2-c}$ units

   Note: The coordinates of the centre and the radius of the circle could also be found by completing the square.

3. **The Perpendicular Bisector of a Chord**
   The perpendicular bisector of any chord AB of a circle centred O **passes through the centre.**
   i.e. $AM = MB$ and $AB \perp OB$



4. **The Tangent of a Circle**
   A tangent to a circle is perpendicular to the radius at the point of contact.



5. **Intersection of Circles**
   C_1 and C_2 are the centres of two circles with radius R_1 units and R_2 respectively, where $R_1 > R_2$.
   - The two circles intersects at two points if $C_1C_2 < R_1 + R_2$.
   - The two circles touch at a point if $C_1C_2 = R_1 + R_2$ or $C_1C_2 = R_1 - R_2$.
   - The two circles do not intersect if $C_1C_2 > R_1 + R_2$.

Example

1. Show that the point $A(1, -2)$ lies inside the circle $x^2+y^2-4x+6y+9=0$.

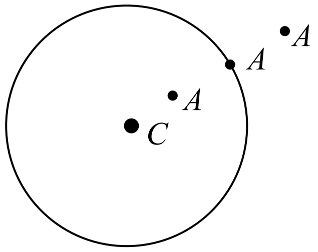
Solution

Let the centre of the circle be C.
$\therefore C(2, -3)$

$$\text{Radius} = \sqrt{2^2+(-3)^2-9}$$
$$= 2 \text{ units}$$
$$AC = \sqrt{(2-1)^2+(-3+2)^2}$$
$$= \sqrt{2} \text{ units} < \text{Radius}$$

Hence, A lies inside the circle.

- If AC < Radius, A lies inside the circle.
- If AC = Radius, A lies on the circle.
- If AC > Radius, A lies outside the circle.



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2. The points $(-1, 5)$ and $(15, 5)$ are on the circumference of a circle whose centre, C, lies above the x-axis. The line $x=17$ is a tangent to the circle. Find
   (i) the radius of the circle,
   (ii) the coordinates of C,
   (iii) the equation of the circle in the form $x^2+y^2+px+qy+r=0$, where p, q and r are integers,
   (iv) the equations of the tangents to the circle parallel to the x-axis.

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3. The equation of a circle, C, is $x^2+y^2-6x-8y+16=0$.
   (i) Find the coordinates of the centre of C and find the radius of C.
   (ii) Show that C touches the y-axis.
   (iii) Find the equation of the circle which is a reflection of C in the y-axis.

Solution

(i) Centre of the circle, $(3, 4)$

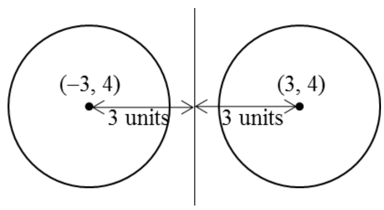
$$\text{Radius} = \sqrt{3^2+4^2-16}$$
$$= 3 \text{ units}$$

(ii) The centre C is 3 units away from the y-axis. Hence, C touches y-axis.

(iii) New centre, $(-3, 4)$

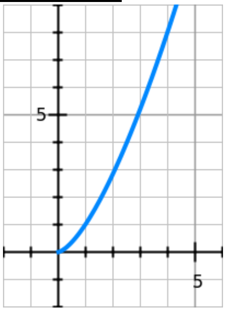
Equation of the circle, $(x+3)^2+(y-4)^2=9$

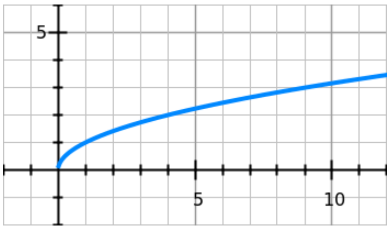
- The centres of the two circles are equidistant from the mirror line.
- The image has the same radius as the original circle.

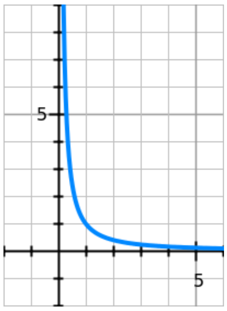


Parabolas

Graphs of *y = x^n*

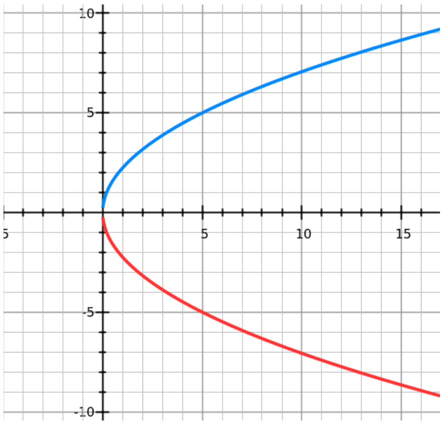


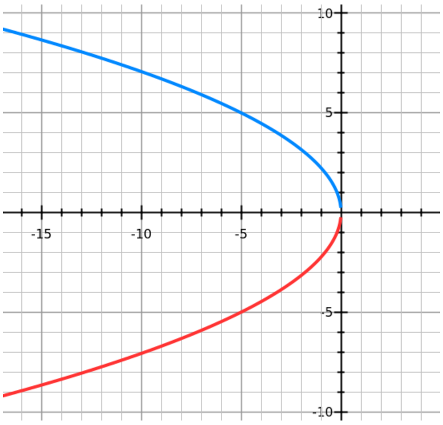




- $y = x^n$ where $x \geq 0$ and $n > 1$
- $y = x^n$ where $x \geq 0$ and $0 < n < 1$
- $y = x^n$ where $x > 0$ and $n < 0$

Graphs of *y² = kx*





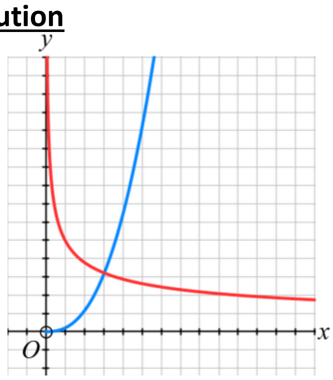
- $y^2 = kx$ where $x \geq 0$ and $k > 0$
- $y^2 = kx$ where $x \leq 0$ and $k < 0$

Example

1. (i) Sketch the graph of $y = \frac{1}{5}x^{\frac{5}{2}}$ for $x > 0$.

(ii) On the same diagram, sketch the graph of $y = 5x^{-\frac{2}{5}}$.

Solution



Exercise

1. A circle with centre C passes through the points $A(-6, 2)$ and $B(10, 2)$.
Given that C lies above the x – axis and the line $x = -8$ is a tangent to the circle,

(i) show that the radius of the circle is 10 units.
(ii) find the coordinates of C.
(iii) find the equation of the circle in the form $x^2 + y^2 + px + qy + r = 0$.
(iv) find the equations of the tangents to the circle parallel to the x-axis.

(i) x-coordinates of $C = \frac{1}{2}(-6+10)$
$= 2$
Radiu of circle $= 2-(-8)$
$= 10 \text{ units}$

(ii) Let $C(2, y)$
$(2+6)^2 + (y-2)^2 = 10^2$
$64 + (y-2)^2 = 100$
$(y-2)^2 = 36$
$y - 2 = \pm 6$
$y = -4\,(\text{NA}), 8$
$\therefore C(2, 8)$

(iii) Equation of the circle, $(x-2)^2 + (y-8)^2 = 100$
$x^2 - 4x + 4 + y^2 - 16y + 64 = 100$
$x^2 + y^2 - 4x - 16y - 32 = 0$

(v) The equations of the tangents to the circle $y = -2$ and $y = 18$.

2. The circle C_1 has centre (3, –2) and radius 4.

(a) State the equation of C_1.

(b) The circle C_2 has equation $x^2 + y^2 - 4x + 6y - 12 = 0$.
  (i) Find the coordinates of the centre of C_2 and the radius of C_2.
  (ii) Calculate the exact distance between the centre of the circle C_1 and the centre of the circle C_2.

(c) The circle C_3 has equation $2x^2 + 2y^2 - 8x + 12y - k = 0$, where k is a constant.
  (i) Show that C_3 is concentric with C_2 for all values of k.
  (ii) Find the range of values of k such that C_3 is smaller than C_2, but larger than C_1.

(a) Equation of the circle, $(x-3)^2 + (y+2)^2 = 16$

(b) (i) Centre of C_2 = (2, –3)

$$\text{Radius} = \sqrt{2^2 + (-3)^2 - (-12)}$$
$$= 5 \text{ unit}$$

  (ii) Distance between the centre of the two circle

$$= \sqrt{(3-2)^2 + (-2+3)^2}$$
$$= \sqrt{2} \text{ unit}$$

(c) $2x^2 + 2y^2 - 8x + 12y - k = 0$

$$x^2 + y^2 - 4x + 6y - \frac{k}{2} = 0$$

  (i) Centre of C_3 = (2, –3).
  Since C_2 and C_3 share a common ccentre, C_3 is concentric with C_2 for all values of k.

  (ii) 4 < Radius of C_3 < 5

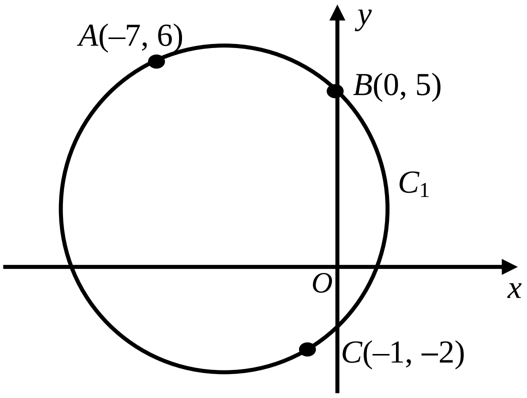
$$4 < \sqrt{2^2 + (-3)^2 - \left(-\frac{k}{2}\right)} < 5$$

$$16 < 13 + \frac{k}{2} < 25$$

$$3 < \frac{k}{2} < 12$$

$$6 < k < 24$$

3.



In the diagram, which is not drawn to scale, A, B and C are points on the circle C_1.

(i) Show that AC is the diameter of the circle C_1 and hence find the centre of C_1.

(ii) Find the equation of C_1 in the form $x^2 + y^2 + px + qy + r = 0$, where p, q and r are integers.

(iii) Given that C_2 is a reflection of the circle C_1 in the line $x = 2$, find the centre of C_2 and the equation of C_2.

(iv) Determine whether C lies inside or outside the circle C_2.

(i) If AC is the diameter of the circle C_1, $m_{AB} \times m_{BC} = -1$

$$m_{AB} = \frac{6-5}{-7}$$
$$= -\frac{1}{7}$$
$$m_{BC} = \frac{5+2}{1}$$
$$= 7$$
$$m_{AB} \times m_{BC} = -\frac{1}{7} \times 7$$
$$= -1$$

$\therefore AC$ is the diameter of the circle C_1.

(ii) Centre of circle = Midpoint of AC

= (–4, 2)

Radius of circle $= \frac{1}{2}\sqrt{6^2 + 8^2}$

$= 5$ unit

Equation of circle, $(x+4)^2 + (y-2)^2 = 25$

$x^2 + y^2 + 8x - 4y - 5 = 0$

(iii) New centre = (8, 2)

Equation of C_2, $(x-8)^2 + (y-2)^2 = 25$

(iv) Distance between C and centre of C_2 $= \sqrt{9^2 + 4^2} = 9.85$ unit

C lies outside C_2.

4. The equation of a circle, C, is $x^2 + y^2 + 4x - 2y - 20 = 0$.

(i) Find the coordinates of the centre of the circle, C and find the radius of the circle, C.
(ii) Show that the centre of the circle, C, lies on the line $y = 7x + 15$.
(iii) The circle, C, is reflected on the y-axis. Find the equation of the image of C.

(i) Centre of C = (–2, 1)
Radius $= \sqrt{(-2)^2 + 1^2 - (-20)}$
$= 5$ unit

(ii) When $x = -2$, $y = 7(-2) + 15 = 1$
$\therefore$ (–2, 1) lies on the line $y = 7x + 15$.

(iii) Centre of the image = (2, 1)
Equation of the image of C, $(x-2)^2 + (y-1)^2 = 25$

---

5. A circle, centre C(4, 1), has a radius of 5 units and a diameter PQ, where Q has coordinates (8, 4).

(a) Write down the equation of the circle.
(b) Find the coordinates of P.
(c) If the circle is reflected in the line $x = 1$, find the equation that represents the reflected circle.

(a) Equation of the image of C, $(x-4)^2 + (y-1)^2 = 25$

(b) Midpoint of PQ = (4, 1)
Let $P(x, y)$.
$\frac{x+8}{2} = 4 \Rightarrow x = 0$
$\frac{y+4}{2} = 1 \Rightarrow y = -2$
$P(0, -2)$

(c) Centre of the image = (–2, 1)
Equation of the image of C, $(x+2)^2 + (y-1)^2 = 25$

---

6. The equation of a circle, C_1, is given by $x^2 + y^2 - 4x - 8y + 11 = 0$.

(i) Find the centre P and the radius of the circle.

(ii) Show that the point $A(4, 6)$ is inside the circle C_1.

(iii) Another circle, C_2, has the point A as its centre. Find the equation of C_2 if C_1 and C_2 have the same radius.

(iv) C_1 intersects the y-axis at two points. Find the y-coordinates of the two points in the form of $a \pm b\sqrt{5}$.

(i) $P(2, 4)$

Radius $= \sqrt{2^2 + 4^2 - 11}$

$= 3$ unit

(ii) $AP = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}$ unit < 3 unit

$\therefore$ A lies inside the circle.

(iii) Equation of C_2, $(x-4)^2 + (y-6)^2 = 9$ or $x^2 + y^2 - 8x - 12y + 43 = 0$

(iv) When $x = 0$, $y^2 - 8y + 11 = 0$

$$y = \frac{8 \pm \sqrt{64 - 44}}{2}$$

$$= \frac{8 \pm 2\sqrt{5}}{2}$$

$$= 4 \pm \sqrt{5}$$

7. A circle, centre C, passes through the points $P(1, -2)$, $Q(9, -2)$ and $R(5, 6)$.

(i) Show that the coordinates of C is (5, 1).
(ii) Find the radius of the circle.
(iii) Find the equation of the circle in the form $x^2 + y^2 + fx + gy + h = 0$, where f, g and h are integers.
(iv) Does the point (10, 4) lie outside, inside or on the circle? Justify your answer.

(i) Midpoint of PQ = (5, −2)
Equation of $\perp$ bisector of PQ, $x = 5$

Midpoint of QR = (7, 2)

$$m_{QR} = \frac{6+2}{5-9}$$
$$= -2$$

Equation of $\perp$ bisector of PQ, $y - 2 = -\frac{1}{2}(x-7)$

$$y = -\frac{x}{2} + \frac{3}{2}$$

At the centre, $x = 5$, $y = -\frac{1}{2}(5) + \frac{3}{2} = 1$

Centre (5, 1)

(ii) Radius $= \sqrt{(5-1)^2 + (1+2)^2}$
$= 5 \text{ unit}$

(iii) Equation of the circle, $(x-5)^2 + (y-1)^2 = 25$

$$x^2 + y^2 - 10x - 2y + 25 + 1 - 25 = 0$$
$$x^2 + y^2 - 10x - 2y + 1 = 0$$

(iv) Distance between (10, 4) and centre (5, 1) $= \sqrt{5^2 + 3^2}$
$= \sqrt{34} \text{ unit} > 5 \text{ unit}$

(10, 4) lies outside the circle.

8. A circle, C_1, has equation $x^2 + y^2 + 4x - 6y = 36$.

(i) Find the radius and the coordinates of the centre of C_1.

A second circle, C_2 has a diameter AB. The point A has coordinates (–5, 5) and the equation of the tangent to C_2 is $3y = 4x - 15$.

(ii) Find the equation of the diameter AB and hence the coordinates of B.

(iii) Find the radius and the coordinates of the centre of C_2.

(iv) Explain why the point (4, 6) lies within only one of the circles C_1 and C_2.

(i) Centre of circle = (–2, 3)

Radius $= \sqrt{(-2)^2 + 3^2 - (-36)}$

$= 7 \text{ unit}$

(ii) $3y = 4x - 15 \Leftrightarrow y = \dfrac{4x}{3} - 5$

$m_{AB} = -\dfrac{3}{4}$

Equation of the diameter AB, $y - 5 = -\dfrac{3}{4}(x + 5)$

$$y = -\frac{3}{4}x + \frac{5}{4}$$

At B, $\dfrac{4x}{3} - 5 = -\dfrac{3}{4}x + \dfrac{5}{4}$

$16x - 60 = -9x + 15$

$25x = 75$

$x = 3$

$y = -1$

$\therefore B(3, -1)$.

(iii) Radius $= \dfrac{1}{2}\sqrt{(-5-3)^2 + (5+1)^2}$

$= 5 \text{ unit}$

Centre $=$ Midpoint of AB

$= \left(\dfrac{-5+3}{2}, \dfrac{5-1}{2} \right)$

$= (-1, 2)$

(iv) Distance between (4, 6) and the centre of circle C_1 $= \sqrt{(4+2)^2 + (6-3)^2}$

$= \sqrt{45} \text{ unit} < 7 \text{ unit}$

Distance between (4, 6) and the centre of circle C_2 $= \sqrt{(4+1)^2 + (6-2)^2}$

$= \sqrt{41} \text{ unit} > 5 \text{ unit}$

The point (4, 6) lies inside C_1 but outside C_2.

Singapore Chinese Girls' School
Secondary 4(IP)
Integrated Mathematics II
Differentiation and its Techniques

Name: ________________________________ (____)   Date: ________________

Class: Sec 4_____

Worksheet 1: Limits and Differentiation

The Limit of a Function

If the values of $\mathrm{f}(x)$ get close to some fixed number *m* when *x* approaches a value *a*, *m* is known as **the limit of f(*x*)** as *x* tends to a.
Mathematically, it is written as $\lim\limits_{x \to a} \mathrm{f}(x) = m$

Example 1
Find the limit of $\mathrm{f}(x) = 2x$ when $x \to 2$.

$$\begin{aligned}\lim_{x \to 2} \mathrm{f}(x) &= \lim_{x \to 2} (2x) \\ &= 4\end{aligned}$$

Example 2
Find the limit of $\mathrm{f}(x) = x^2$ as $x \to \infty$.

$$\begin{aligned}\lim_{x \to \infty} \mathrm{f}(x) &= \lim_{x \to \infty} (x^2) \\ &= \infty\end{aligned}$$

Example 3
Find the limit of $\mathrm{f}(x) = \dfrac{4}{x^3}$ as $x \to 3$.

$$\begin{aligned}\lim_{x \to 3} \mathrm{f}(x) &= \lim_{x \to 3} \left(\frac{4}{x^3}\right) \\ &= \frac{4}{27}\end{aligned}$$

Example 4
Find the limit of $\mathrm{f}(x) = \dfrac{3}{x}$ as $x \to \infty$.

$$\begin{aligned}\lim_{x \to \infty} \mathrm{f}(x) &= \lim_{x \to \infty} \left(\frac{3}{x}\right) \\ &= 0\end{aligned}$$

Example 5
Find the limit of $\mathrm{f}(x) = \dfrac{x^2 - 4}{x - 2}$ when $x \to 2$.

$$\begin{aligned}\lim_{x \to 2} \mathrm{f}(x) &= \lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2}\right) \\ &= 4\end{aligned}$$

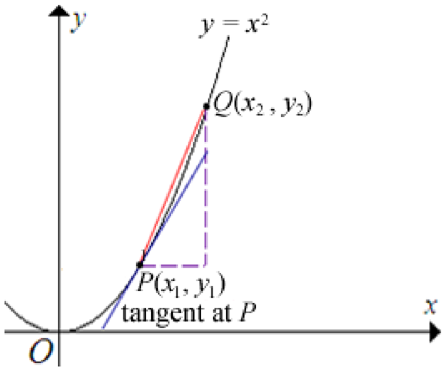
Example 6
Find the limit of $\mathrm{f}(x) = \dfrac{3 - 4x}{1 - 2x}$ as $x \to \infty$.

$$\begin{aligned}\lim_{x \to \infty} \mathrm{f}(x) &= \lim_{x \to \infty} \left(\frac{3 - 4x}{1 - 2x}\right) \\ &= 2\end{aligned}$$

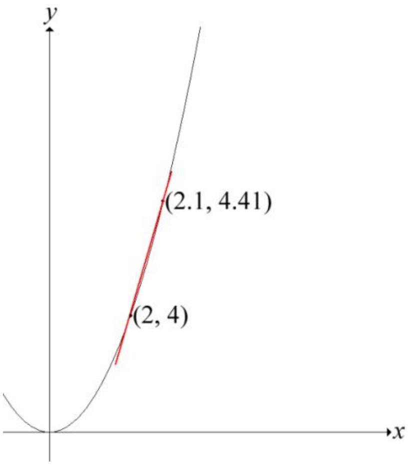
The Gradient of a Curve at a Point

In the diagram, $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on the curve $y = x^2$.

Therefore, the gradient of the straight line PQ $= \dfrac{y_2 - y_1}{x_2 - x_1}$.

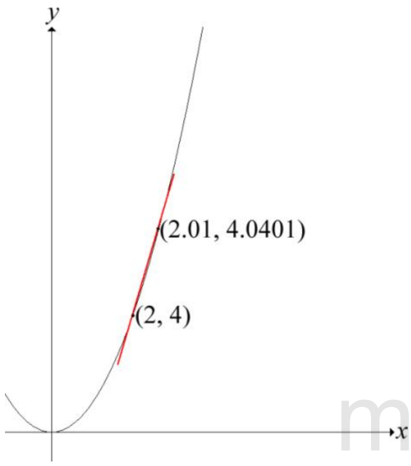


Find the gradient of the line segment PQ in each of the figure below.



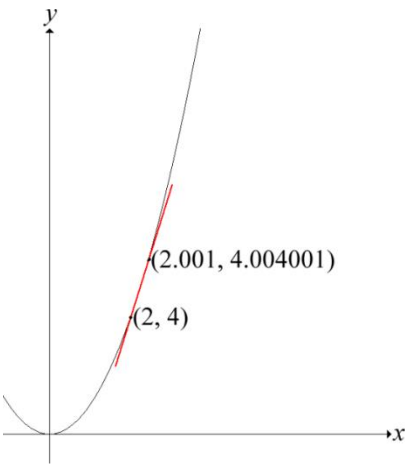
Gradient of PQ $= \dfrac{4.41 - 4}{2.1 - 2}$

$= 4.1$



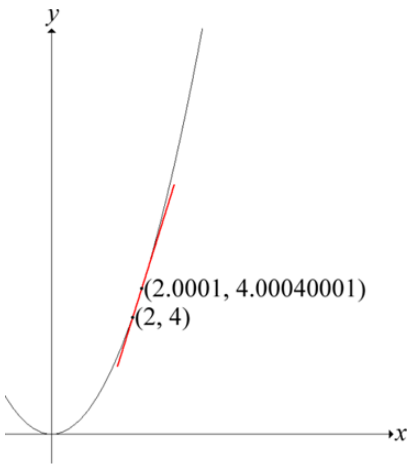
Gradient of PQ $= \dfrac{4.041 - 4}{2.01 - 2}$

$= 4.01$



Gradient of PQ $= \dfrac{4.004001 - 4}{2.001 - 2}$

$= 4.001$



Gradient of PQ $= \dfrac{4.00040001 - 4}{2.0001 - 2}$

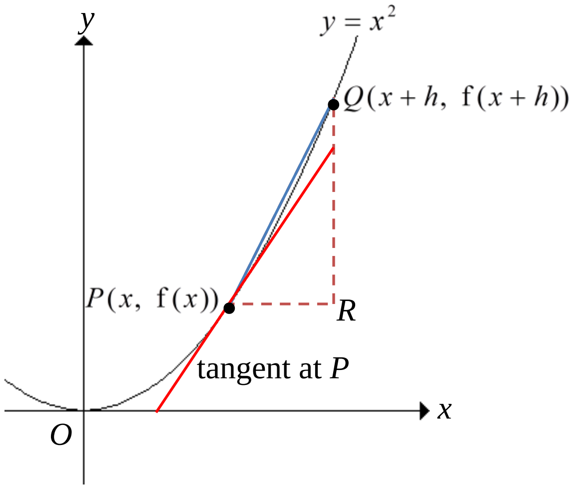
$= 4.0001$

The above results shows that as the point Q gets closer to the point P, the gradient of the line segment PQ gets closer to (tends to) a value of 4 which is the gradient of the tangent at P.

In general, the gradient of the tangent at any point $P = \lim\limits_{x_2 \to x_1} \left(\dfrac{y_2 - y_1}{x_2 - x_1} \right)$

The Gradient Function of a Curve

A function is defined by $f(x) = x^2$ for all real values of x. In the diagram, $P(x,\ f(x))$ and $Q(x+h,\ f(x+h))$ are two points on the curve $y = f(x)$ where Q is just a little further along the graph.



From the diagram,

$$\begin{aligned} QR &= f(x+h) - f(x) \\ &= (x+h)^2 - x^2 \\ &= x^2 + 2xh + h^2 - x^2 \\ &= 2xh + h^2 \end{aligned}$$

$$\begin{aligned} PR &= (x+h) - x \\ &= h \end{aligned}$$

$$\begin{aligned} \text{Gradient of the line } PQ \text{ segment} &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{QR}{PR} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h \end{aligned}$$

As Q gets closer and closer to P, the line segment PQ will get closer and closer to the tangent at P. Hence, the gradient of the line segment PQ gets closer to the tangent at P. If Q goes all the way to touch P (i.e. $h = 0$), then we would have the **exact** slope of the tangent at P.

$$\begin{aligned} \text{i.e. Gradient of the tangent at } P &= \lim_{h \to 0} (\text{Gradient of } PQ) \ (\text{denoted by } \frac{dy}{dx}) \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

This process of obtaining the gradient function of a curve is known as **differentiation from the first principles.** It gives the instantaneous rate of change of y with respect to x.

Differentiation

The process of obtaining $\frac{dy}{dx}$ (or $f'(x)$) of a given function $y = f(x)$ is called **differentiation.**

By the first principles, the gradient of the tangent at a point P on any curve, $y = f(x)$ is

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The function $\frac{dy}{dx}$ is also known as

- the gradient function
- the derived function
- the (first) derivative
- the differential coefficient of y with respect x.

Some Results Obtained From First Principles

1. If $y = a$ and a is a constant, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

2. **The Power Rule**

   If $y = ax^n$, where a is a constant and n is a rational number, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$.

3. **The Sum Rule and Difference Rule**

   If $y = ax^m \pm bx^n$, where a and b are constants and n is a rational number, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = amx^{m-1} \pm bnx^{n-1}$.

4. **The Constant Multiple Rule**

   If $\mathrm{f}(x)$ is a function and k is a constant, then $\dfrac{\mathrm{d}}{\mathrm{d}x}[k\mathrm{f}(x)] = k \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)]$.

Example 7

Find the following.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(4x^5)$ $= 4(5)x^4$ $= 20x^4$	(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(-\dfrac{4}{5}x^5\right)$ $= 5\left(-\dfrac{4}{5}x^4\right)$ $= -4x^4$
(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}(3x^{-7})$ $= -7(3)x^{-8}$ $= -21x^{-8}$ or $-\dfrac{21}{x^8}$	(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}(5)$ $= 0$
(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{7}{16x^4}\right)$ $= \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{7}{16}x^{-4}\right)$ $= -4\left(\dfrac{7}{16}x^{-5}\right)$ $= -\dfrac{7}{4}x^{-5}$ or $-\dfrac{7}{4x^5}$	(f) $\dfrac{\mathrm{d}}{\mathrm{d}x}(2\sqrt{x})$ $= \dfrac{\mathrm{d}}{\mathrm{d}x}(2x^{\frac{1}{2}})$ $= x^{-\frac{1}{2}}$ or $\dfrac{1}{\sqrt{x}}$

(g) $\frac{d}{dx}(\sqrt[3]{x^2})$

$= \frac{d}{dx}(x^{\frac{2}{3}})$

$= \frac{2}{3}x^{-\frac{1}{3}}$ or $\frac{2}{3\sqrt[3]{x}}$

(h) $\frac{d}{dx}(3x^2\sqrt{x})$

$= \frac{d}{dx}(3x^{\frac{5}{2}})$

$= \frac{15}{2}x^{\frac{3}{2}}$ or $\frac{15}{2}x\sqrt{x}$

(i) $\frac{d}{dx}(3x^2+4x-1)$

$= 6x+4$

(j) $\frac{d}{dx}(-4x^3+5x^2-12)$

$= -12x^2+10x$

(k) $\frac{d}{dx}\left(\frac{\pi^2x^2}{3}-\frac{2x^4}{5}\right)$

$= \frac{2\pi^2x}{3}-\frac{8x^3}{5}$

(l) $\frac{d}{dx}\left(\frac{3x^8}{20}-\frac{\pi x^9}{11}\right)$

$= \frac{6x^7}{5}-\frac{9\pi x^8}{11}$

(m) $\frac{d}{dx}\left(4x+\frac{2}{x}\right)$

$= \frac{d}{dx}(4x+2x^{-1})$

$= 4-2x^{-2}$ or $4-\frac{2}{x^2}$

(n) $\frac{d}{dx}\left(9x^2-\frac{3}{x^2}\right)$

$= \frac{d}{dx}(9x^2-3x^{-2})$

$= 18x-3(-2)x^{-3}$

$= 18x+6x^{-3}$ or $18x+\frac{6}{x^3}$

(o) $\frac{d}{dx}[x^2(3x^2+4x-1)]$

$= \frac{d}{dx}(3x^4+4x^3-x^2)$

$= 12x^3+12x^2-2x$

(p) $\frac{d}{dx}\left(\frac{-4x^3+5x^2-12}{2x}\right)$

$= \frac{d}{dx}\left(-2x^2+\frac{5}{2}x-6x^{-1}\right)$

$= -4x+\frac{5}{2}-(-6x^{-2})$

or $-4x+\frac{5}{2}+\frac{6}{x^2}$

Example 8

(a) For each of the expression of y, obtain an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(i) $\begin{aligned} y &= \frac{2x^2+4x}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x^2+4x}{x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(2x+4) \\ &= 2 \end{aligned}$	(ii) $\begin{aligned} y &= \frac{x^2-6x+4}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(x-6+4x^{-1}) \\ &= 1-4x^{-2} \text{ or } 1+\frac{4}{x^2} \end{aligned}$
(iii) $\begin{aligned} y &= (x+1)(2x-1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(2x^2+x-1) \\ &= 4x+1 \end{aligned}$	(iv) $\begin{aligned} y &= x(\sqrt{x}-2) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{3}{2}}-2x\right) \\ &= \frac{3}{2}x^{\frac{1}{2}}-2 \text{ or } \frac{3}{2}\sqrt{x}-2 \end{aligned}$

(b) For each of the expression of $\mathrm{f}(x)$, obtain an expression for $\mathrm{f}'(x)$.

(i) $\begin{aligned} \mathrm{f}(x) &= (3-\sqrt{x})(3+\sqrt{x}) \\ \mathrm{f}'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(9-x) \\ &= -1 \end{aligned}$	(ii) $\begin{aligned} \mathrm{f}(x) &= \frac{5(2x+1)(x-1)}{2x} \\ &= \frac{5(2x^2+x-1)}{2x} \\ &= 5x-\frac{5}{2}-\frac{5}{2}x^{-1} \\ \mathrm{f}'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(5x-\frac{5}{2}-\frac{5}{2}x^{-1}\right) \\ &= 5+\frac{5}{2}x^{-2} \text{ or } 5+\frac{5}{2x^2} \end{aligned}$
(iii) $\begin{aligned} \mathrm{f}(x) &= 4x^2(3-\sqrt{x}) \\ \mathrm{f}'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(12x^2-4x^{\frac{5}{2}}\right) \\ &= 24x-10x^{\frac{3}{2}} \end{aligned}$	(iv) $\begin{aligned} \mathrm{f}(x) &= \frac{3(2\sqrt{x}-1)(2\sqrt{x}+1)}{2x} \\ &= \frac{3(4x-1)}{2x} \\ &= 6-\frac{3}{2}x^{-1} \\ \mathrm{f}'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(6-\frac{3}{2}x^{-1}\right) \\ &= \frac{3}{2}x^{-2} \text{ or } \frac{3}{2x^2} \end{aligned}$

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Assignment 2: Derivative of *y* = *ax*ⁿ and *y* = *ax*ᵐ + *bx*ⁿ

1. Find $\frac{dy}{dx}$ of each of the following :

(a) $y=\frac{3}{4}x^{12}$ $\frac{dy}{dx}=12\times\frac{3}{4}x^{11}$ $=9x^{11}$	(b) $y=\frac{4}{x^3}$ $\frac{dy}{dx}=\frac{d}{dx}(4x^{-3})$ $=-12x^{-4}$
(c) $y=\frac{2}{3\sqrt[3]{x^2}}$ $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2}{3}x^{-\frac{2}{3}}\right)$ $=-\frac{2}{3}\left(\frac{2}{3}x^{-\frac{5}{3}}\right)$ $=-\frac{4}{9}x^{-\frac{5}{3}}$ or $\frac{4}{9\sqrt[3]{x^5}}$	(d) $y=\frac{2x^4+5x^3-4x^2+9x}{x}$ $y=2x^3+5x^2-4x+9$ $\frac{dy}{dx}=6x^2-10x-4$
(e) $y=2x\,(3x^2-4\sqrt{x}-2)$ $\frac{dy}{dx}=\frac{d}{dx}(6x^3-8x^{\frac{3}{2}}-4x)$ $=18x^2-12x^{\frac{1}{2}}-4$ $=18x^2-12\sqrt{x}-4$	(f) $y=\frac{x^2+4}{2x^2}$ $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{2}+2x^{-2}\right)$ $=-4x^{-3}$

(g) $y=\dfrac{2x^2-5x+3}{3\sqrt{x}}$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{5}{3}x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)$$

$$=x^{\frac{1}{2}}-\frac{5}{6}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$$

$$=\frac{6x^2-5x-3}{6x\sqrt{x}}\ \text{(optional)}$$

(h) $y=\sqrt{x}(2-x)$

$$y=2x^{\frac{1}{2}}-x^{\frac{3}{2}}$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(2x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)$$

$$=-\left(\frac{1}{2}\right)\times 2x^{-\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}$$

$$=-x^{-\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}\ \text{or}\ -\frac{1}{\sqrt{x}}-\frac{3}{2}\sqrt{x}$$

(i) $y=\dfrac{(x+1)(2x-3)}{3x}$

$$y=\frac{2x^2-x-3}{3x}$$

$$=\frac{2}{3}x-\frac{1}{3}-x^{-1}$$

$$\frac{dy}{dx}=\frac{2}{3}+x^{-2}$$

$$=\frac{2}{3}+\frac{1}{x^2}$$

(j) $y=1-\dfrac{2}{x}-\dfrac{4}{5}x$

$$\frac{dy}{dx}=\frac{d}{dx}\left((1-2x^{-1}-\frac{4}{5}x\right)$$

$$=2x^{-2}-\frac{4}{5}\ \text{or}\ \frac{2}{x^2}-\frac{4}{5}$$

(k) $y=\dfrac{\pi\sqrt{x}}{2}+\dfrac{3}{\sqrt{x}}=\dfrac{\pi}{2}x^{\frac{1}{2}}+3x^{-\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{\pi}{2}x^{-\frac{1}{2}}\right)+\left(-\frac{1}{2}\right)\left(3x^{-\frac{3}{2}}\right)$$

$$=\frac{\pi}{4}x^{-\frac{1}{2}}-\frac{3}{2}x^{-\frac{3}{2}}$$

$$=\frac{\pi}{4\sqrt{x}}-\frac{3}{2x\sqrt{x}}$$

(l) $y=(1+2\sqrt{x})(1-3\sqrt{x})$

$$=1-\sqrt{x}-6x$$

$$=1-x^{\frac{1}{2}}-6x$$

$$\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{1}{2}}-6$$

$$=-\frac{1}{2\sqrt{x}}-6$$

(m) * $y=(x^3+1)(x^2+1)(x+1)$

$$=(x^5+x^3+x^2+1)(x+1)$$

$$=x^6+x^5+x^4+2x^3+x^2+x+1$$

$$\frac{dy}{dx}=6x^5+5x^4+4x^3+6x^2+2x+1$$

2* Given that $y = ax^n$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ from the first principles.

Let $\mathrm{f}(x) = ax^n$

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h\to 0} \frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h} \\
&= \lim_{h\to 0} \frac{a(x+h)^n - ax^n}{h} \\
&= \lim_{h\to 0} \frac{a\left[x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n\right] - ax^n}{h} \\
&= \lim_{h\to 0} \frac{\left[ax^n + a\binom{n}{1}x^{n-1}h + a\binom{n}{2}x^{n-2}h^2 + \cdots + ah^n - ax^n\right]}{h} \\
&= \lim_{h\to 0} \frac{\left[a\binom{n}{1}x^{n-1}h + a\binom{n}{2}x^{n-2}h^2 + \cdots + ah^n\right]}{h} \\
&= \lim_{h\to 0} \left[a\binom{n}{1}x^{n-1} + a\binom{n}{2}x^{n-2}h + \cdots + ah^{n-1}\right] \\
&= a\binom{n}{1}x^{n-1} \\
&= anx^{n-1}
\end{aligned}
$$

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Worksheet 3: Gradient Function

Gradient of a Curve at Any Point

For a curve $y = \mathrm{f}(x)$,

(a) the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $A(x_1, y_1)$

- represents the gradient of the tangent at the point A.
- measures the rate of change of y with respect to x at the point A.

(b) the value of $-\dfrac{1}{\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)}$ at $A(x_1, y_1)$

- represents the gradient of the normal at the point A.

Example 1

Calculate the gradient of the tangent to the curve $y = 3x^2 + 4x - 1$ at the point $(-1, 0)$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x + 4$$

At $(-1, 0)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6(-1) + 4$

$$= -2$$

Example 2

Calculate the gradient of the tangent to the curve $y = x^2 - \dfrac{x}{5}$, at the given point where $x = 3$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{1}{5}$$

When $x = 3$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2(3) - \dfrac{1}{5}$

$$= \frac{29}{5}$$

Example 3

Find the gradient of the point where the curve $y=\dfrac{x-4}{x}$ crosses the x-axis .

$$\begin{aligned} y &= \frac{x-4}{x} \\ &= 1-4x^{-1} \end{aligned}$$

$$\frac{dy}{dx}=\frac{4}{x^2}$$

On the x-axis , $y=0$, $\quad \dfrac{x-4}{x}=0$

$$x=4$$

When $x=4$, $\dfrac{dy}{dx}=\dfrac{4}{4^2}$

$$=\frac{1}{4}$$

$\therefore$ Gradient of the tangent at (4, 0) $=\dfrac{1}{4}$

Example 4

Find the coordinates of the points on the curve $y=\dfrac{9x^2+1}{x}$ where the tangent to the curve is parallel to the x-axis. (Pg 358 Ex14.1 Q14)

$$y=9x+x^{-1}$$

$$\frac{dy}{dx}=9-\frac{1}{x^2}$$

When $\dfrac{dy}{dx}=0$, $\quad 9-\dfrac{1}{x^2}=0$

$$x^2=\frac{1}{9}$$

$$x=\pm\frac{1}{3}$$

$\therefore$ The required point is $\left(-\dfrac{1}{3},\ -6\right)$ and $\left(\dfrac{1}{3},\ 6\right)$.

Example 5

Find the gradient of the curve at the point(s) where the curve $y=\dfrac{x+1}{2x}$ intersects the line $y=x$.

$$y=\frac{x+1}{2x}=\frac{1}{2}+\frac{1}{2}x^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2x^2}$$

At point of intersection, $x=\dfrac{x+1}{2x}$

$$2x^2=x+1$$
$$2x^2-x-1=0$$
$$(2x+1)(x-1)=0$$
$$x=-\frac{1}{2},\ \ 1$$

When $x=-\dfrac{1}{2}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1}{2\left(-\frac{1}{2}\right)^2}$

$$=-2$$

When $x=1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1}{2(1)^2}$

$$=-\frac{1}{2}$$

Example 6

A curve that has $\dfrac{\mathrm{d}y}{\mathrm{d}x}=kx+3$, where k is a constant, passes through the point (1, 9). At $x=5$, the gradient of the normal to the curve is $-\dfrac{1}{13}$. Find

(a) the value of k,
(b) the equation of the tangent at (1, 9).

Gradient of the normal to the curve $=-\dfrac{1}{13}$ $\Rightarrow$ Gradient of the tangent $=13$

(a) When $x=5$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=13$,

$$5k+3=13$$
$$5k=10$$
$$k=2$$

(b) At (1, 9), $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2(1)+3=5$

Equation of tangent, $y-9=5(x-1)$

$$y=5x+4$$

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Assignment 4: Gradient Function

1. Calculate the gradient of the tangent to the curve $y = 4x^2 - 6x + 1$ at the point (2, 5).

$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x - 6$

At (2, 5), $\frac{\mathrm{d}y}{\mathrm{d}x} = 8(2) - 6$

$= 10$

---

2. Find the coordinates of the point on the curve $y = x^3 - 3x^2 + 6x + 2$ at which the gradient is 3.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 6$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$, $3x^2 - 6x + 6 = 3$

$3x^2 - 6x + 3 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1$

$y = 6$

Coordinates of the required point is (1, 6)

---

3. The curve $y = ax^2 + \frac{b}{x}$ has gradients 2 and -1 at $x = 1$ and $x = 4$ respectively. Find the value of a and of b. (Pg358Q10(b))

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax - \frac{b}{x^2}$

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$, $2a - b = 2$ ……(1)

When $x = 4$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, $8a - \frac{b}{16} = -1$

$128a - b = -16$ ……(2)

(2) − (1) $126a = -18$

$a = -\frac{1}{7}$

$b = -\frac{16}{7}$

4. Find the coordinates of the points on the curve $y = \dfrac{9x^2 + 1}{x}$ where the tangent is parallel to the x-axis. (Pg358Q13)

$$y = 9x + x^{-1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 9 - x^{-2}$$
$$= 9 - \frac{1}{x^2}$$

When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad 9 - \dfrac{1}{x^2} = 0$

$$\frac{1}{x^2} = 9$$
$$x = \pm\frac{1}{3}$$

When $x = \dfrac{1}{3}$, $\quad y = 9\left(\dfrac{1}{3}\right) + 3$

$$= 6$$

When $x = -\dfrac{1}{3}$, $\quad y = 9\left(-\dfrac{1}{3}\right) - 3$

$$= -6$$

The points are $\left(\dfrac{1}{3},\ 6\right)$ and $\left(\dfrac{1}{3},\ -6\right)$.

---

5. The equation of a curve is $y = \dfrac{\pi\sqrt{x}}{2} + \dfrac{3}{\sqrt{x}}$. Find gradient of the tangent to the curve at $x = 1$.

(Pg358Q14)

$$y = \frac{\pi}{2}x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \times \frac{\pi}{2}x^{-\frac{1}{2}} + 3\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$
$$= \frac{\pi}{4\sqrt{x}} - \frac{3}{2x\sqrt{x}}$$

When $x = 1$, $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\pi}{4} - \dfrac{3}{2}$ or $\dfrac{\pi - 6}{4}$ or -0.715

6. Given that $y = x^3 - 3x^2 + 4x$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Hence, show that the gradient of the curve is positive for all values of x.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - 6x + 4 \\ &= 3x^2 - 6x + 3 + 1 \\ &= 3(x^2 - 2x + 1) + 1 \\ &= 3(x-1)^2 + 1\end{aligned}$$

Since $(x-1)^2 \geq 0$, $\quad 3(x-1)^2 + 1 > 0$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} > 0$ for all values of x.

Alternative Method

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 4$

Discriminant $= (-6)^2 - 4(3)(4)$

$= -12$

Since the coefficient of $x^2 = 3 > 0$, $3x^2 - 6x + 4 > 0$ for all x.

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} > 0$ for all values of x.

---

7. Show that the tangent to the curve $y = \frac{3}{x} - \frac{4}{x^2}$ at the point $\left(2, \frac{1}{2}\right)$ passes through the origin.

$$\begin{aligned}y &= \frac{3}{x} - \frac{4}{x^2} \\ &= 3x^{-1} - 4x^{-2}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3(-1)x^{-2} - 4(-2)x^{-3} \\ &= -\frac{3}{x^2} + \frac{8}{x^3}\end{aligned}$$

When $x = 2$, $\quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{2^2} + \frac{8}{2^3}$

$= \frac{1}{4}$

$y = \frac{3}{2} - \frac{4}{4}$

$= \frac{1}{2}$

Equation of the tangent, $\quad y - \frac{1}{2} = \frac{1}{4}(x - 2)$

$y = \frac{x}{4}$

The y-intercept is $y = 0$.

Hence, the tangent at the point $\left(2, \frac{1}{2}\right)$ passes through the origin.

8. A curve has the equation $y = x^3 + px + q$, where p and q are constants. The gradient of the curve at the point (3, 16) is 20.
(i) Find the value of p and of q.
(ii) Find the coordinates of the other point on the curve where the gradient is 20.

$$y = x^3 + px + q$$
$$\frac{dy}{dx} = 3x^2 + p$$

(i) At (3, 16), $\frac{dy}{dx} = 20$, $\quad 3(3)^2 + p = 20$

$p = -7$

At (3, 16), $y = 16$, $\quad 16 = 3^3 - 7(3) + q$

$q = 10$

(ii) When $\frac{dy}{dx} = 20$, $\quad 3x^2 - 7 = 20$

$3x^2 = 27$

$x^2 = 9$

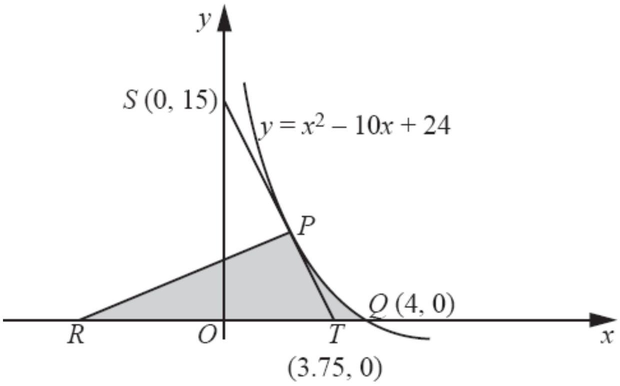
$x = \pm 3$

When $x = -3$, $\quad y = (-3)^3 - 7(-3) + 10$

$= 4$

The other point is (–3, 4).

9. The diagram shows part of the curve $y = x^2 - 10x + 24$ cutting the x-axis at $Q(4, 0)$. The tangent to the curve at the point P on the curve meets the coordinate axes at $S(0, 15)$ and at $T(3.75, 0)$.



(i) Find the coordinates of P.

The normal to the curve at P meets the x-axis at R.

(ii) Find the coordinates of R. (N2005)

(i) Gradient of tangent at P = Gradient of tangent at ST

$$= \frac{15}{-3.75}$$

$$= -4$$

When $\frac{dy}{dx} = -4$, $\quad 2x - 10 = -4$

$$x = 3$$

$$y = 3^2 - 10(3) + 24$$

$$= 3$$

$\therefore P(3, 3)$

(ii) Gradient of normal at P $= \frac{1}{4}$

Equation of normal, $\quad y - 3 = \frac{1}{4}(x - 3)$

$$y = \frac{1}{4}x + 2\frac{1}{4}$$

At R, $y = 0$, $\quad \frac{1}{4}x + 2\frac{1}{4} = 0$

$$x = -9$$

$\therefore R\ (-9, 0)$

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Worksheet 5: Chain Rule - Derivative of a Composite Function

Differentiation of a Composite Function

If $y = au^n$ where u is a function of x and a and n are constants, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$
$$= anu^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}$$

Example 1

Differentiate the following with respect to x.

(a) $y = 4(x+1)^3$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 4(3)(x+1)^2$ $= 12(x+1)^2$	(b) $y = (1+3x^2)^3$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 3(1+3x^2)^2(6x)$ $= 18x(1+3x^2)^2$
(c) $y = \frac{4}{2x+7}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[4(2x+7)^{-1}]$ $= 4(-1)(2x+7)^{-2}(2)$ $= -\frac{8}{(2x+7)^2}$	(d) $y = \left(2 - \frac{2}{x^2}\right)^4$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[(2-2x^{-2})^4]$ $= 4(2-2x^{-2})^3(4x^{-3})$ $= \frac{16}{x^3}\left(2-\frac{2}{x^2}\right)^3$
(e) $y = \frac{2}{\sqrt{x^2-2}}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[2(x^2-2)^{-\frac{1}{2}}]$ $= 2\left(-\frac{1}{2}\right)(x^2-2)^{-\frac{3}{2}}(2x)$ $= -\frac{2x}{(x^2-2)^{-\frac{3}{2}}}$	(f) $y = \sqrt{x^3-2x-3}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(x^3-2x-3)^{-\frac{1}{2}}(3x^2-2)$ $= \frac{3x^2-2}{2\sqrt{x^3-2x-3}}$

(g) $y=(1-3x)^5(1+3x)^5$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[(1-3x)^5(1+3x)^5\right]\\ &= \frac{d}{dx}(1-9x^2)^5\\ &= 5(1-9x^2)^4(-18x)\\ &= -90x(1-9x^2)^4\end{aligned}$$

(h) $y=\left(\dfrac{2x^2-x}{\sqrt{x}}\right)^{10}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(2x^{\frac{3}{2}}-x^{\frac{1}{2}})^{10}\\ &= 10\left[\frac{3}{2}(2x^{\frac{1}{2}})-\frac{1}{2}x^{-\frac{1}{2}}\right]\left(\frac{2x^2-x}{\sqrt{x}}\right)^9\\ &= 10\left(3\sqrt{x}-\frac{1}{2\sqrt{x}}\right)\left(\frac{2x^2-x}{\sqrt{x}}\right)^9 \quad \text{Or} \quad 5\left(\frac{6x-1}{\sqrt{x}}\right)\left(\frac{2x^2-x}{\sqrt{x}}\right)^9\end{aligned}$$

(i)
$$\begin{aligned}y &= (x+2)^2\sqrt{x+2}\\ &= (x+2)^{\frac{5}{2}}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x+2)^{\frac{5}{2}}\\ &= \frac{5}{2}(x+2)^{\frac{3}{2}}\end{aligned}$$

Example 2

(i) Show that $\dfrac{x}{1+x}=1-\dfrac{1}{1+x}$.

(ii) Hence differentiate $\dfrac{x}{1+x}$ with respect to x.

(i)
$$\begin{aligned}\frac{x}{1+x} &= \frac{1+x-1}{1+x}\\ &= 1-\frac{1}{1+x}\end{aligned}$$

(ii)
$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{1+x}\right) &= (-1)(-1)(1+x)^{-2}\\ &= \frac{1}{(1+x)^2}\end{aligned}$$

Example 3

Find the gradient of the tangent to the curve $y=\left(\dfrac{x}{2}-1\right)^6$ at the point where the curve meets the x-axis.

$$y=\left(\frac{x}{2}-1\right)^6$$

$$\frac{dy}{dx}=6\left(\frac{x}{2}-1\right)^5\left(\frac{1}{2}\right)$$

$$=2\left(\frac{x}{2}-1\right)^5$$

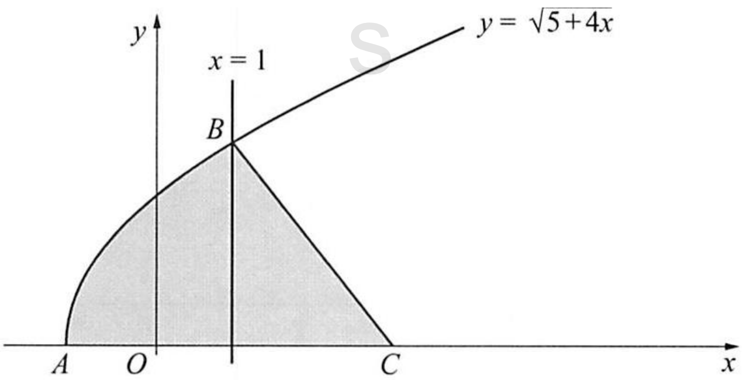
When $y=0$, $\quad \left(\dfrac{x}{2}-1\right)^6=0$

$x=2$

When $x=2$, $\quad \dfrac{dy}{dx}=0$

Example 4

The diagram shows part of the curve $y=\sqrt{5+4x}$, meeting the x-axis at the point A and the line $x=1$ at the point B. The normal to the curve at B meets the x-axis at the point C. Find the coordinates of C.



$$y=\sqrt{5+4x}$$

$$\frac{dy}{dx}=\frac{1}{2}(5+4x)^{-\frac{1}{2}}(4)$$

$$=\frac{2}{\sqrt{5+4x}}$$

When $x=1$, $\quad y=3$

$$\frac{dy}{dx}=\frac{2}{3}$$

Equation of BC, $\quad y-3=-\dfrac{3}{2}(x-1)$

$$y=-\frac{3}{2}x+\frac{9}{2}$$

At C, $\quad -\dfrac{3}{2}x+\dfrac{9}{2}=0$

$3x=9$

$x=3$

$\therefore C(3,\ 0)$.

Singapore Chinese Girls' School
Secondary 4(IP)
Integrated Mathematics II
Differentiation and its Techniques

Name: ______________________________ (____) Date: ________________

Class: Sec 4_____

Assignment 6: Chain Rule - Derivative of a Composite Function

1. Differentiate the following with respect to *x*, leaving your as a single fraction with positive indices where applicable.

(a) $y=\left(\frac{3}{5}x^3+2x^2\right)^5$ $\frac{dy}{dx}=5\left(\frac{3}{5}x^3+2x^2\right)^4\left(\frac{9}{5}x^2+4x\right)$ or $(9x^2+20x)\left(\frac{3}{5}x^3+2x^2\right)^4$	(b) $y=\frac{3}{2(3-4x^3)^2}$ $\frac{dy}{dx}=\frac{d}{dx}\left[\frac{3}{2}(3-4x^3)^{-2}\right]$ $=\frac{3}{2}(-2)(3-4x^3)^{-3}(-12x^2)$ $=\frac{36x^2}{(3-4x^3)^3}$
(c) $y=\sqrt{3x^3-1}$ $\frac{dy}{dx}=\frac{d}{dx}(3x^3-1)^{\frac{1}{2}}$ $=\frac{1}{2}(3x^3-1)^{-\frac{1}{2}}(9x^2)$ $=\frac{9x^2}{2\sqrt{3x^3-1}}$	(d) $y=\sqrt[3]{(2x-7)^2}$ $\frac{dy}{dx}=\frac{d}{dx}(2x-7)^{\frac{2}{3}}$ $=\frac{2}{3}(2x-7)^{-\frac{1}{3}}(2)$ $=\frac{4}{3}(2x-7)^{-\frac{1}{3}}$ $=\frac{4}{3\sqrt[3]{2x-7}}$
(e) $y=(5x^2-2)^{\frac{3}{2}}$ $\frac{dy}{dx}=\frac{d}{dx}(5x^2-2)^{\frac{3}{2}}$ $=\frac{3}{2}(5x^2-2)^{\frac{1}{2}}(10x)$ $=15x(5x^2-2)^{\frac{1}{2}}$ $=15x\sqrt{5x^2-2}$	(f) $y=(\sqrt{x}-2x)^4$ $\frac{dy}{dx}=\frac{d}{dx}(\sqrt{x}-2x)^4$ $=4(\sqrt{x}-2x)^3\left(\frac{1}{2}x^{-\frac{1}{2}}-2\right)$ $=(\sqrt{x}-2x)^3\left(\frac{2}{\sqrt{x}}-8\right)$ $=\frac{(\sqrt{x}-2x)^3(2-8\sqrt{x})}{\sqrt{x}}$

2. Find the coordinates of the point on the curve $y = \sqrt[3]{x^2 - 2x + 5}$ at which $\dfrac{dy}{dx} = 0$.

(Pg363Q11)

$$y = (x^2 - 2x + 5)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{1}{3}(x^2 - 2x + 5)^{-\frac{2}{3}}(2x - 2)$$

$$= \frac{2(x-1)}{3\sqrt[3]{(x^2 - 2x + 5)^2}}$$

When $\dfrac{dy}{dx} = 0$,

$$\frac{2(x-1)}{3\sqrt[3]{(x^2 - 2x + 5)^2}} = 0$$

$$2(x-1) = 0$$

$$x = 1$$

$$y = (1 - 2 + 5)^{\frac{1}{3}}$$

$$= \sqrt[3]{4}$$

The required point is $(1, \sqrt[3]{4})$ or $(1, 1.59)$

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3. The curve $y = \sqrt{(a-x)^3}$ has a gradient of $-\dfrac{1}{3}$ at $x = 2$. Find the value of a. (Pg362Q12)

$$y = (a - x)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2}(a - x)^{\frac{1}{2}}(-1)$$

$$= -\frac{3}{2}\sqrt{a - x}$$

When $x = 2$, $\dfrac{dy}{dx} = -\dfrac{1}{3}$,

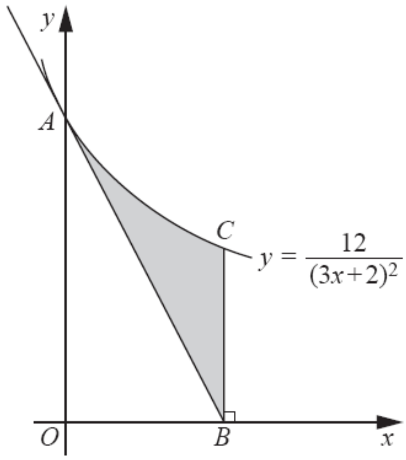
$$-\frac{3}{2}\sqrt{a-2} = -\frac{1}{3}$$

$$\frac{9}{4}(a-2) = \frac{1}{9}$$

$$a - 2 = \frac{4}{81}$$

$$a = \frac{166}{81}$$

4. The diagram shows part of the curve $y = \dfrac{12}{(3x+2)^2}$, intersecting the y-axis at A. The tangent to the curve at A meets the x-axis at B. The point C lies on the curve and BC is parallel to the y-axis.
Find the x-coordinate of B. **(N05)**



$$y = \frac{12}{(3x+2)^2}$$
$$= 12(3x+2)^{-2}$$

$$\frac{dy}{dx} = -24(3x+2)^{-3}(3)$$
$$= -\frac{72}{(3x+2)^3}$$

At A, $x = 0$, $\qquad \dfrac{dy}{dx} = -\dfrac{72}{8}$
$= -9$
$y = \dfrac{12}{4} = 3$

Equation of AB, $\qquad y = -9x + 3$

At B, $y = 0$, $\qquad -9x + 3 = 0$
$x = \dfrac{1}{3}$

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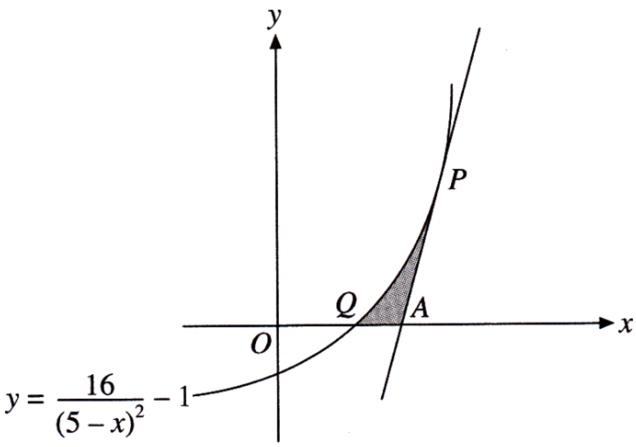
5. The diagram shows part of the curve $y = \dfrac{16}{(5-x)^2} - 1$, cutting the x-axis at Q. The tangent at the point P on the curve cuts the x-axis at A. Given that the gradient of this tangent is 4, calculate the coordinates of P. **(N07)**



$$y = 16(5-x)^{-2} - 1$$
$$\frac{dy}{dx} = 16(-2)(5-x)^{-3}(-1)$$
$$= \frac{32}{(5-x)^3}$$

At P, $\dfrac{dy}{dx} = 4$, $\qquad \dfrac{32}{(5-x)^3} = 4$
$(5-x)^3 = 8$
$5 - x = 2$
$x = 3$
$y = 3$

$\therefore P(3,\ 3)$

6* Given that $y=\dfrac{1}{\sqrt{x^2+3}}$, Show that $(x^2+3)\dfrac{dy}{dx}+xy=0.$ (Pg363Q14)

$$y=\frac{1}{\sqrt{x^2+3}}$$
$$=(x^2+3)^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=-\frac{1}{2}(x^2+3)^{-\frac{3}{2}}(2x)$$
$$=-\frac{x}{\sqrt{(x^2+3)^3}}$$
$$=-\frac{x}{(x^2+3)\sqrt{x^2+3}}$$
$$=-\frac{xy}{x^2+3}$$
$$(x^2+3)\frac{dy}{dx}=-xy$$
$$\therefore (x^2+3)\frac{dy}{dx}+xy=0$$

7* Given that $f(x)=\sqrt{1+\sqrt{x}}$, where $x\geq 0$, show that $f'(x)=\dfrac{1}{4\sqrt{x+x\sqrt{x}}}$. (Pg363Q17)

$$f(x)=\sqrt{1+\sqrt{x}}$$
$$=(1+x^{\frac{1}{2}})^{\frac{1}{2}}$$

$$f'(x)=\frac{d}{dx}(1+x^{\frac{1}{2}})^{\frac{1}{2}}$$
$$=\frac{1}{2}[1+x^{\frac{1}{2}}]^{-\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$=\frac{1}{2}\times\frac{1}{\sqrt{1+\sqrt{x}}}\times\frac{1}{2\sqrt{x}}$$
$$=\frac{1}{4\sqrt{x+x\sqrt{x}}}$$

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Worksheet 7: Product Rule

Differentiation of Products

Sometimes a function y is given as the product of two other functions u and v. The derivative of the derivative of y could be found by using the following rule:

> **Product Rule**
> If $y = uv$ where u and v are two differentiable functions, then
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}.$$

Example 1

Differentiate the following expressions of y with respect to x.

(a) $y = (x+1)(2x-5)$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (2x-5)\frac{\mathrm{d}}{\mathrm{d}x}(x+1) + (x+1)\frac{\mathrm{d}}{\mathrm{d}x}(2x-5)\\ &= (2x-5)(1) + 2(x+1)\\ &= 2x-5+2x+2\\ &= 4x-3\end{aligned}$$

(b) $y = (x+2)(x^2-1)$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}(x+2) + (x+2)\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\\ &= (x^2-1)(1) + (x+2)(2x)\\ &= x^2-1+2x^2+4x\\ &= 3x^2+4x-1\end{aligned}$$

(c) $y = (2x-1)^2(3x-5)^3$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (3x-5)^3\frac{\mathrm{d}}{\mathrm{d}x}(2x-1)^2 + (2x-1)^2\frac{\mathrm{d}}{\mathrm{d}x}(3x-5)^3\\ &= (3x-5)^3(2)(2x-1)(2) + (2x-1)^2(3)(3x-5)^2(3)\\ &= 4(3x-5)^3(2x-1) + 9(2x-1)^2(3x-5)^2\\ &= (2x-1)(3x-5)^2(12x-20+18x-9)\\ &= (2x-1)(3x-5)^2(30x-29)\end{aligned}$$

(d) $y=(x+1)\sqrt{2x-5}$ (Express $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a single fraction)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(2x-5)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(x+1)+(x+1)\frac{\mathrm{d}}{\mathrm{d}x}(2x-5)^{\frac{1}{2}}$$

$$=(2x-5)^{\frac{1}{2}}+\frac{1}{2}(x+1)(2x-5)^{-\frac{1}{2}}(2)$$

$$=(2x-5)^{-\frac{1}{2}}(2x-5+x+1)$$

$$=(2x-5)^{-\frac{1}{2}}(3x-4)\ \text{ or }\ \frac{3x-4}{\sqrt{2x-5}}$$

(e) $y=(2x+3)\sqrt{4x-3}$ (Express $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a single fraction)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(4x-3)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(2x+3)+(2x+3)\frac{\mathrm{d}}{\mathrm{d}x}(4x-3)^{\frac{1}{2}}$$

$$=2(4x-3)^{\frac{1}{2}}+\frac{1}{2}(2x+3)(4x-3)^{-\frac{1}{2}}(4)$$

$$=(4x-3)^{-\frac{1}{2}}[2(4x-3)+2(2x+3)]$$

$$=12x(4x-3)^{-\frac{1}{2}}\ \text{ or }\ \frac{12x}{\sqrt{4x-3}}$$

(f) $y=x^2\sqrt{2x-1}$ (Express $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a single fraction)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2x(2x-1)^{\frac{1}{2}}+x^2\left[\frac{1}{2}(2x-1)^{-\frac{1}{2}}(2)\right]$$

$$=(2x-1)^{-\frac{1}{2}}[2x(2x-1)+x^2]$$

$$=(2x-1)^{-\frac{1}{2}}(5x^2-2x)\ \text{ or }\ \frac{5x^2-2x}{\sqrt{2x-1}}$$

(g) $y=x(1-\sqrt{x})^3$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(1-\sqrt{x})^3+3x(1-\sqrt{x})^2\left(-\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$=(1-\sqrt{x})^3-\frac{3\sqrt{x}}{2}(1-\sqrt{x})^2$$

$$=(1-\sqrt{x})^2\left(1-\sqrt{x}-\frac{3\sqrt{x}}{2}\right)$$

$$=(1-\sqrt{x})^2\left(1-\frac{5\sqrt{x}}{2}\right)\ \text{ or }\ \frac{1}{2}(2-5\sqrt{x})(1-\sqrt{x})^2$$

Example 2

Find

(i) the gradient of the curve $y = x(x-2)^2 + 3$ at the point (2, 3),

(ii) the coordinates of the points on the curve at which the gradient is -1.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x-2)^2 + 2x(x-2)$$
$$= (x-2)(3x-2)$$

(i) When $x = 2$, gradient, $\frac{\mathrm{d}y}{\mathrm{d}x} = (2-2)(6-2) = 0$

(ii) When gradient, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, $(x-2)(3x-2) = -1$

$$3x^2 - 8x + 5 = 0$$
$$(3x-5)(x-1) = 0$$
$$x = \frac{5}{3},\quad 1$$
$$y = 3\frac{5}{27},\quad 4$$

Coordinates of points are $\left(1\frac{2}{3},\quad 3\frac{5}{27}\right)$ and $(1,\quad 4)$ or $\left(\frac{5}{3},\quad \frac{86}{27}\right)$ and $(1,\quad 4)$.

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Assignment 8: Product Rule

1. Differentiate the following expressions of y with respect to x. For (b) to (f), express $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a single fraction. **(Pg365Q2-4)**

(a) $y = x^2(1-4x)^3$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (1-4x)^3\frac{\mathrm{d}}{\mathrm{d}x}(x^2) + x^2\frac{\mathrm{d}}{\mathrm{d}x}(1-4x)^3\\ &= 2x(1-4x)^3 + 3x^2(1-4x)^2(-4)\\ &= 2x(1-4x)^3 - 12x^2(1-4x)^2\\ &= 2x(1-4x)^2(1-4x-6x)\\ &= 2x(1-4x)^2(1-10x)\end{aligned}$$

(b) $y = (x^2+1)\sqrt{x+1}$

$$\begin{aligned} y &= (x^2+1)(x+1)^{\frac{1}{2}}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (x+1)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(x^2+1) + (x^2+1)\frac{\mathrm{d}}{\mathrm{d}x}(x+1)^{\frac{1}{2}}\\ &= 2x(x+1)^{\frac{1}{2}} + \frac{1}{2}(x^2+1)(x+1)^{-\frac{1}{2}}(1)\\ &= \frac{1}{2}(x+1)^{-\frac{1}{2}}[4x(x+1)+(x^2+1)]\\ &= \frac{1}{2}(x+1)^{-\frac{1}{2}}(5x^2+4x+1) \text{ or } \frac{5x^2+4x+1}{2\sqrt{x+1}}\end{aligned}$$

(c) $y = x\sqrt{1+2x}$

$$\begin{aligned} y &= x(1+2x)^{\frac{1}{2}}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (1+2x)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(x) + x\frac{\mathrm{d}}{\mathrm{d}x}(1+2x)^{\frac{1}{2}}\\ &= (1+2x)^{\frac{1}{2}} + x\times\frac{1}{2}(1+2x)^{-\frac{1}{2}}(2)\\ &= (1+2x)^{\frac{1}{2}}(1+2x+x)\\ &= (1+2x)^{\frac{1}{2}}(1+3x) \text{ or } \frac{1+3x}{\sqrt{1+2x}}\end{aligned}$$

(d) $y=(4x-1)\sqrt{3x^2+1}$

$y=(4x-1)(3x^2+1)^{\frac{1}{2}}$

$$\frac{dy}{dx}=(3x^2+1)^{\frac{1}{2}}\frac{d}{dx}(4x-1)+(4x-1)\frac{d}{dx}(3x^2+1)^{\frac{1}{2}}$$

$$=(3x^2+1)^{\frac{1}{2}}(4)+\frac{1}{2}(4x-1)(3x^2+1)^{-\frac{1}{2}}(6x)$$

$$=4(3x^2+1)^{-\frac{1}{2}}[4(3x^2+1)+3x(4x-1)]$$

$$=(3x^2+1)^{-\frac{1}{2}}(24x^2-3x+4)\ \text{ or }\ \frac{24x^2-3x+4}{\sqrt{3x^2+1}}$$

(e) $y=(x^{\frac{3}{2}}+1)\sqrt{x+1}$

$y=(x^{\frac{3}{2}}+1)(x+1)^{\frac{1}{2}}$

$$\frac{dy}{dx}=(x+1)^{\frac{1}{2}}\frac{d}{dx}(x^{\frac{3}{2}}+1)+(x^{\frac{3}{2}}+1)\frac{d}{dx}(x+1)^{\frac{1}{2}}$$

$$=(x+1)^{\frac{1}{2}}\left(\frac{3}{2}x^{\frac{1}{2}}\right)+(x^{\frac{3}{2}}+1)\times\frac{1}{2}(x+1)^{-\frac{1}{2}}$$

$$=(x+1)^{-\frac{1}{2}}\left[\frac{3}{2}x^{\frac{1}{2}}(x+1)+\frac{1}{2}(x^{\frac{3}{2}}+1)\right]$$

$$=\frac{1}{2}(x+1)^{-\frac{1}{2}}(4x^{\frac{3}{2}}+3x^{\frac{1}{2}}+1)\ \text{ or }\ \frac{4x\sqrt{x}+3\sqrt{x}+1}{2\sqrt{x+1}}$$

(f) $y=x\sqrt{3-x^2}$

$y=x(3-x^2)^{\frac{1}{2}}$

$$\frac{dy}{dx}=(3-x^2)^{\frac{1}{2}}\frac{d}{dx}(x)+x\frac{d}{dx}(3-x^2)^{\frac{1}{2}}$$

$$=(3-x^2)^{\frac{1}{2}}(1)+x\left(\frac{1}{2}\right)(3-x^2)^{-\frac{1}{2}}(-2x)$$

$$=(3-x^2)^{-\frac{1}{2}}(3-x^2-x^2)$$

$$=(3-x^2)^{-\frac{1}{2}}(3-2x^2)\ \text{ or }\ \frac{3-2x^2}{\sqrt{3-x^2}}$$

(g)* $y = \frac{1}{x}(x+1)(x+2)^3$
$= (1+x^{-1})(x+2)^3$

$$\frac{dy}{dx} = (x+2)^3(-x^{-2}) + (1+x^{-1})(3)(x+2)^2$$
$$= (x+2)^2[(x+2)(-x^{-2}) + 3(1+x^{-1})]$$
$$= (x+2)^2(-x^{-1} - 2x^{-2} + 3 + 3x^{-1})$$
$$= (x+2)^2(3+2x^{-1} - 2x^{-2}) \text{ or } \frac{(x+2)^2(3x^2+2x-2)}{x^2}$$

(h)* $y = x^2(x-1)\sqrt{5+6x}$

$$= (x^3 - x^2)(5+6x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = (5+6x)^{\frac{1}{2}}(3x^2 - 2x) + (x^3 - x^2)\left(\frac{1}{2}\right)(5+6x)^{-\frac{1}{2}}(6)$$
$$= (5+6x)^{-\frac{1}{2}}[(5+6x)(3x^2-2x) + 3(x^3 - x^2)]$$
$$= (5+6x)^{-\frac{1}{2}}(15x^2 - 10x + 18x^3 - 12x^2 - 2x + 3x^3 - 3x^2)$$
$$= (5+6x)^{-\frac{1}{2}}(21x^3 - 10x) \text{ or } \frac{21x^3 - 10x}{\sqrt{5+6x}}$$

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2. Calculate the gradients of the curve $y = (2x+1)^3(x-1)$ at the points where it crosses the x-axis.

$$\frac{dy}{dx} = (x-1)[3(2x+1)^2(2)] + (2x+1)^3$$
$$= 6(x-1)(2x+1)^2 + (2x+1)^3$$
$$= (2x+1)^2(6x - 6 + 2x + 1)$$
$$= (2x+1)^2(8x-5)$$

When $y = 0$, $(2x+1)^3(x-1) = 0$

$x = -\frac{1}{2},\ 1$

When $x = -\frac{1}{2}$, $\frac{dy}{dx} = 0$

When $x = 1$, $\frac{dy}{dx} = (3)^2(8-5)$

$= 27$

3. Find the gradients of the curve $y = x\sqrt{4-x^2}$ at the points where it crosses the straight line $y = x$. (Pg366Q9)

At point of intersection,
$$\begin{aligned} x &= x\sqrt{4-x^2} \\ x^2 &= x^2(4-x^2) \\ x^2(4-x^2-1) &= 0 \\ x^2(3-x^2) &= 0 \\ x &= -\sqrt{3},\ 0, \sqrt{3} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= (4-x^2)^{\frac{1}{2}}\frac{d}{dx}(x) + x\frac{d}{dx}(4-x^2)^{\frac{1}{2}} \\ &= (4-x^2)^{\frac{1}{2}} + \frac{1}{2}x(-2x)(4-x^2)^{-\frac{1}{2}} \\ &= (4-x^2)^{-\frac{1}{2}}(4-x^2-x^2) \\ &= (4-x^2)^{-\frac{1}{2}}(4-2x^2) \text{ or } \frac{4-2x^2}{\sqrt{4-x^2}} \end{aligned}$$

When $x = -\sqrt{3}$, $\frac{dy}{dx} = \frac{4-2(3)}{\sqrt{4-3}} = -2$

When $x = 0$, $\frac{dy}{dx} = \frac{4}{\sqrt{4}} = 2$

When $x = \sqrt{3}$, $\frac{dy}{dx} = \frac{4-2(3)}{\sqrt{4-3}} = -2$

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4. The equation of the curve is $y = 3x\sqrt{5-x^2}$. Find the x-coordinates of the points where $\frac{dy}{dx} = 0$.

$$\begin{aligned} \frac{dy}{dx} &= (5-x^2)^{\frac{1}{2}}\frac{d}{dx}(3x) + 3x\frac{d}{dx}(5-x^2)^{\frac{1}{2}} \\ &= 3\sqrt{5-x^2} + \frac{1}{2}(3x)(5-x^2)^{-\frac{1}{2}}(-2x) \\ &= 3\sqrt{5-x^2} - \frac{3x^2}{\sqrt{5-x^2}} \\ &= \frac{3(5-x^2)-3x^2}{\sqrt{5-x^2}} \\ &= \frac{15-6x^2}{\sqrt{5-x^2}} \end{aligned}$$

When $\frac{dy}{dx} = 0$,
$$\begin{aligned} \frac{15-6x^2}{\sqrt{5-x^2}} &= 0 \\ 15-6x^2 &= 0 \\ 2x^2 &= 5 \\ x^2 &= 2.5 \\ x &= \pm 1.58 \text{ or } \pm\frac{\sqrt{10}}{2} \end{aligned}$$

5*. Given that $y = a(x-b)\sqrt{x+1}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x}{\sqrt{x+1}}$, find the value of *a* and of *b*.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\sqrt{x+1} + a(x-b)\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}}$$

$$= a\sqrt{x+1} + \frac{a(x-b)}{2\sqrt{x+1}}$$

$$= \frac{2a(x+1) + a(x-b)}{2\sqrt{x+1}}$$

$$= \frac{3ax + (2a - ab)}{2\sqrt{x+1}}$$

$$\frac{3ax + (2a-ab)}{2\sqrt{x+1}} = \frac{3x}{\sqrt{x+1}}$$

$\therefore 3ax + (2a - ab) = 3x$

Comparing coefficient of *x*,

$$\frac{3a}{2} = 3$$

$a = 2$

Comparing constant term,

$$\frac{2a - ab}{2} = 0$$

$2(2) - 2b = 0$

$b = 2$

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Worksheet 9: Quotient Rule

If a function y is given as the quotient of two other differentiable functions u and v where $v \neq 0$ for all values of x, the derivative of the derivative of y could be found by using the following rule:

Quotient Rule If $y = \dfrac{u}{v}$ where u and v are two differentiable functions and $v \neq 0$ for all values of x, then $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2} .$$

Example 1

Differentiate the following expressions of y with respect to x.

(a) $y = \dfrac{5x}{2x+1}$, $x \neq -\dfrac{1}{2}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(2x+1)\dfrac{\mathrm{d}}{\mathrm{d}x}(5x) - 5x\dfrac{\mathrm{d}}{\mathrm{d}x}(2x+1)}{(2x+1)^2}\\ &= \frac{5(2x+1)-5x(2)}{(2x+1)^2}\\ &= \frac{5}{(2x+1)^2}\end{aligned}$$

(b) $y = \dfrac{1-x}{1-2x}$, $x \neq \dfrac{1}{2}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(1-2x)\dfrac{\mathrm{d}}{\mathrm{d}x}(1-x) - (1-x)\dfrac{\mathrm{d}}{\mathrm{d}x}(1-2x)}{(1-2x)^2}\\ &= \frac{-(1-2x)-(1-x)(-2)}{(1-2x)^2}\\ &= \frac{-1+2x+2-2x}{(1-2x)^2}\\ &= \frac{1}{(1-2x)^2}\end{aligned}$$

(c) $y = \dfrac{2x-3}{x+5}, \;\; x \neq -5$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x+5)\dfrac{d}{dx}(2x-3)-(2x-3)\dfrac{d}{dx}(x+5)}{(x+5)^2} \\ &= \frac{2(x+5)-(2x-3)(1)}{(x+5)^2} \\ &= \frac{2x+10-2x+3}{(x+5)^2} \\ &= \frac{13}{(x+5)^2}\end{aligned}$$

(d) $y = \dfrac{x^2}{x+3}, \;\; x \neq -3$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x+3)\dfrac{d}{dx}(x^2)-x^2\dfrac{d}{dx}(x+3)}{(x+3)^2} \\ &= \frac{2x(x+3)-x^2(1)}{(x+3)^2} \\ &= \frac{x^2+6x}{(x+3)^2}\end{aligned}$$

(e) $y = \dfrac{1-x^2}{1+x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+x^2)\dfrac{d}{dx}(1-x^2)-(1-x^2)\dfrac{d}{dx}(1+x^2)}{(2-3x)^2} \\ &= \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2} \\ &= \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2} \\ &= -\frac{4x}{(1+x^2)^2}\end{aligned}$$

(f) $y = \dfrac{1-3x}{\sqrt{1+x}}$, $x \neq -1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-3(1+x)^{\frac{1}{2}} - (1-3x)\left(\frac{1}{2}\right)(1+x)^{-\frac{1}{2}}}{(\sqrt{1+x})^2}$$

$$= \frac{1}{2}(1+x)^{-\frac{3}{2}}[-6(1+x) - (1+3x)]$$

$$= \frac{1}{2}(1+x)^{-\frac{3}{2}}(-7-3x) \text{ or } \frac{-7-3x}{2(1+x)^{\frac{3}{2}}}$$

(g) $y = \dfrac{\sqrt{x}}{1+x}$, $x \neq -1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1+x)\frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{1}{2}}) - x^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - x^{\frac{1}{2}}}{(1+x)^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1+x-2x)}{(1+x)^2}$$

$$= \frac{1-x}{2\sqrt{x}(1+x)^2}$$

Example 2

Calculate the x-coordinates of the points on the curve $y = \dfrac{3x-2}{\sqrt{4+x^2}}$ for which $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(4+x^2)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(3x-2) - (3x-2)\frac{\mathrm{d}}{\mathrm{d}x}(4+x^2)^{\frac{1}{2}}}{(4+x^2)}$$

$$= \frac{3(4+x^2)^{\frac{1}{2}} - (3x-2)\left(\frac{1}{2}\right)(4+x^2)^{-\frac{1}{2}}(2x)}{(4+x^2)}$$

$$= \frac{3(4+x^2)^{-\frac{1}{2}}[3(4+x^2) - x(3x-2)]}{(4+x^2)}$$

$$= \frac{12+2x}{(4+x^2)^{\frac{3}{2}}}$$

When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad 12 + 2x = 0$

$$x = -6$$

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Assignment 10: Quotient Rule

1. Differentiate the following expressions of y with respect to x. **(Pg368Q1-2, TYS)**

(a) $y = \dfrac{2x-1}{x+5}$; $x \neq -5$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x+5)\dfrac{\mathrm{d}}{\mathrm{d}x}(2x-1)-(2x-1)\dfrac{\mathrm{d}}{\mathrm{d}x}(x+5)}{(x+5)^2}\\ &= \frac{(x+5)(2)-(2x-1)(1)}{(x+5)^2}\\ &= \frac{2x+10-2x+1}{(x+5)^2}\\ &= \frac{11}{(x+5)^2}\end{aligned}$$

(b) $y = \dfrac{2x+3}{x-4}$; $x \neq 4$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x-4)\dfrac{\mathrm{d}}{\mathrm{d}x}(2x+3)-(2x+3)\dfrac{\mathrm{d}}{\mathrm{d}x}(x-4)}{(x-4)^2}\\ &= \frac{(x-4)(2)-(2x+3)(1)}{(x-4)^2}\\ &= \frac{2x-8-2x-3}{(x-4)^2}\\ &= -\frac{11}{(x-4)^2}\end{aligned}$$

(c) $y = \dfrac{x}{x^2 - 3}$; $x^2 \neq 3$

$$\frac{dy}{dx} = \frac{(x^2-3)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2-3)}{(x^2-3)^2}$$
$$= \frac{(x^2-3)(1) - x(2x)}{(x^2-3)^2}$$
$$= \frac{x^2 - 3 - 2x^2}{(x^2-3)^2}$$
$$= \frac{-x^2-3}{(x^2-3)^2}$$

(d) $y = \dfrac{3x^2}{1-2x}$; $x \neq \dfrac{1}{2}$

$$\frac{dy}{dx} = \frac{(1-2x)\frac{d}{dx}(3x^2) - 3x^2\frac{d}{dx}(1-2x)}{(1-2x)^2}$$
$$= \frac{(1-2x)(6x) - 3x^2(-2)}{(1-2x)^2}$$
$$= \frac{6x - 6x^2}{(1-2x)^2}$$

(e) $y = \dfrac{x^2+1}{2x-1}$; $x \neq \dfrac{1}{2}$

$$\frac{dy}{dx} = \frac{(2x-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(2x-1)}{(2x-1)^2}$$
$$= \frac{(2x-1)(2x) - (x^2+1)(2)}{(2x-1)^2}$$
$$= \frac{4x^2 - 2x - 2x^2 - 2}{(2x-1)^2}$$
$$= \frac{2x^2 - 2x - 2}{(2x-1)^2}$$

(f) $y = \dfrac{2x^3}{1-x}$; $x \neq 1$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1-x)\dfrac{d}{dx}(2x^3) - 2x^3\dfrac{d}{dx}(1-x)}{(1-x)^2} \\
&= \frac{(1-x)(6x^2) - 2x^3(-1)}{(1-x)^2} \\
&= \frac{6x^2 - 6x^3 + 2x^3}{(1-x)^2} \\
&= \frac{6x^2 - 4x^3}{(1-x)^2}
\end{aligned}$$

(g) $y = \dfrac{x}{\sqrt{1-x}}$; $x \neq 1$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1-x)^{\frac{1}{2}}\dfrac{d}{dx}(x) - x\dfrac{d}{dx}(1-x)^{\frac{1}{2}}}{(\sqrt{1-x})^2} \\
&= \frac{(1-x)^{\frac{1}{2}} - x\left(\dfrac{1}{2}\right)(1-x)^{-\frac{1}{2}}(-1)}{(1-x)} \\
&= \frac{\dfrac{1}{2}(1-x)^{-\frac{1}{2}}(2-2x+x)}{(1-x)} \\
&= \frac{2-x}{2(1-x)\sqrt{1-x}} \text{ or } \frac{2-x}{2(1-x)^{\frac{3}{2}}}
\end{aligned}$$

(h) $y = \dfrac{5x}{\sqrt{1-x^2}}$; $x^2 \neq 1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1-x^2)^{\frac{1}{2}}\dfrac{\mathrm{d}}{\mathrm{d}x}(5x) - 5x\dfrac{\mathrm{d}}{\mathrm{d}x}(1-x^2)^{\frac{1}{2}}}{(\sqrt{1-x^2})^2}$$

$$= \frac{5(1-x^2)^{\frac{1}{2}} - 5x\left(\dfrac{1}{2}\right)(1-x^2)^{-\frac{1}{2}}(-2x)}{(\sqrt{1-x^2})^2}$$

$$= \frac{(1-x^2)^{-\frac{1}{2}}[5(1-x^2)+5x^2]}{(\sqrt{1-x^2})^2}$$

$$= \frac{5-5x^2+5x^2}{(1-x^2)\sqrt{1-x^2}}$$

$$= \frac{5}{(1-x^2)\sqrt{1-x^2}} \text{ or } \frac{5}{(1-x^2)^{\frac{3}{2}}}$$

2. Calculate the x-coordinate of the point on the curve $y=\sqrt{\dfrac{1-x}{x^2+3}}$ for which $\dfrac{dy}{dx}=0$.

(Pg321Q7)

$$y=\left(\frac{1-x}{x^2+3}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{(x^2+3)^{\frac{1}{2}}\times\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)-(1-x)^{\frac{1}{2}}\times\frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x)}{(x^2+3)^2}$$

$$=\frac{\frac{1}{2}(1-x)^{-\frac{1}{2}}(x^2+3)^{-\frac{1}{2}}[-(x^2+3)-2x(1-x)]}{(x^2+3)^2}$$

$$=\frac{(1-x)^{-\frac{1}{2}}(x^2+3)^{-\frac{1}{2}}(x^2-2x+3)}{2(1-x)^{\frac{1}{2}}(x^2+3)^{\frac{3}{2}}}$$

When $\dfrac{dy}{dx}=0$, $\quad \dfrac{x^2-2x-3}{2(x^2+3)}\sqrt{\dfrac{1}{(1-x)(x^2+3)^2}}=0$

$$\therefore x^2-2x-3=0$$
$$(x-3)(x+1)=0$$
$$x=-1,\ 3\,(\text{NA})$$

ms

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3. Find the coordinates of the point on the curve $y=\dfrac{3x^2-4}{3x^2+2}$ at which the tangent is parallel to the line $y=-5$.

$$y=\frac{3x^2+2-6}{3x^2+2}$$
$$=1-6(3x^2+2)^{-1}$$
$$\frac{dy}{dx}=-6(-1)(3x^2+2)^{-2}(6x)$$
$$=\frac{36x}{(3x^2+2)^2}$$

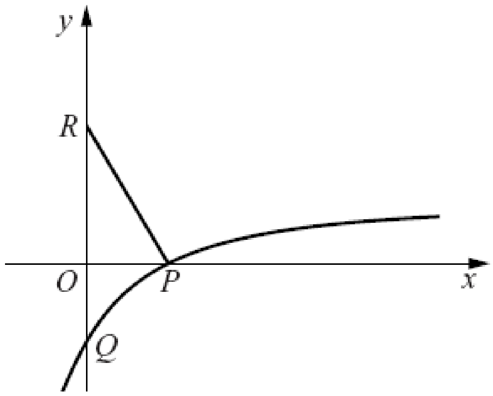
$$\frac{dy}{dx}=\frac{(3x^2+2)(6x)-(3x^2-4)(6x)}{(3x^2+2)^2}$$
$$=\frac{18x^2+12x-18x^2+24x}{(3x^2+2)^2}$$
$$=\frac{36x}{(3x^2+2)^2}$$

When $\dfrac{dy}{dx}=0$, $\quad \dfrac{36x}{(3x^2+2)^2}=0$

$$x=0$$
$$y=-2$$

The required point is (0, –2)

4. The diagram shows part of the curve $y = \dfrac{2x-6}{x+2}$ crossing the x-axis at P and the y-axis at Q. The normal to the curve at P meets the y-axis at R.



(i) Given that $\dfrac{dy}{dx} = \dfrac{k}{(x+2)^2}$, evaluate k.

(ii) Find the length of RQ. (N02)

(i) $\dfrac{dy}{dx} = \dfrac{(x+2)(2)-(2x-6)}{(x+2)^2}$

$= \dfrac{10}{(x+2)^2}$

$\therefore k = 10$

(ii) At P, $y = 0$, $\quad 2x - 6 = 0$

$x = 3$

When $x = 3$, $\quad y = 0$, $\quad \dfrac{dy}{dx} = \dfrac{2}{5}$

Gradient of normal $= -\dfrac{5}{2}$

Equation of tangent, $\quad y = -\dfrac{5}{2}(x-3)$

$y = -\dfrac{5}{2}x + \dfrac{15}{2}$

$\therefore R\left(0,\ \dfrac{15}{2}\right)$

At Q, $x = 0$, $\quad y = -3$

$\therefore RQ = 10\dfrac{1}{2}$ unit

5*. A curve has the equation $y=\sqrt{\dfrac{x-a}{b-x}}$, where $a<x<b$, and a and b are constants. Show that the gradient of the curve at $x=\dfrac{a+b}{2}$ is $\dfrac{2}{b-a}$. (Pg369Q8)

$$y=\left(\frac{x-a}{b-x}\right)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(\frac{x-a}{b-x}\right)^{-\frac{1}{2}}\left[\frac{(b-x)(1)-(x-a)(-1)}{(b-x)^2}\right]$$

$$=\frac{1}{2}\sqrt{\frac{b-x}{x-a}}\left[\frac{(b-a)}{(b-x)^2}\right]$$

$$=\frac{(b-a)}{2\sqrt{(x-a)(b-x)^3}}$$

When $x=\dfrac{a+b}{2}$,

$$x-a=\frac{a+b}{2}-a$$

$$=\frac{b-a}{2}$$

$$b-x=b-\frac{a+b}{2}$$

$$=\frac{b-a}{2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(b-a)}{2\sqrt{\left(\frac{b-a}{2}\right)\left(\frac{b-a}{2}\right)^3}}$$

$$=\frac{(b-a)}{2\left(\frac{b-a}{2}\right)^2}$$

$$=\frac{(b-a)}{2}\times\frac{4}{(b-a)^2}$$

$$=\frac{2}{b-a}$$

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Worksheet 11 : Equations of Tangent and Normal

Consider a point A (x_1, y_1) on a curve defined by $y = \mathrm{f}(x)$,

(a) the gradient at A, m = value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at (x_1, y_1),

(b) the equation of the tangent at A is $y - y_1 = m(x - x_1)$

(c) the equation of the normal at A is $y - y_1 = -\dfrac{1}{m}(x - x_1)$

Example 1

Find the equations of the tangent and the normal to the curve $y = x + \dfrac{2}{x}$ at $x = 1$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{2}{x^2}$$

When $x = 1$,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{2}{1} = -1$$
$$y = 3$$

Equation of tangent,
$$y - 3 = -(x - 1)$$
$$y = 4 - x$$

Equation of normal,
$$y - 3 = x - 1$$
$$y = x + 2$$

Example 2

Find the equation of the normal to the curve $y=\dfrac{3x+1}{1-x}$ for $x\neq 1$, at the point where the curve crosses the x-axis. **(Pg377Q5)**

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1-x)\frac{d}{dx}(3x+1)-(3x+1)\frac{d}{dx}(1-x)}{(1-x)^2}\\ &= \frac{3(1-x)-(3x+1)(-1)}{(1-x)^2}\\ &= \frac{3-3x+3x+1}{(1-x)^2}\\ &= \frac{4}{(1-x)^2}\end{aligned}$$

When $y=0$, $\quad \dfrac{3x+1}{1-x}=0$

$$x=-\frac{1}{3}$$

At $\left(-\dfrac{1}{3},0\right)$, $\quad \dfrac{dy}{dx}=\dfrac{4}{\left(1+\frac{1}{3}\right)^2}$

$$=\frac{9}{4}$$

Equation of normal, $\quad y=-\dfrac{4}{9}\left(x+\dfrac{1}{3}\right)$

$$y=-\frac{4}{9}x-\frac{4}{27}$$

Example 3

Find the equation of the tangent to the curve $y=\dfrac{(x+6)^2}{x-9}$ at the point where the curve crosses the y-axis.

$$\begin{aligned}\frac{dy}{dx} &= \frac{2(x+6)(x-9)-(x+6)^2}{(x-9)^2}\\ &= \frac{(x+6)(2x-18-x-6)}{(x-9)^2}\\ &= \frac{(x+6)(x-24)}{(x-9)^2}\end{aligned}$$

When $x=0$, $\quad \dfrac{dy}{dx}=\dfrac{2(6)(-9)-6^2}{(-9)^2}=-\dfrac{16}{9}$

$$y=-4$$

Equation of tangent, $\quad y=-\dfrac{16}{9}x-4$

Example 4

Given that $y = ax^5$, where $a \neq 0$, and $\frac{dy}{dx} = \frac{3}{2}$ at (1, b), find

(i) the value of a and of b,

(ii) the equation of the tangent to the curve at the origin. **(Pg377Q8)**

$\frac{dy}{dx} = 5ax^4$

(i) At (1, b), $\frac{dy}{dx} = \frac{3}{2}$, $\quad 5a = \frac{3}{2}$

$a = \frac{3}{10}$

At (1, b), $y = b$, $\quad b = \frac{3}{10}$

(ii) At (0, 0), $\quad \frac{dy}{dx} = 0$

$\therefore$ Equation of tangent at the origin is $y = 0$.

Example 5

Find the equation of the normal to the curve $y = 3x - \frac{4}{(x-1)^2}$ which is parallel to the line $4y + x - 3 = 0$.

$y = 3x - 4(x-1)^{-2}$

$\frac{dy}{dx} = 3 - 4(-2)(x-1)^{-3}$

$= 3 + \frac{8}{(x-1)^3}$

$4y + x - 3 = 0 \quad \Rightarrow y = -\frac{x}{4} + \frac{3}{4}$

$\Rightarrow$ gradient of normal = $-\frac{1}{4}$

$\Rightarrow$ gradient of tangent = 4

When $\frac{dy}{dx} = 4$, $\quad 3 + \frac{8}{(x-1)^3} = 4$

$\frac{8}{(x-1)^3} = 1$

$(x-1)^3 = 8$

$x - 1 = 2$

$x = 3$

When $x = 3$, $\quad y = 8$

Equation of normal, $\quad y - 8 = -\frac{1}{4}(x-3)$

$y = 8\frac{3}{4} - \frac{x}{4}$

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Assignment 12: Equations of Tangent and Normal

1. Find the equations of the tangents to the curve $y = \dfrac{x^2+5}{x+1}$ at $x = 1$ and $x = 3$. Find the coordinates of the points where these tangents meet.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x(x+1)-(x^2+5)}{(x+1)^2}$$
$$= \frac{x^2+2x-5}{(x+1)^2}$$

When $x = 1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1^2+2-5}{2^2}$
$$= -\frac{1}{2}$$
$$y = 3$$

Equation of tangent, $y - 3 = -\dfrac{1}{2}(x-1)$
$$y = -\frac{1}{2}x + \frac{7}{2}$$

When $x = 3$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3^2+6-5}{4^2}$
$$= \frac{5}{8}$$
$$y = \frac{3^2+5}{4} = \frac{7}{2}$$

Equation of tangent, $y - \dfrac{7}{2} = \dfrac{5}{8}(x-3)$
$$y = \frac{5}{8}x + \frac{13}{8}$$

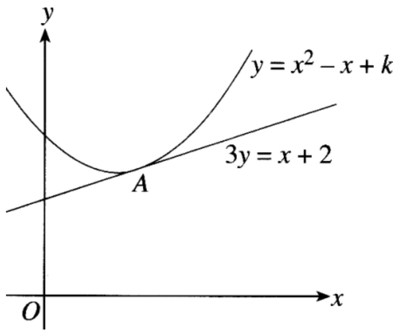
At point of intersection, $-\dfrac{1}{2}x + \dfrac{7}{2} = \dfrac{5}{8}x + \dfrac{13}{8}$
$$\frac{5}{8}x + \frac{1}{2}x = \frac{7}{2} - \frac{13}{8}$$
$$\frac{9}{8}x = \frac{15}{8}$$
$$x = \frac{5}{3}$$
$$y = -\frac{1}{2}\left(\frac{5}{3}\right) + \frac{7}{2}$$
$$= \frac{8}{3}$$

$\therefore$ The point of intersection is $\left(\dfrac{5}{3}, \dfrac{8}{3}\right)$

2. In the diagram, the line $3y = x + 2$ is tangent to the curve $y = x^2 - x + k$ at the point A. Find

(a) the coordinates of A,

(b) the value of the constant k.



(a) $3y = x + 2 \Rightarrow y = \dfrac{x}{3} + \dfrac{2}{3}$

$$\frac{dy}{dx} = 2x - 1$$

At A, $\dfrac{dy}{dx} = \dfrac{1}{3}$, $\quad 2x - 1 = \dfrac{1}{3}$

$$2x = \frac{4}{3}$$

$$x = \frac{2}{3}$$

$$y = \frac{8}{9}$$

$$\therefore A\left(\frac{2}{3}, \frac{8}{9}\right)$$

(b) At $A\left(\dfrac{2}{3}, \dfrac{8}{9}\right)$, $\quad \dfrac{8}{9} = \left(\dfrac{2}{3}\right)^2 - \dfrac{2}{3} + k$

$$k = \frac{8}{9} - \left(\frac{2}{3}\right)^2 + \frac{2}{3}$$

$$= \frac{10}{9}$$

3. The tangent and the normal to the curve $y = 4\sqrt{x+2}$ at the point $P(7,12)$ cut the x-axis at the point M and N respectively. Calculate the area of the triangle PMN.

$$y = 4(x+2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2(x+2)^{-\frac{1}{2}}$$

$$= \frac{2}{\sqrt{x+2}}$$

At $P(7,12)$, $\quad \frac{dy}{dx} = \frac{2}{\sqrt{7+2}}$

$$= \frac{2}{3}$$

Equation of tangent, $\quad y - 12 = \frac{2}{3}(x-7)$

$$y = \frac{2}{3}x + \frac{22}{3}$$

At M, $y = 0$, $\quad \frac{2}{3}x + \frac{22}{3} = 0$

$$x = -11$$

$\therefore$ $M(-11,0)$.

Equation of normal, $\quad y - 12 = -\frac{3}{2}(x-7)$

$$y = -\frac{3}{2}x + \frac{45}{2}$$

At N, $y = 0$, $\quad -\frac{3}{2}x + \frac{45}{2} = 0$

$$x = 15$$

$\therefore$ $N(0,15)$.

Area of the triangle PMN $= \frac{1}{2} \times 26 \times 12$

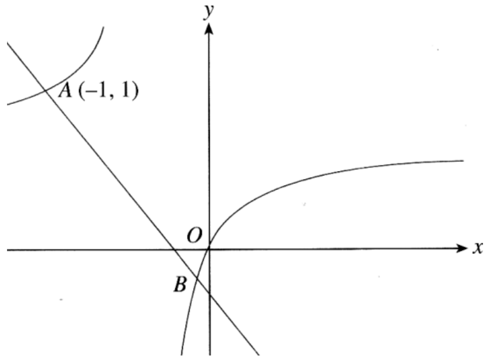
$$= 156 \text{ unit}^2$$

4. The diagram shows part of the curve $y = \frac{2x}{3x+1}$. The normal to the curve at the point $A(-1,1)$ meets the curve again at point B.

Find

(a) the equation of the normal

(b) the coordinates of B.



(a) $\frac{dy}{dx} = \frac{2(3x+1)-2x(3)}{(3x+1)^2}$

$= \frac{2}{(3x+1)^2}$

At $A(-1,1)$, $\frac{dy}{dx} = \frac{2}{(-3+1)^2}$

$= \frac{1}{2}$

Equation of the normal, $y-1=-2(x+1)$

$y=-2x-1$

(b) At B, $-2x-1=\frac{2x}{3x+1}$

$(-2x-1)(3x+1)=2x$

$-6x^2-5x-1=2x$

$6x^2+7x+1=0$

$(6x+1)(x+1)=0$

$x=-\frac{1}{6},\ --1$ (NA)

$y=-\frac{2}{3}$

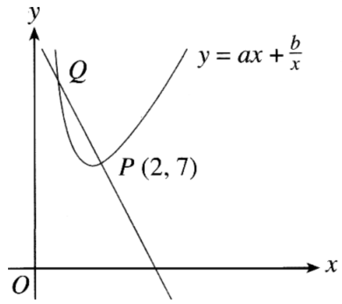
$\therefore B\left(-\frac{1}{6},-\frac{2}{3}\right)$

5. The diagram shows part of the curve $y = ax + \frac{b}{x}$. The line $y + 2x = 11$ is the normal to the curve at the point $P(2, 7)$ and this normal meets the curve again at point Q.

Find

(a) the value of a and of b.

(b) the coordinates of Q.



(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = a - \frac{b}{x^2}$

$y + 2x = 11 \Rightarrow y = 11 - 2x$

Gradient of tangent at $P = \frac{1}{2}$

At $P(2, 7)$, $\quad 7 = 2a + \frac{b}{2}$

$b = 14 - 4a \quad \ldots\ldots(1)$

At $P(2, 7)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$, $\quad a - \frac{b}{4} = \frac{1}{2} \quad \ldots\ldots(2)$

Sub (1) into (2) $\quad a - \frac{(14 - 4a)}{4} = \frac{1}{2}$

$4a - 14 + 4a = 2$

$8a = 16$

$a = 2$

$b = 6$

(b) At Q, $\quad 11 - 2x = 2x + \frac{6}{x}$

$11x - 2x^2 = 2x^2 + 6$

$4x^2 - 11x + 6 = 0$

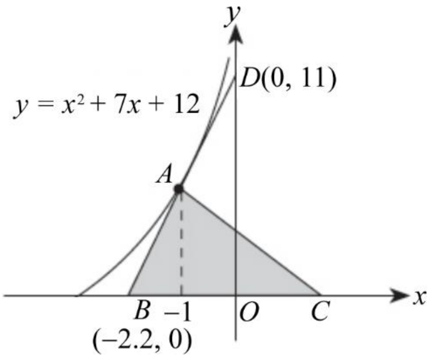
$(4x - 3)(x - 2) = 0$

$x = \frac{3}{4}, 2$ (NA)

$y = \frac{19}{2}$

$\therefore Q\left(\frac{3}{4}, \frac{19}{2}\right)$.

6. The diagram, not drawn to scale, shows part of the curve $y = x^2 + 7x + 12$. The tangent to the curve at the point A meets the coordinate axes at $B(-2.2, 0)$ and at $D(0, 11)$.



(i) Find the coordinates of A.

The normal to the curve at A meets the x-axis at C. Find
(ii) the coordinates of C,
(iii) the area of the triangle ABC. **(P378Q14)**

(i) At A, $x = -1$, $\quad y = 1 - 7 + 12$

$= 6$

$\therefore A(-1, 6)$

(ii) $\dfrac{dy}{dx} = 2x + 7$

At A, $x = -1$, $\quad \dfrac{dy}{dx} = \dfrac{11}{2.2} = 5$

Gradient of normal at $A = -\dfrac{1}{5}$

Equation of AC, $\quad y - 6 = -\dfrac{1}{5}(x+1)$

$y = -\dfrac{x}{5} + \dfrac{29}{5}$

At C, $y = 0$, $\quad -\dfrac{x}{5} + \dfrac{29}{5} = 0$

$x = 29$

$\therefore C(29, 0)$

(iii) Area of triangle $ABC = \dfrac{1}{2} \times (29 + 2.2) \times 6$

$= 93.6 \text{ unit}^2$

7*. The tangent at the point $P(a, b)$ on the curve $y = \frac{ab}{x}$ meets the x-axis and the y-axis at points at Q and R respectively. Show that $PQ = RP$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ab}{x^2}$$

Gradient of tangent at P $= -\frac{ab}{a^2}$

$$= -\frac{b}{a}$$

Equation of tangent at P, $\quad y - b = -\frac{b}{a}(x - a)$

$$y = -\frac{b}{a}x + b + b$$

$$y = -\frac{b}{a}x + 2b$$

At Q, $y = 0$, $\quad \frac{b}{a}x = 2b$

$$x = 2a$$

$\therefore$ $Q(2a,\ 0)$

At R, $x = 0$, $\quad y - b = -\frac{b}{a}(-a)$

$$y = 2b$$

$\therefore$ $R(0, 2b)$

Midpoint of $QR = \left(\frac{2a + 0}{2},\ \ \frac{0 + 2b}{2}\right)$

$$= (a,\ b)$$

$$= \text{Coordinates of } P$$

Hence, $PQ = RP$.

Alternative Method

$PQ = \sqrt{a^2 + b^2}$ unit

$RP = \sqrt{a^2 + b^2}$ unit

Hence, $PQ = RP$.

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 12

Name: ______________________________ ( ) Date: ______________

Class: Sec 4______

Revision 12 Techniques of Differentiation

1. Differentiate the following with respect to x.

(a) $y=\frac{2}{5}\sqrt{x}-\frac{5}{2x^2}$

$$\frac{dy}{dx}=\frac{2}{5}\left(\frac{1}{2}\right)x^{-\frac{1}{2}}-\frac{5}{2}(-2)x^{-3}$$

$$=\frac{1}{5}x^{-\frac{1}{2}}+5x^{-3} \text{ or } \frac{1}{5\sqrt{x}}+\frac{5}{x^3}$$

(b) $y=\frac{6}{1-2x}$

$$\frac{dy}{dx}=6(-1)(1-2x)^{-2}(-2)$$

$$=12(1-2x)^{-2} \text{ or } \frac{12}{(1-2x)^2}$$

(c) $y=(3-2x)^{10}$

$$\frac{dy}{dx}=10(3-2x)^9(-2)$$

$$=20(3-2x)^9$$

(d) $y=\frac{8}{(1-3x)^3}$

$$\frac{dy}{dx}=8(-3)(1-3x)^{-4}(-3)$$

$$=72(1-3x)^{-4} \text{ or } \frac{72}{(1-3x)^4}$$

(e) $y=\frac{21}{1+x^2}$

$$\frac{dy}{dx}=21(-1)(1+x^2)^{-2}(2x)$$

$$=-42x(1+x^2)^{-2} \text{ or } -\frac{42x}{(1+x^2)^2}$$

(f) $y=\frac{3}{4}\left(\frac{2x}{5}-1\right)^{15}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{4}(15)\left(\frac{2x}{5}-1\right)^{14}\left(\frac{2}{5}\right)$$

$$=\frac{9}{2}\left(\frac{2x}{5}-1\right)^{14}$$

(g) $y=3x(4-x^2)^3$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3(4-x^2)^3+3x(3)(4-x^2)^2(-2x)$$

$$=3(4-x^2)^3-18x^2(4-x^2)^2$$

$$=3(4-x^2)^2(4-7x^2)$$

(h) $y=\frac{2x-1}{1+x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2(1+x)-(2x-1)}{(1+x)^2}$$

$$=\frac{3}{(1+x)^2}$$

(i) $y=\frac{7x-2}{1+x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{7(1+x)-(7x-2)}{(1+x)^2}$$

$$=\frac{9}{(1+x)^2}$$

(j) $y=\frac{2x^2-1}{1+x^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4x(1+x^2)-2x(2x^2-1)}{(1+x^2)^2}$$

$$=\frac{6x}{(1+x^2)^2}$$

(k) $y=(x-2)\sqrt{x+4}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(x+4)^{\frac{1}{2}}+(x-2)\left(\frac{1}{2}\right)(x+4)^{-\frac{1}{2}}$$

$$=\frac{1}{2}(x+4)^{-\frac{1}{2}}(3x+6)$$

$$=\frac{3x+6}{2\sqrt{x+4}}$$

(l) $y=(3x+2)\sqrt{2x-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3(2x-1)^{\frac{1}{2}}+(3x+2)\left(\frac{1}{2}\right)(2x-1)^{-\frac{1}{2}}(2)$$

$$=(2x-1)^{-\frac{1}{2}}(6x-3+3x+2)$$

$$=(2x-1)^{-\frac{1}{2}}(9x-1)$$

$$=\frac{9x+5}{\sqrt{2x-1}}$$

(m) $y=\sqrt{1+\sqrt{x}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}(1+x^{\frac{1}{2}})^{-\frac{1}{2}}\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$$

$$=\frac{1}{4}x^{-\frac{1}{2}}(1+x^{\frac{1}{2}})^{-\frac{1}{2}}$$

$$=\frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}\text{ or }\frac{1}{4\sqrt{x+x\sqrt{x}}}$$

(n) $y=\sqrt{\dfrac{2x-7}{3x+7}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{1}{2}(3x+7)^{\frac{1}{2}}(2x-7)^{-\frac{1}{2}}(2)-\frac{1}{2}(2x-7)^{\frac{1}{2}}(3x+7)^{-\frac{1}{2}}(3)}{3x+7}$$

$$=\frac{\frac{1}{2}(2x-7)^{-\frac{1}{2}}(3x+7)^{-\frac{1}{2}}(6x+14-6x+21)}{3x+7}$$

$$=\frac{35}{2(2x-7)^{\frac{1}{2}}(3x+7)^{\frac{3}{2}}}$$

(o) $y=x\tan(x+\pi)$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\tan(x+\pi)+x\sec^2 x(x+\pi)$$

(p) $y=\dfrac{x\sin 3x}{1+x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(1+x)(\sin 3x+3x\cos x)-x\sin 3x}{(1+x)^2}$$

(q) $y = \cos^2\left(4x + \frac{\pi}{3}\right)$

$$\frac{dy}{dx} = 2\cos\left(4x + \frac{\pi}{3}\right)\left[-\sin\left(4x + \frac{\pi}{3}\right)\right](4)$$

$$= -8\cos\left(4x + \frac{\pi}{3}\right)\sin\left(4x + \frac{\pi}{3}\right)$$

(r) $y = \sin x \cos^3 x$

$$\frac{dy}{dx} = \cos x \cos^3 x + \sin x(3)\cos^2 x(-\sin x)$$

$$= \cos^4 x - 3\sin^2 x \cos^2 x$$

(s) $y = \dfrac{\sin x}{1 + \cos x}$

$$\frac{dy}{dx} = \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{1+\cos x}{(1+\cos x)^2}$$

$$= \frac{1}{1+\cos x}$$

(t) $y = e^{x^2 + \frac{1}{x}}$

$$\frac{dy}{dx} = e^{x^2 + \frac{1}{x}}\left(2x - \frac{1}{x^2}\right)$$

(u) $y = \dfrac{e^{3x}}{x}$

$$\frac{dy}{dx} = \frac{3xe^{3x} - e^{3x}}{x^2}$$

$$= \frac{e^{3x}(3x-1)}{x^2}$$

(v) $y = \ln\sqrt{\dfrac{6x}{2x-3}} = \dfrac{1}{2}\ln 6 + \dfrac{1}{2}\ln x - \dfrac{1}{2}\ln(2x-3)$

$$\frac{dy}{dx} = \frac{1}{2x} - \frac{2}{2(2x-3)}$$

$$= \frac{1}{2x} - \frac{1}{2x-3} \text{ or } -\frac{3}{2x(2x-3)}$$

$$=$$

(w) $y = 3\ln(5x^2 - 4x)$

$$\frac{dy}{dx} = \frac{3(10x-4)}{5x^2-4x}$$
$$= \frac{6(5x-2)}{x(5x-4)}$$

(x) $y = x^3 \ln(2x+1)$

$$\frac{dy}{dx} = 3x^2 \ln(2x+1) + x^3\left(\frac{2}{2x+1}\right)$$
$$= 3x^2 \ln(2x+1) + \frac{2x^3}{2x+1}$$

(y) $y = \ln\left(\frac{x+1}{2x-3}\right) = \ln(x+1) - \ln(2x-3)$

$$\frac{dy}{dx} = \frac{1}{x+1} - \frac{2}{2x-3}$$
$$= -\frac{5}{(x+1)(2x-3)}$$

(z) $e^{3x+1}(x^3+3x)$

$$\frac{dy}{dx} = 3e^{3x+1}(x^3+3x) + e^{3x+1}(3x^2+3)$$
$$= 3e^{3x+1}(x^3+x^2+3x+1)$$

---

2. A curve has the equation $y = x + \frac{4}{x^2}$. Find

(i) an expression for $\frac{dy}{dx}$,

(ii) the value of k for which $y = 2x + k$ is a tangent to the curve.

Solution

(i) $\frac{dy}{dx} = 1 - \frac{8}{x^3}$

(ii) When $\frac{dy}{dx} = 2$, $1 - \frac{8}{x^3} = 2$

$$\frac{8}{x^3} = -1$$
$$x^3 = -8$$
$$x = -2$$
$$y = -1$$

On the line $y = 2x + k$, $-1 = -4 + k$

$$k = 3$$

3. The equation of a curve is $y = \dfrac{x}{2-3x}$. Find an expression for $\dfrac{dy}{dx}$.
Hence, find the equation of the normal to the curve at the point $x = 1$.

Solution

$$\frac{dy}{dx} = \frac{(2-3x) - x(-3)}{(2-3x)^2}$$
$$= \frac{2}{(2-3x)^2}$$

When $x = 1$, $\dfrac{dy}{dx} = 2$

$y = -1$

Equation of normal, $y + 1 = -\dfrac{1}{2}(x-1)$ On

$y = -\dfrac{x}{2} - \dfrac{1}{2}$

---

4. P is the point (4, 7) on the curve $y = x^2 - 6x + 15$.
Find the equation of the normal at P.
The tangent at another point Q is parallel to the normal at P.
Calculate the x-coordinate of Q. **[5]**

Solution

$$\frac{dy}{dx} = 2x - 6$$

When $x = 4$, $\dfrac{dy}{dx} = 2$

Equation of normal, $y - 7 = -\dfrac{1}{2}(x-4)$ On

$y = 9 - \dfrac{x}{2}$

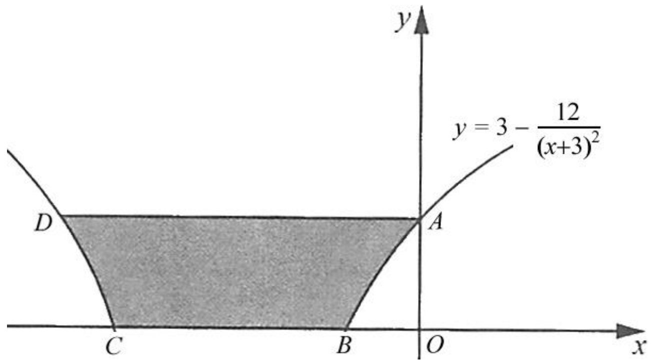
At Q, $\dfrac{dy}{dx} = -\dfrac{1}{2}$, $2x - 6 = -\dfrac{1}{2}$

$2x = \dfrac{11}{2}$

$x = \dfrac{11}{4}$

---

5. The diagram shows part of the curve $y = 3 - \dfrac{12}{(x+3)^2}$ intersecting the y-axis at the point A and intersecting the x-axis at the points B and C. The point D lies on the curve and AD is parallel to the x-axis.



(i) Find the coordinates of A and D.
(ii) Find the equation of the tangent at D.

Solution

(i) At A, $x = 0$, $\quad y = 3 - \dfrac{12}{9} = \dfrac{5}{3}$

$\therefore A\left(0, \dfrac{5}{3}\right)$

At D, $y = \dfrac{5}{3}$, $\quad 3 - \dfrac{12}{(x+3)^2} = \dfrac{5}{3}$

$$\frac{12}{(x+3)^2} = \frac{4}{3}$$

$$(x+3)^2 = 9$$

$$x + 3 = \pm 3$$

$$x = -6, 0 \text{ (NA)}$$

$\therefore D\left(-6, \dfrac{5}{3}\right)$

(ii) $\dfrac{dy}{dx} = \dfrac{24}{(x+3)^3}$

At $D\left(-6, \dfrac{5}{3}\right)$, $\quad \dfrac{dy}{dx} = \dfrac{24}{-3^3} = -\dfrac{8}{9}$

Equation of tangent at D, $\quad y - \dfrac{5}{3} = -\dfrac{8}{9}(x+6)$

$$y = -\frac{8x}{9} - \frac{11}{3}$$

6. (i) Express $y=\dfrac{5x+10}{(x-2)(x+3)}$ in partial fractions.

(ii) Hence, find the gradient of the normal to the curve $y=\dfrac{5x+10}{(x-2)(x+3)}$ at the point $\left(3, \dfrac{25}{6}\right)$.

Solution

(i) Let $\dfrac{5x+10}{(x-2)(x+3)}=\dfrac{A}{x-2}+\dfrac{B}{x+3}$

$5x+10=A(x+3)+B(x-2)$

Let $x=2$, $\quad 20=5A$

$A=4$

Let $x=-3$, $\quad -5=-5B$

$B=1$

$\therefore \dfrac{5x+10}{(x-2)(x+3)}=\dfrac{4}{x-2}+\dfrac{1}{x+3}$

(ii) $\dfrac{dy}{dx}=-\dfrac{4}{(x-2)^2}-\dfrac{1}{(x+3)^2}$

At $\left(3, \dfrac{25}{6}\right)$, $\quad \dfrac{dy}{dx}=-4-\dfrac{1}{36}$

$=-\dfrac{145}{36}$

Gradient of the normal $=-\dfrac{36}{145}$

---

7. The equation of a curve is $y=\sin 2x+2\cos 2x$. Find the x-coordinate, where $\dfrac{\pi}{2}<x<\pi$, of the point at which the tangent to the curve is parallel to the x-axis.

$\dfrac{dy}{dx}=2\cos 2x-4\sin 2x$

When $x=0$, $\dfrac{dy}{dx}=0$, $\quad 2\cos 2x-4\sin 2x=0$

$4\sin 2x=2\cos 2x$

$\tan 2x=\dfrac{1}{2}$

Basic $\angle=0.46364$

$2x=\pi+0.46364$

$x=1.80$

8. The equation of a curve is $y = \frac{e^{2x}}{2} - 2e^{x}$. Find the x-coordinate of the point on the curve where the gradient of the tangent is 15.

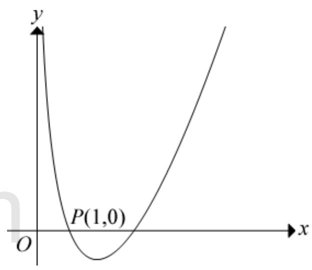
$$\frac{dy}{dx} = e^{2x} - 2e^{x}$$

When $\frac{dy}{dx} = 15$,

$$e^{2x} - 2e^{x} = 15$$
$$e^{2x} - 2e^{x} - 15 = 0$$
$$(e^{x} - 5)(e^{x} + 3) = 0$$
$$e^{x} = 5,\ -3\ (\text{NA})$$
$$x = \ln 5$$
$$= 1.61$$

---

9. The diagram shows part of the curve $y = (2x + c)\ln x$, crossing the x-axis at $P(1, 0)$. The tangent to the curve at P is parallel to the line $2y + 8x - 7 = 0$. Find the value of c.



$$2y + 8x - 7 = 0 \Rightarrow y = -4x + \frac{7}{2}$$

$$\frac{dy}{dx} = 2\ln x + \frac{2x + c}{x}$$

At $P(1, 0)$, $\frac{dy}{dx} = -4$, $\quad 2 + c = -4$

$$c = -6$$

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 16

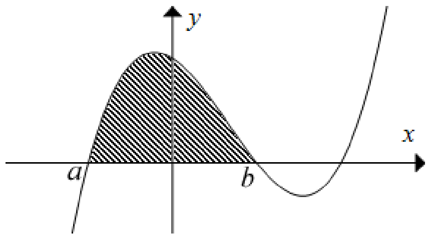
Name: ________________________________ (____)   Date: ________________

Class: Sec 4______

Revision 16: Application of Integration

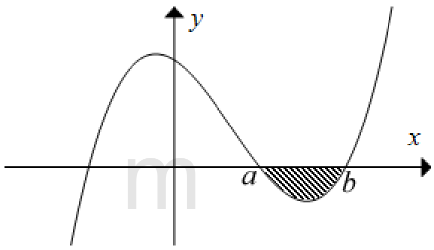
1. The area of the region bounded by the curve $y = \mathrm{f}(x)$, from $x = a$ to $x = b$ is given by

$$\int_a^b y \, \mathrm{d}x$$



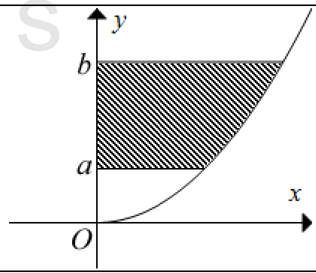
2. The area of the region bounded by the curve $y = \mathrm{f}(x)$, from $x = b$ to $x = c$ is given by

$$\left| \int_a^b y \, \mathrm{d}x \right| \text{ or } -\int_a^b y \, \mathrm{d}x$$



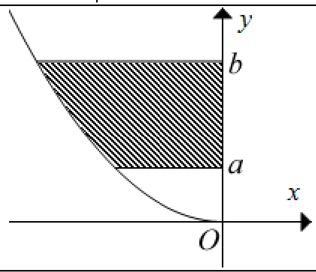
3. The area of the region bounded by the curve $y = \mathrm{f}(x)$, from $y = a$ to $y = b$ is given by

$$\int_a^b x \, \mathrm{d}y$$



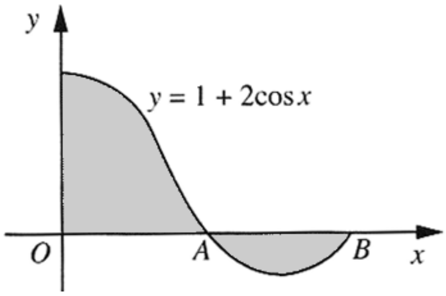
4. The area of the region bounded by the curve $y = \mathrm{f}(x)$, from $x = b$ to $x = c$ is given by

$$\left| \int_a^b x \, \mathrm{d}y \right| \text{ or } -\int_a^b x \, \mathrm{d}y$$



Example

1. The diagram shows part of the curve $y = 1 + 2\cos x$, meeting the x-axis at the points A and B.
   (i) Show that the x-coordinate of A is $\frac{2\pi}{3}$ and find the x-coordinate of B.
   (ii) Find the total area of the shaded regions.



Solution

(i) At A, $1 + 2\cos x = 0$

$$\cos x = -\frac{1}{2}$$

$$\text{Basic } \angle = \frac{\pi}{3}$$

$$x = \frac{2\pi}{3}, \ \frac{4\pi}{3}$$

x-coordinates of $A = \frac{2\pi}{3}$

x-coordinates of $B = \frac{4\pi}{3}$

(ii) Area of shaded region

$$= \int_0^{\frac{2\pi}{3}} (1 + 2\cos x)\mathrm{d}x + \left| \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 2\cos x)\mathrm{d}x \right|$$

$$= \left[x + 2\sin x\right]_0^{\frac{2\pi}{3}} + \left| \left[x + 2\sin x\right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \right|$$

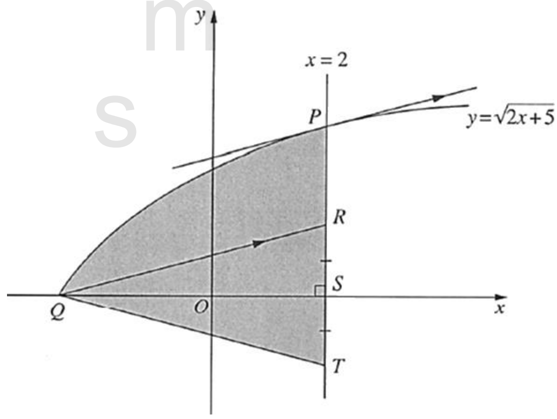
$$= \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} + \left| \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} - \left(\frac{2\pi}{3} + 2\sin\frac{2\pi}{3}\right) \right|$$

$$= 3.8264 + 1.3697$$

$$= 5.20 \text{ unit}^2$$

---

2. The diagram shows part of the curve $y = \sqrt{2x+5}$ passing through the point P and meeting the x-axis at the point Q. The line $x = 2$ passes through P and intersects the x-axis at the point S. Lines from Q meet $x = 2$ at the points R and T such that QR is parallel to the tangent to the curve at P, and $RS = ST$. Find
   (i) the equation of QR,
   (ii) the area of the shaded region.



Solution

(i) At Q, $y = 0$, $\sqrt{2x+5} = 0$

$$\therefore Q\left(-\frac{5}{2}, \ 0\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(2x+5)^{-\frac{1}{2}}(2)$$

$$= \frac{1}{\sqrt{2x+5}}$$

At P, $x = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{2(2)+5}}$

$$= \frac{1}{3}$$

Equation of QR, $y = \frac{1}{3}\left(x + \frac{5}{2}\right)$

$$y = \frac{x}{3} + \frac{5}{6}$$

(ii) At R, $x = 2$, $y = \frac{2}{3} + \frac{5}{6} = \frac{3}{2}$

Area of shaded region

$$= \int_{-2.5}^{2} \sqrt{2x+5}\ \mathrm{d}x + \frac{1}{2} \times \left(2 + \frac{5}{2}\right) \times \frac{3}{2}$$

$$= \left[\frac{(2x+5)^{\frac{3}{2}}}{\frac{3}{2} \times 2}\right]_{-2.5}^{2} + 3\frac{3}{8}$$

$$= \frac{9^{\frac{3}{2}}}{3} + 3\frac{3}{8}$$

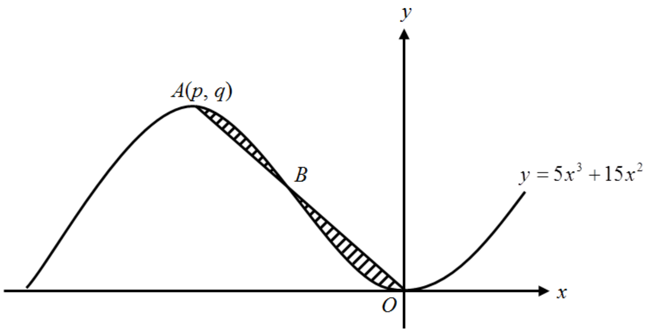
$$= 12\frac{3}{8} \text{ unit}^2$$

Exercise

1. The diagram below shows part of the curve $y = 5x^3 + 15x^2$. The line ABO cuts the curve at the maximum point of the curve $A(p, q)$ at B where $x = -1$, and at the origin.

(a) Find
(i) the coordinates of the maximum point $A(p, q)$,
(ii) the area of the shaded region.

(b) A point (x, y) such that $p < x < 0$ moves along the curve $y = 5x^3 + 15x^2$. Find the value(s) of y when the rate of decrease of y is 10 times the rate of increase of x.



Solution

(a) (i) $\dfrac{dy}{dx} = 15x^2 + 30x$

At A, $x > 0$, $\dfrac{dy}{dx} = 0$, $\quad 15x^2 + 30x = 0$

$$15x(x+2) = 0$$
$$x = 0, -2$$
$$y = 0, 20$$

$\therefore A(-2, 20)$.

(ii) At B, $\quad y = 10$

$\therefore B(-1, 10)$.

Area of shaded region

$$= \int_{-2}^{-1} (5x^3 + 15x^2)dx - \frac{1}{2} \times (20+10) \times 1 + \frac{1}{2} \times 10 \times 1 - \int_{-1}^{0} (5x^3 + 15x^2)dx$$

$$= \left[\frac{5x^4}{4} + 5x^3\right]_{-2}^{-1} - 15 + 5 - \left[\frac{5x^4}{4} + 5x^3\right]_{-1}^{0}$$

$$= \frac{5}{4} - 5 - (20 - 40) - 10 + \left(\frac{5}{4} - 5\right)$$

$$= \frac{5}{2} \text{ unit}^2$$

(b) $\dfrac{dy}{dt} = -10\dfrac{dx}{dt} \Rightarrow \dfrac{dy}{dx} = -10$

When $\dfrac{dy}{dx} = -10$, $\quad 15x^2 + 30x = -10$

$$3x^2 + 6x - 2 = 0$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(2)}}{6}$$

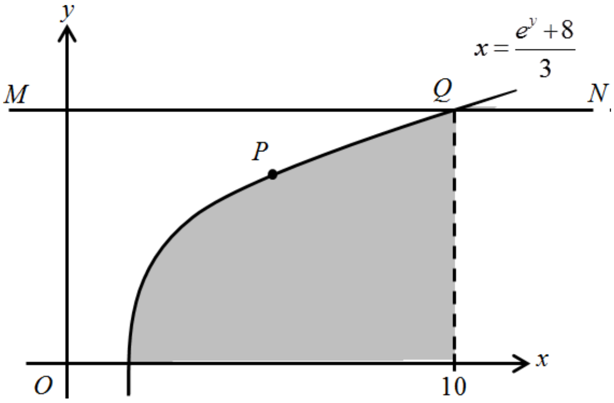
$$= \frac{-3 \pm \sqrt{3}}{3}$$

$$y = 2.30, 17.7$$

2. The diagram shows part of the curve $x = \dfrac{e^y + 8}{3}$ intersecting the horizontal line MN at point Q with x-coordinate = 10. The point P lies on the curve and the tangent at P is parallel to the line $5y = x + 6$.

(i) Find the x-coordinate of P.

(ii) Find the area of the shaded region.



Solution

(i) $x = \dfrac{e^y + 8}{3}$

$3x = e^y + 8$

$e^y = 3x - 8$

$y = \ln(3x - 8)$

$\dfrac{dy}{dx} = \dfrac{3}{3x - 8}$

At P, $\dfrac{dy}{dx} = \dfrac{1}{5}$, $\dfrac{3}{3x-8} = \dfrac{1}{5}$

$3x - 8 = 15$

$x = \dfrac{23}{3}$

(ii) At Q, $x = 10$, $y = \ln 22$

Area of shaded region

$= 10 \ln 22 - \displaystyle\int_0^{\ln 22} \left(\frac{e^y + 8}{3} \right) dy$

$= 10 \ln 22 - \left[\dfrac{e^y + 8y}{3} \right]_0^{\ln 22}$

$= 10 \ln 22 - \left[\dfrac{e^{\ln 22} + 8 \ln 22}{3} - \dfrac{1}{3} \right]$

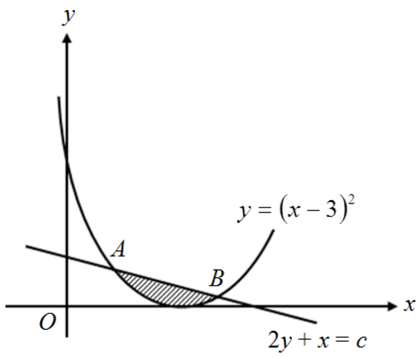
$= 17.7 \text{ unit}^2$

3. The diagram shows part of the curve $y=(x-3)^2$ intersecting the line $2y+x=c$, where c is a constant, at points A and B. The tangent at B is perpendicular to the line $2y+x=c$.

(i) Find the x-coordinate of B.

(ii) Hence show that $c = 6$.

(iii) Using your answer to part (ii), find the area of the shaded region.



Solution

(i) $\dfrac{dy}{dx}=2(x-3)$

At B, $\dfrac{dy}{dx}=2$, $\quad 2(x-3)=2$

$$2x=8$$
$$x=4$$

(ii) At B, $x=4$, $\quad y=(4-3)^2=1$

$$c=2+4$$
$$=6$$

(iii) At A, $\quad 2(x-3)^2+x=6$

$$2x^2-12x+18+x-6=0$$
$$2x^2-11x+12=0$$
$$(2x-3)(x-4)=0$$
$$x=\frac{3}{2},\ 4$$
$$y=\frac{9}{4},\ 1$$

Area of shaded region

$$=\frac{1}{2}\times\left(\frac{9}{4}+1\right)\times\left(4-\frac{3}{2}\right)-\int_{\frac{3}{2}}^{4}(x-3)^2\,dx$$

$$=\frac{65}{16}-\left[\frac{1}{3}(x-3)^3\right]_{\frac{3}{2}}^{4}$$

$$=\frac{65}{16}-\frac{1}{3}\left[1-\left(-\frac{3}{2}\right)^3\right]$$

$$=2\frac{29}{48}\text{ or } 2.60\text{ unit}^2$$

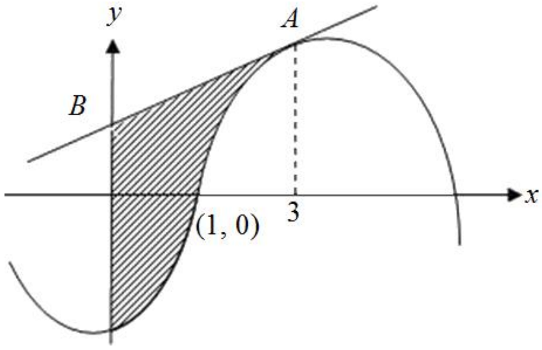
4. The diagram shows part of the curve $y = -x^3 + 5x^2 + k$ which cuts the x-axis at (1, 0). The tangent to the curve at A where $x = 3$ cuts the y-axis at B.

Find

(i) the value of k,

(ii) the coordinates of B,

(iii) the area of the shaded region.



Solution

(i) At (1, 0), $\quad 0 = -1 + 5 + k \quad 2(x-3) = 2$

$\qquad k = -4$

(ii) $y = -x^3 + 5x^2 - 4$

$$\frac{dy}{dx} = -3x^2 + 10x$$

At A, $x = 3$, $\quad \dfrac{dy}{dx} = -3(3)^2 + 10(3)$

$\qquad = 3$

$\qquad y = -3^3 + 5(3)^2 - 4$

$\qquad = 14$

Equation of AB, $\quad y - 14 = 3(x - 3)$

$\qquad y = 3x + 5$

$\therefore B(0, 5).$

(iii) Area of shaded region

$$= \frac{1}{2} \times (5 + 14) \times 3 - \int_0^1 (-x^3 + 5x^2 - 4)dx - \int_1^3 (-x^3 + 5x^2 - 4)dx$$

$$= \frac{57}{2} - \int_0^3 (-x^3 + 5x^2 - 4)dx$$

$$= \frac{57}{2} - \left[-\frac{x^4}{4} + \frac{5x^3}{3} - 4x \right]_0^3$$

$$= \frac{57}{2} - \left(-\frac{81}{4} + 45 - 12 \right)$$

$$= \frac{63}{4} \text{ or } 15.75 \text{ unit}^2$$

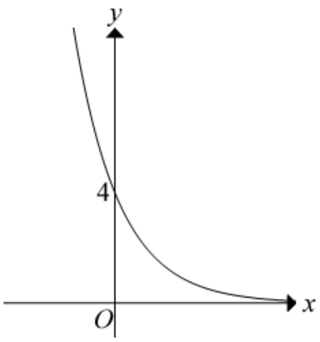
5. (i) Sketch the curve $y=\dfrac{4}{e^{2x}}$.

(ii) Find the equation of the tangent to the curve at point $x = 0$.

(iii) Hence, find the exact area of the region bounded by the curve, $y=\dfrac{4}{e^{2x}}$, the tangent at $x=0$, the line $x = 1$ and the x-axis.

Solution

(i)



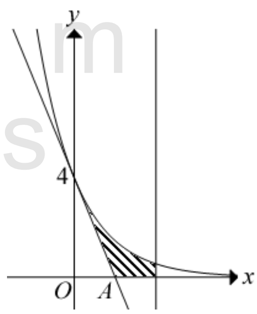
(ii) $\dfrac{dy}{dx}=-8e^{-2x}$

When $x=0$, $\dfrac{dy}{dx}=-8$

Equation of tangent, $y=-8x+4$

(iii) At A, $y=0$, $-8x+4=0$

$x=\dfrac{1}{2}$



Area of bounded region

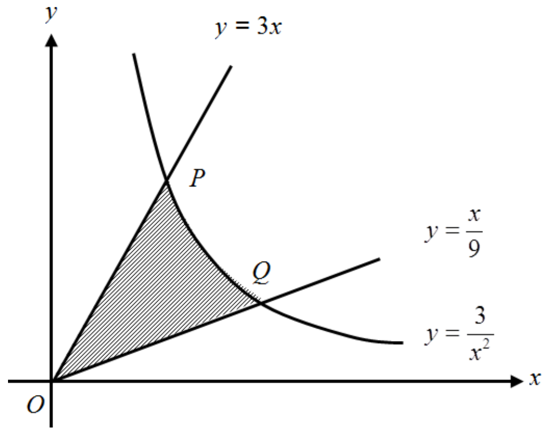
$=\int_0^1 4e^{-2x}dx-\dfrac{1}{2}\times\dfrac{1}{2}\times 4$

$=\left[-2e^{-2x}\right]_0^1-1$

$=-2e^{-1}+2-1$

$=\left(1-\dfrac{2}{e^2}\right)\text{unit}^2$

6. The diagram shows part of the curve $y = \frac{3}{x^2}$. The line $y = 3x$ intersects the curve at P and the line $y = \frac{x}{9}$ intersects at Q. Find



(i) the coordinates of P and Q.
(ii) the area of the shaded region.

Solution

(i) At P,

$$3x = \frac{3}{x^2}$$
$$x^3 = 1$$
$$x = 1$$
$$y = 3$$

$\therefore P(1, 3)$.

At Q,

$$\frac{x}{9} = \frac{3}{x^2}$$
$$x^3 = 27$$
$$x = 3$$
$$y = \frac{1}{3}$$

$\therefore B\left(3, \frac{1}{3}\right)$.

(ii) Area of bounded region

$$= \frac{1}{2} \times 1 \times 3 + \int_1^3 3x^{-2}\,dx - \frac{1}{2} \times 3 \times \frac{1}{3}$$
$$= \frac{3}{2} + \left[-\frac{3}{x}\right]_1^3 - \frac{1}{2}$$
$$= 1 + (-1 + 3)$$
$$= 3 \text{ unit}^2$$

7. (a) The figure shows part of the curve $y^2 = x + 9$ and part of the line $2y = 6 - x$.
Find
(i) the coordinates of *A*, *B* and *C*,
(ii) the area of the shaded region.

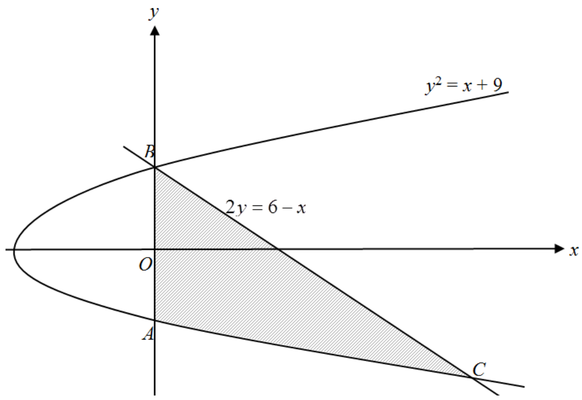
(b) Sketch the graph of $y = \dfrac{10}{x}$ and hence briefly explain why $\dfrac{15}{2} < \displaystyle\int_2^5 \dfrac{10}{x}\,\mathrm{d}x < 15$.

Solution

(a) (i) At *A* and *B*, $y^2 = 9$

$y = \pm 3$

$\therefore A(0, -3)$ and $B(0, 3)$.



At *C*, $\left(\dfrac{6-x}{2}\right)^2 = x + 9$

$36 - 12x + x^2 = 4x + 36$

$x^2 - 16x = 0$

$x(x - 16) = 0$

since $x > 0$, $x = 16$

$y = -5$

$\therefore C(16, -5)$

(ii) Area of bounded region $= \dfrac{1}{2} \times 6 \times 3 - \displaystyle\int_0^{16} -(x+9)^{\frac{1}{2}}\,\mathrm{d}x - \dfrac{1}{2} \times 10 \times 5$

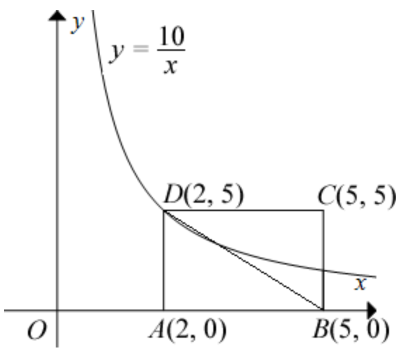
$= 9 + \displaystyle\int_0^{16} (x+9)^{\frac{1}{2}}\,\mathrm{d}x - 25$

$= \left[\dfrac{2}{3}(x+9)^{\frac{3}{2}}\right]_0^{16} - 16$

$= \dfrac{2}{3}(125 - 27) - 16$

$= 49\dfrac{1}{3}$ unit ^{2}

(b)



Area of triangle *ABD* < Area of region *ABCEA* < Area of rectangle *ABCD*

$\dfrac{1}{2} \times 3 \times 5 < \displaystyle\int_2^5 \dfrac{10}{x}\,\mathrm{d}x < 3 \times 5$

$\therefore \dfrac{15}{2} < \displaystyle\int_2^5 \dfrac{10}{x}\,\mathrm{d}x < 15$

8. (a) Given that $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x = 4$ and $\int_0^4 \mathrm{f}(x)\,\mathrm{d}x = 11$, evaluate $\int_2^4 [\mathrm{f}(x) - 3\tan^2 x]\,\mathrm{d}x$.

Solution

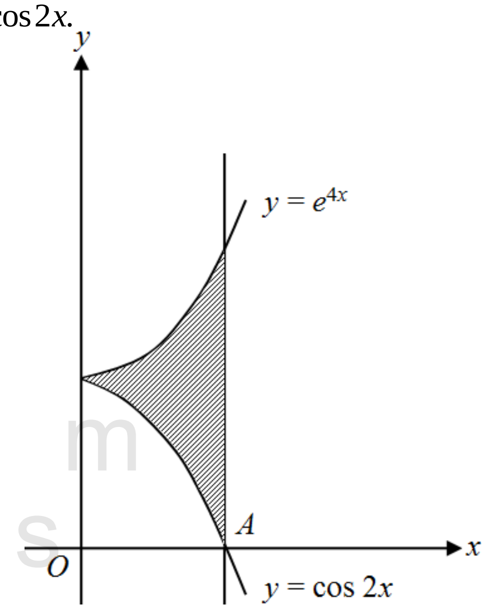
$$\int_2^4 [\mathrm{f}(x) - 3\tan^2 x]\,\mathrm{d}x = \int_0^4 \mathrm{f}(x)\,\mathrm{d}x - \int_0^2 \mathrm{f}(x)\,\mathrm{d}x - \int_2^4 3(\sec^2 x - 1)\,\mathrm{d}x$$

$$= 11 - 4 - 3[\tan x - x]_2^4$$

$$= 7 - 3(\tan 4 - 4 - \tan 2 + 2)$$

$$= 2.97$$

(b) The diagram shows part of the graphs of $y = e^{4x}$ and $y = \cos 2x$.



Find
(i) the coordinates of A,
(ii) the area of the shaded region.

Solution

(i) At A, $\cos 2x = 0$

$$2x = \frac{\pi}{2}$$

$$x = \frac{\pi}{4}$$

$$\therefore A\left(\frac{\pi}{4}, 0\right)$$

(ii) Area of the shaded region

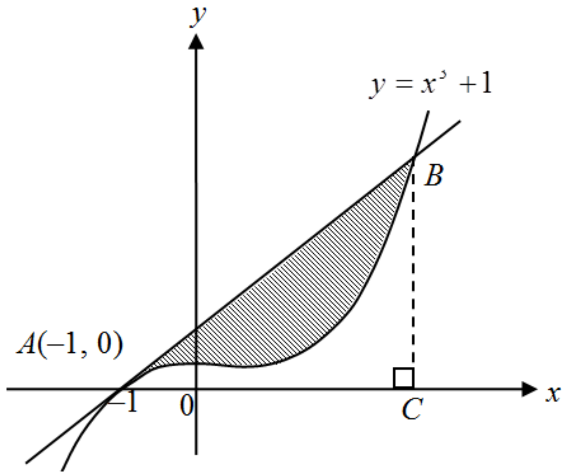
$$= \int_0^{\frac{\pi}{4}} e^{4x}\,\mathrm{d}x - \int_0^{\frac{\pi}{4}} \cos 2x\,\mathrm{d}x$$

$$= \left[\frac{e^{4x}}{4} - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}e^{\pi} - \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{4}$$

$$= 5.04 \text{ unit}^2$$

9. The diagram shows part of the curve $y = x^3 + 1$. The tangent at $A(-1, 0)$ meets the curve again at B.



Find
(i) the equation of AB,
(ii) the coordinates of B,
(iii) the area of the triangle ABC.
(iv) the area of the shaded region in the figure.

Solution

(i) $\dfrac{dy}{dx} = 3x^2$

At A, $\dfrac{dy}{dx} = 3$

Equation of AB, $y = 3(x+1)$

$y = 3x + 3$

(ii) At B, $x^3 + 1 = 3x + 3$

$x^3 - 3x - 2 = 0$

Let $f(x) = x^3 - 3x - 2$

$f(2) = 8 - 6 - 2 = 0$

$x - 2$ is a factor of $f(x)$.

Let $f(x) = (x-2)(x^2 + bx + 1)$

Comparing coefficient of x, $-2b + 1 = -3$

$2b = 4$

$b = 2$

$f(x) = (x-2)(x^2 + 2x + 1)$

$= (x-2)(x+1)^2$

$f(x) = 0 \quad \Rightarrow x = 2, -1$

$\Rightarrow y = 9, 1$

$\therefore B(2, 9)$

(iii) Area of triangle ABC

$= \dfrac{1}{2} \times 3 \times 9$

$= 13.5 \text{ unit}^2$

(iv) Area of shaded region

$= 13.5 - \int_{-1}^{2} (x^3 + 1)\, dx$

$= 13.5 - \left[\dfrac{x^4}{4} + x \right]_{-1}^{2}$

$= 13.5 - \left[4 + 2 - \left(\dfrac{1}{4} - 1 \right) \right]$

$= 6.75 \text{ unit}^2$

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 15

Name: ________________________________ (____)   Date: ________________

Class: Sec 4______

Revision 15: Integration

1. Integration of algebraic functions

If $m \neq -1$, $n \neq -1$, a, b, m and n are constants, then

(a) $\int ax^n \mathrm{d}x = \dfrac{ax^{n+1}}{n+1} + C$,

(b) $\int (ax^m + bx^n)\mathrm{d}x = \dfrac{ax^{m+1}}{m+1} + \dfrac{bx^{n+1}}{n+1} + C$,

(c) $\int (ax+b)^n \mathrm{d}x = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$ where C is an arbitrary constant,

(d) $\int [\mathrm{f}(x) \pm \mathrm{g}(x)]\mathrm{d}x = \int \mathrm{f}(x)\mathrm{d}x \pm \int \mathrm{g}(x)\mathrm{d}x$.

Note

An indefinite integral represents a family of **identical** curves with different y-intercepts.

2. Integration of Trigonometric Functions

(a) $\int \sin x \mathrm{d}x = -\cos x + C$,

(b) $\int \cos x \mathrm{d}x = \sin x + C$,

(c) $\int \sec^2 x \mathrm{d}x = \tan x + C$,

(d) $\int \sin(ax+b)\mathrm{d}x = -\dfrac{1}{a}\cos(ax+b) + C$,

(e) $\int \cos(ax+b)\mathrm{d}x = \dfrac{1}{a}\sin(ax+b) + C$

(f) $\int \sec^2(ax+b)\mathrm{d}x = \dfrac{1}{a}\tan(ax+b) + C$

where C is an arbitrary constant.

3. Integral of Exponential and Reciprocal Functions

(a) $\int e^x \mathrm{d}x = e^x + C$,

(b) $\int e^{ax+b}\mathrm{d}x = \dfrac{1}{a}e^{ax+b} + C$,

(c) $\int \dfrac{1}{x}\mathrm{d}x = \ln x + C$,

(d) $\int \dfrac{1}{(ax+b)}\mathrm{d}x = \dfrac{1}{a}\ln(ax+b) + C$,

(e) $\int \dfrac{1}{(ax+b)^n}\mathrm{d}x = \dfrac{(ax+b)^{-n+1}}{a(-n+1)} + C$

where C is an arbitrary constant.

4. Integral – the Reverse Process of Differentiation

If $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x) = \mathrm{f}(x)$, then $\int \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(x) + C$, where C is an arbitrary constant.

Example

1. (i) Differentiate $x^3 \ln x$ with respect to x.
   (ii) Hence find $\int 5x^2 \ln x \, \mathrm{d}x$.

Solution

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^3 \ln x) \;= 3x^2 \ln x + x^3\left(\dfrac{1}{x}\right)$

$\qquad = 3x^2 \ln x + x^2$

(ii) $\int (3x^2 \ln x + x^2)\,\mathrm{d}x = x^3 \ln x + C_1$

$\int 3x^2 \ln x \,\mathrm{d}x = x^3 \ln x - \int x^2 \,\mathrm{d}x + C_1$

$\qquad = x^3 \ln x - \dfrac{x^3}{3} + C_1$

$\int x^2 \ln x \,\mathrm{d}x = \dfrac{x^3}{3}\ln x - \dfrac{x^3}{9} + \dfrac{C_1}{3}$

$\int 5x^2 \ln x \,\mathrm{d}x = \dfrac{5x^3}{3}\ln x - \dfrac{5x^3}{9} + C$

- Apply the concept integration is the reverse process of differentiation. i.e. if $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x) = \mathrm{f}(x)$, then $\int \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(x) + C$.
- Apply the unitary method, i.e. find the value of a single unit ($\int x^2 \ln x \,\mathrm{d}x$) and then derive the value of the required multiple ($\int 5x^2 \ln x \,\mathrm{d}x$).

2. (i) Prove the identity $\sin^2\theta\cos^2\theta = \dfrac{1}{8}(1-\cos 4\theta)$.
   (ii) Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sin^2\theta\cos^2\theta \,\mathrm{d}\theta$.

Solution

(i) LHS $= \sin^2\theta\cos^2\theta$

$= \dfrac{1}{4}(2\sin\theta\cos\theta)^2$

$= \dfrac{1}{4}\sin^2 2\theta$

$= \dfrac{1}{4}\times\dfrac{1}{2}(1-\cos 4\theta)$

$= \dfrac{1}{8}(1-\cos 4\theta)$

$=$ RHS

- Apply $\sin 2\theta = 2\sin\theta\cos\theta$.
- Apply $\cos 2\theta = 1 - 2\sin^2\theta$.

(ii) $\int_0^{\frac{\pi}{3}} \sin^2\theta\cos^2\theta \,\mathrm{d}\theta \;= \int_0^{\frac{\pi}{3}} \dfrac{1}{8}(1-\cos 4\theta)\,\mathrm{d}\theta$

$= \left[\dfrac{1}{8}(\theta - \dfrac{1}{4}\sin 4\theta)\right]_0^{\frac{\pi}{3}}$

$= \dfrac{1}{8}\left(\dfrac{\pi}{3} - \dfrac{1}{4}\sin\dfrac{4\pi}{3}\right)$

$= 0.158$

3. (i) Express $\dfrac{8x^2+3x+1}{x(2x+1)^2}$ in partial fractions.

(ii) Hence find $\displaystyle\int \frac{8x^2+3x+1}{x(2x+1)^2}\,\mathrm{d}x$.

Solution

(i) Let $\dfrac{8x^2+3x+1}{x(2x+1)^2} = \dfrac{A}{x} + \dfrac{B}{2x+1} + \dfrac{C}{(2x+1)^2}$

$\therefore 8x^2+3x+1 = A(2x+1)^2 + Bx(2x+1) + Cx$

Let $x=0$, $\quad A=1$

Let $x=-\dfrac{1}{2}$, $\quad 8\left(-\dfrac{1}{2}\right)^2 + 3\left(-\dfrac{1}{2}\right) + 1 = \left(-\dfrac{1}{2}\right)C$

$\dfrac{3}{2} = -\dfrac{C}{2}$

$C=-3$

Comparing coefficient of x^2, $\quad 4A+2B=8$

$B=2$

$\therefore \dfrac{8x^2+3x+1}{x(2x+1)^2} = \dfrac{1}{x} + \dfrac{2}{2x+1} - \dfrac{3}{(2x+1)^2}$

(ii) $\displaystyle\int \frac{8x^2+3x+1}{x(2x+1)^2}\,\mathrm{d}x$

$\displaystyle = \int \left[\frac{1}{x} + \frac{2}{2x+1} - \frac{3}{(2x+1)^2}\right]\mathrm{d}x$

$\displaystyle = \ln x + \frac{2\ln(2x+1)}{2} - \frac{3(2x+1)^{-1}}{(-1)(2)} + C$

$\displaystyle = \ln x + \ln(2x+1) + \frac{3}{2(2x+1)} + C$

- There is a **<u>repeated factor</u>** in the denominator of one on the terms; **<u>its integral is not a ln function</u>**.
- $\displaystyle\int \frac{3}{(2x+1)^2}\,\mathrm{d}x = \frac{3(2x+1)^{-2+1}}{-2+1} + C$

Exercise

1. Integrate each of the following.

(a) $\displaystyle\int\left(2x^4+x-\frac{3}{x^2}\right)\mathrm{d}x=\frac{2x^5}{5}+\frac{x^2}{2}+\frac{3}{x}+C$

(b) $$\begin{aligned}\int(2x-1)^2\,\mathrm{d}x&=\frac{(2x-1)^3}{3(2)}+C\\&=\frac{(2x-1)^3}{6}+C\end{aligned}$$

(c) $\displaystyle\int\left(x+\frac{2}{x}\right)\mathrm{d}x=\frac{x^2}{2}+2\ln x+C$

(d) $$\begin{aligned}\int\sqrt{4x-2}\,\mathrm{d}x&=\frac{(4x-2)^{\frac{3}{2}}}{\frac{3}{2}\times 4}+C\\&=\frac{(4x-2)^{\frac{3}{2}}}{6}+C\end{aligned}$$

(e) $$\begin{aligned}\int\frac{3}{2\sqrt{2x-3}}\,\mathrm{d}x&=\frac{\frac{3}{2}(2x-3)^{\frac{1}{2}}}{\frac{1}{2}\times 2}+C\\&=\frac{3}{2}(2x-3)^{\frac{1}{2}}+C\end{aligned}$$

(f) $$\begin{aligned}\int\left[\frac{1}{(x-1)}-\frac{3}{(x-1)^2}\right]\mathrm{d}x&=\ln(x-1)-\frac{3(x-1)^{-1}}{-1}+C\\&=\ln(x-1)+\frac{3}{x-1}+C\end{aligned}$$

(g) $\displaystyle\int(3\sin 2x+4\cos x)\,\mathrm{d}x=-\frac{3\cos 2x}{2}+4\sin x+C$

(h) $\displaystyle\int 6\sin\left(3x+\frac{\pi}{4}\right)\mathrm{d}x=-2\cos\left(3x+\frac{\pi}{4}\right)+C$

(i) $\displaystyle\int\sec^2(4x+5)\mathrm{d}x=\frac{1}{4}\tan(4x+5)+C$

(j) $$\begin{aligned}\int\tan^2 x\,\mathrm{d}x&=\int(\sec^2 x-1)\mathrm{d}x\\&=\tan x-x+C\end{aligned}$$

(k) $\int \sin^2 x \, \mathrm{d}x \; = \int \frac{1}{2}(1-\cos 2x)\,\mathrm{d}x$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(l) $\int \cos^2 x \, \mathrm{d}x \; = \int \frac{1}{2}(\cos 2x + 1)\,\mathrm{d}x$

$$= \frac{1}{2}\left(\frac{\sin 2x}{2} + x\right) + C$$

$$= \frac{\sin 2x}{4} + \frac{x}{2} + C$$

(m) $\int (e^x + e^{-x})\,\mathrm{d}x = e^x - e^{-x} + C$

(n) $\int (e^{\frac{x}{2}} + 3e^{-\frac{x}{2}})\,\mathrm{d}x = \frac{e^{\frac{x}{2}}}{\frac{1}{2}} + \frac{3e^{-\frac{x}{2}}}{-\frac{1}{2}} + C$

$$= 2e^{\frac{x}{2}} - 6^{-\frac{x}{2}} + C$$

---

2. The gradient of a curve, at any point (x, y) on it, is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{5-2x}$. Given that the curve passes through the point $A(2, 7)$, find the equation of the curve.

Solution

$$y = \int \frac{1}{5-2x}\mathrm{d}x$$

$$= \frac{\ln(5-2x)}{-2} + C$$

$$= -\frac{1}{2}\ln(5-2x) + C$$

At $A(2, 7)$, $\quad C + \frac{1}{2}\ln 1 = 7$

$$C = 7$$

$$\therefore y = -\frac{1}{2}\ln(5-2x) + 7$$

---

3. The gradient function of the normal to a curve is given as $\dfrac{x^2}{9-2x^3}$.

If the curve passes through the point (3, 5), find the equation of the curve.

Solution

$$\frac{dy}{dx} = -\frac{9-2x^3}{x^2}$$

$$= -9x^{-2} + 2x$$

$$y = \int (2x - 9x^{-2})\,dx$$

$$= x^2 + 9x^{-1} + C$$

$$= x^2 + \frac{9}{x} + C$$

At $A(3, 5)$, $\quad 5 = \dfrac{9}{3} + 9 + C$

$$C = -7$$

$$\therefore y = x^2 + \frac{9}{x} - 7$$

---

4. (i) Differentiate $\ln(\sin x)$ with respect to x.

(ii) Using your result from part (i), evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cot x\,dx$.

Solution

(i) $$\frac{d}{dx}[\ln(\sin x)] = \frac{\cos x}{\sin x}$$

$$= \cot x$$

(ii) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cot x\,dx$$

$$= \left[2\ln(\sin x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\ln\left(\sin\frac{\pi}{2}\right) - 2\ln\left(\sin\frac{\pi}{6}\right)$$

$$= 1.39$$

---

5. It is given that $y = (x+1)(2x-3)^{\frac{3}{2}}$.

(i) Show that $\dfrac{dy}{dx}$ can be written in the form $kx\sqrt{2x-3}$ and state the value of k.

Hence

(ii) evaluate $\displaystyle\int_2^6 x\sqrt{2x-3}\,dx$.

Solution

(i) $$\frac{dy}{dx} = (2x-3)^{\frac{3}{2}} + \frac{3}{2}(x+1)(2x-3)^{\frac{1}{2}}(2)$$

$$= (2x-3)^{\frac{1}{2}}\left[(2x-3) + 3(x+1)\right]$$

$$= 5x\sqrt{2x-3}$$

$$\therefore k = 5$$

(ii) $$\int_2^6 x\sqrt{2x-3}\,dx = \left[\frac{1}{5}(x+1)(2x-3)^{\frac{3}{2}}\right]_2^6$$

$$= \frac{1}{5}(7)(9)^{\frac{3}{2}} - \frac{1}{5}(3)(1)^{\frac{3}{2}}$$

$$= 37.2 \text{ or } \frac{186}{5}$$

6. (i) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\cos x}{1-\sin x}\right)$ can be written in the form $\dfrac{k}{1-\sin x}$ and state the value of k.

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}}\left(\frac{\sqrt{2}}{1-\sin x}\right)\mathrm{d}x$.

Solution

(i)
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin x(1-\sin x)-\cos x(-\cos x)}{(1-\sin x)^2}$$
$$=\frac{-\sin x+\sin^2 x+\cos^2 x}{(1-\sin x)^2}$$
$$=\frac{1-\sin x}{(1-\sin x)^2}$$
$$=\frac{1}{1-\sin x}$$
$$\therefore k=1$$

(ii)
$$\int_0^{\frac{\pi}{4}}\left(\frac{\sqrt{2}}{1-\sin x}\right)\mathrm{d}x$$
$$=\sqrt{2}\left[\frac{\cos x}{1-\sin x}\right]_0^{\frac{\pi}{4}}$$
$$=\sqrt{2}\left(\frac{\cos\frac{\pi}{4}}{1-\sin\frac{\pi}{4}}-\frac{\cos 0}{1-\sin 0}\right)$$
$$=2$$

---

7. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{9-4x^2}\right)$ and hence evaluate $\displaystyle\int_0^1\frac{x}{(9-4x^2)^2}\,\mathrm{d}x$.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{9-4x^2}\right)=-(9-4x^2)^{-2}(8x)$$
$$=-\frac{8x}{(9-4x^2)^2}$$

$$\int_0^1\frac{x}{(9-4x^2)^2}\,\mathrm{d}x=\left[\frac{1}{8(9-4x^2)}\right]_0^1$$
$$=\frac{1}{8(5)}-\frac{1}{72}$$
$$=\frac{1}{90}$$

---

8. (i) Differentiate $(e^{2x}+1)\ln(e^{2x}+1)$ with respect to x.

(ii) Hence, evaluate $\displaystyle\int_0^1 e^{2x}\ln(e^{2x}+1)\mathrm{d}x$.

Solution

(i)
$$\frac{\mathrm{d}}{\mathrm{d}x}[(e^{2x}+1)\ln(e^{2x}+1)]=2e^{2x}\ln(e^{2x}+1)+\frac{(e^{2x}+1)(2e^{2x})}{e^{2x}+1}$$
$$=2e^{2x}+2e^{2x}\ln(e^{2x}+1)$$

(ii)
$$\int_0^1[2e^{2x}+2e^{2x}\ln(e^{2x}+1)]\mathrm{d}x=\left[(e^{2x}+1)\ln(e^{2x}+1)\right]_0^1$$
$$\left[e^{2x}\right]_0^1+\int_0^1 2e^{2x}\ln(e^{2x}+1)]\mathrm{d}x=(e^2+1)\ln(e^2+1)-2\ln 2$$
$$\int_0^1 2e^{2x}\ln(e^{2x}+1)]\mathrm{d}x=(e^2+1)\ln(e^2+1)-2\ln 2-(e^2-1)$$

$$\int_0^1 e^{2x}\ln(e^{2x}+1)]\mathrm{d}x = \frac{1}{2}[(e^2+1)\ln(e^2+1) - 2\ln 2 - (e^2-1)]$$

$$= 5.03$$

9. (i) Differentiate $x\sin x$ with respect to x.

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} x\cos x \,\mathrm{d}x$.

Solution

(i) $\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x) = \sin x + x\cos x$

(ii) $\int_0^{\frac{\pi}{2}} (\sin x + x\cos x)\,\mathrm{d}x = [x\sin x]_0^{\frac{\pi}{2}}$

$$[-\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = \frac{\pi}{2}\sin\frac{\pi}{2}$$

$$1 + \int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = \frac{\pi}{2} - 1$$

$$= 0.571$$

---

10. (i) Given that $y = (2x+3)\sqrt{4x-3}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form $\frac{kx}{\sqrt{4x-3}}$ and state the value of k.

(ii) Hence evaluate $\int_1^7 \frac{x}{\sqrt{4x-3}}\,\mathrm{d}x$.

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2(4x-3)^{\frac{1}{2}} + \frac{1}{2}(2x+3)(4x-3)^{-\frac{1}{2}}(4)$

$$= (4x-3)^{-\frac{1}{2}}[2(4x-3) + 2(2x+3)]$$

$$= 12(4x-3)^{-\frac{1}{2}}$$

$$= \frac{12}{\sqrt{4x-3}}$$

$\therefore k = 12$

(ii) $\int_1^7 \frac{12x}{\sqrt{4x-3}}\,\mathrm{d}x = \left[(2x+3)\sqrt{4x-3}\right]_1^7$

$$= \left[(2x+3)\sqrt{4x-3}\right]_1^7$$

$$\int_1^7 \frac{x}{\sqrt{4x-3}}\,\mathrm{d}x = \frac{1}{5}(7)(9)^{\frac{3}{2}} - \frac{1}{5}(3)(1)^{\frac{3}{2}}$$

$$= 37.2 \text{ or } \frac{186}{5}$$

11. (i) Differentiate xe^{5x} with respect to x.

(ii) Hence evaluate $\int_0^1 4xe^{5x}\,dx$.

(i) $\dfrac{d}{dx}(xe^{5x}) = e^{5x} + 5xe^{5x}$

(ii) $\int_0^1 (e^{5x} + 5xe^{5x})dx = [xe^{5x}]_0^1$

$$\left[\frac{e^{5x}}{5}\right]_0^1 + 5\int_0^1 xe^{5x}dx = e^5$$

$$5\int_0^1 xe^{5x}dx = e^5 - \left(\frac{e^5}{5} - \frac{1}{5}\right)$$

$$\int_0^1 xe^{5x}dx = \frac{4e^5+1}{25}$$

$$\int_0^1 4xe^{5x}dx = 4\left(\frac{4e^5+1}{25}\right)$$

$$= 95.1$$

12. (i) Given that $y = x\sqrt{6+3x^2}$, show that $\dfrac{dy}{dx} = \dfrac{6+6x^2}{\sqrt{6+3x^2}}$.

(ii) Hence, evaluate $\displaystyle\int_1^5 \frac{1+x^2}{\sqrt{6+3x^2}}\,dx$.

(i) $\dfrac{d}{dx}(x\sqrt{6+3x^2}) = (6+3x^2)^{\frac{1}{2}} + x\left(\dfrac{1}{2}\right)(6+3x^2)^{-\frac{1}{2}}(6x)$

$$= (6+3x^2)^{-\frac{1}{2}}(6+3x^2+3x^2)$$

$$= \frac{6+6x^2}{\sqrt{6+3x^2}}$$

(ii) $\displaystyle\int_1^5 \frac{6+6x^2}{\sqrt{6+3x^2}}dx = [x\sqrt{6+3x^2}]_1^5$

$$\int_1^5 \frac{1+x^2}{\sqrt{6+3x^2}}dx = \frac{1}{6}(5\sqrt{81} - \sqrt{9})$$

$$= 7$$

---

13. (i) Express $\dfrac{x-5}{3x^2-10x-8}$ in partial fractions.

(ii) Hence, evaluate $\displaystyle\int_6^8 \frac{x-5}{6x^2-20x-16}\,dx$.

(i) $\dfrac{x-5}{3x^2-10x-8} = \dfrac{x-5}{(3x-2)(x+4)}$

Let $\dfrac{x-5}{3x^2-10x-8}=\dfrac{A}{3x+2}+\dfrac{B}{x-4}$.

$x-5=A(x-4)+B(3x+2)$

Let $x=4$, $\quad 14B=-1$

$$B=-\frac{1}{14}$$

Let $x=-\dfrac{2}{3}$, $\quad -\dfrac{14A}{3}=-\dfrac{17}{3}$

$$A=\frac{17}{14}$$

$$\frac{x-5}{3x^2-10x-8}=\frac{17}{14(3x+2)}-\frac{1}{14(x-4)}$$

(ii) $\displaystyle\int_6^8 \frac{x-5}{6x^2-20x-16}\,dx = \int_6^8 \frac{x-5}{2(3x^2-10x-8)}\,dx$

$$= \frac{1}{2}\int_6^8 \left[\frac{17}{14(3x+2)} - \frac{1}{14(x-4)}\right] dx$$

$$= \frac{1}{2}\left[\frac{17}{14(3)}\ln(3x+2) - \frac{1}{14}\ln(x-4)\right]_6^8$$

$$= \frac{1}{2}\left[\frac{17}{42}\ln 26 - \frac{1}{14}\ln 4 - \left(\frac{17}{42}\ln 20 - \frac{1}{14}\ln 2\right)\right]$$

$$= 0.0283$$

---

14. (i) Express $\dfrac{1-2x}{x(x+1)^2}$ in the form $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$ where A, B and C are constants.

(ii) Hence or otherwise find the exact value of $\displaystyle\int_2^3 \frac{1-2x}{x(x+1)^2}\,dx$. Express your answer in the form $\ln a + b$, where a and b are real numbers.

(i) Let $\dfrac{1-2x}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$.

$1-2x = A(x+1)^2 + Bx(x+1) + Cx$

Let $x = 0$, $\quad A = 1$

Let $x = -1$, $\quad -C = 3$

$C = -3$

Comparing coefficient of x, $\quad A + B = 0$

$B = -A = -1$

$\therefore \dfrac{1-2x}{x(x+1)^2} = \dfrac{1}{x} - \dfrac{1}{x+1} - \dfrac{3}{(x+1)^2}$

(ii) $\displaystyle\int_2^3 \frac{1-2x}{x(x+1)^2}\,dx = \int_2^3 \left[\frac{1}{x} - \frac{1}{x+1} - \frac{3}{(x+1)^2}\right] dx$

$$= \left[\ln x - \ln(x+1) + \frac{3}{x+1}\right]_2^3$$

$$= \left[\ln\left(\frac{x}{x+1}\right) + \frac{3}{x+1}\right]_2^3$$

$$= \ln\left(\frac{3}{4}\right) + \frac{3}{4} - \ln\left(\frac{2}{3}\right) - 1$$

$$= \ln\left(\frac{3}{4}\times\frac{3}{2}\right) - \frac{1}{4}$$

$$= \ln\left(\frac{9}{8}\right) - \frac{1}{4}$$

15. Given that $y = \dfrac{\cos 2x}{1+\sin 2x}$. Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form $\dfrac{k}{1+\sin 2x}$ and state the value of k.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-2\sin 2x(1+\sin 2x) - \cos 2x(2\cos 2x)}{(1+\sin 2x)^2} \\
&= \frac{-2\sin 2x - 2\sin^2 2x - 2\cos^2 2x}{(1+\sin 2x)^2} \\
&= \frac{-2\sin 2x - 2}{(1+\sin 2x)^2} \\
&= \frac{-2(1+\sin 2x)}{(1+\sin 2x)^2} \\
&= \frac{-2}{1+\sin 2x}
\end{aligned}$$

$\therefore k = -2$

---

16. (i) Given that $\dfrac{3x^2+4x-20}{(2x+1)(x^2+4)} = \dfrac{A}{2x+1} + \dfrac{Bx+C}{x^2+4}$, where A, B and C are constants, find the value of A, of B and show that $C = 0$.

(i) Differentiate $\ln(x^2+4)$ with respect to x.

(ii) Using the results from parts (i) and (ii), find $\displaystyle\int \frac{3x^2+4x-20}{(2x+1)(x^2+4)}\,\mathrm{d}x$.

(i) $\dfrac{3x^2+4x-20}{(2x+1)(x^2+4)} = \dfrac{A}{2x+1} + \dfrac{Bx+C}{x^2+4}$

$3x^2+4x-20 = A(x^2+4) + (Bx+C)(2x+1)$

Let $x = -\dfrac{1}{2}$, $\quad 3\left(-\dfrac{1}{2}\right)^2 + 4\left(-\dfrac{1}{2}\right) - 20 = A\left[\left(-\dfrac{1}{2}\right)^2 + 4\right]$

$-\dfrac{85}{4} = \dfrac{17A}{4}$

$A = -5$

Let $x = 0$, $\quad -20 = 4A + C$

$C = 0$

Comparing coefficient of x^2, $\quad A + 2B = 3$

$2B = 8$

$B = 4$

$\therefore \dfrac{3x^2+4x-20}{(2x+1)(x^2+4)} = -\dfrac{5}{2x+1} + \dfrac{4x}{x^2+4}$

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}[\ln(x^2+4)] = \dfrac{2x}{x^2+4}$

(ii) $\displaystyle\int \frac{3x^2+4x-20}{(2x+1)(x^2+4)}\,\mathrm{d}x = \int\left(-\frac{5}{2x+1}+\frac{4x}{x^2+4}\right)\mathrm{d}x$

$$= -\frac{5}{2}\ln(2x+1) + 2\ln(x^2+4) + C$$

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 17

Name: ________________________________ (____)   Date: ________________

Class: Sec 4______

Revision 17: Kinematics

1. If the displacement of a particle moving in a straight line from a fixed point is s m after time t s, and its velocity and acceleration are v ms^{-1} and a ms^{-2} respectively, then

(a) the velocity function, $v = \int a \, \mathrm{d}t$,

(b) the displacement function, $s = \int v \, \mathrm{d}t$,

(c) the displacement after t seconds, $s = \int_0^t v \, \mathrm{d}t$,

(d) the distance travelled in the n^{th} seconds, $s = \int_{n-1}^{n} v \, \mathrm{d}t$ if $v \neq 0$ throughout the n^{th} second.

2. When $v = \dfrac{\mathrm{d}s}{\mathrm{d}t} = 0,$ the particle

(a) is instantaneously at rest and attempting to change its direction of motion (making a U-turn),

(b) has a maximum or minimum displacement.

3. When $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = 0,$ the particle has a maximum or minimum velocity.

4. When two particles collide into each other, they share a common displacement.

Example

1. Two particles, P and Q, leaves a point O at the same time and travel in the same direction along the same straight line. Particle P starts with a velocity of 9 m/s and moves with a constant acceleration of $1.5\,\text{m/s}^2$. Particle Q starts from rest and moves with an acceleration of $a\,\text{m/s}^2$, where $a = 1 + \frac{t}{2}$ and t seconds is the time since leaving O. Find

   (i) the velocity of each particle in terms of t,
   (ii) the distance travelled by each particle in terms of t.
   Hence find
   (iii) the distance from O at which Q collides with P,

Solution

(i) $V_P = 9 + 1.5t$

$$V_Q = \int (1 + \frac{t}{2})\mathrm{d}t$$

$$= t + \frac{t^2}{4} + C$$

When $t = 0,\ v = 0,\ C = 0$

$$\therefore V_Q = t + \frac{t^2}{4}$$

(ii) $s_P = \int (9 + 1.5t)\mathrm{d}t$

$$= 9t + \frac{3t^2}{4} + C$$

When $t = 0,\ s_P = 0,\ C = 0$

$$\therefore s_P = 9t + \frac{3t^2}{4}$$

$$s_Q = \int (t + \frac{t^2}{4})\mathrm{d}t$$

$$= \frac{t^2}{2} + \frac{t^3}{12} + C$$

When $t = 0,\ s_Q = 0,\ C = 0$

$$\therefore s_Q = \frac{t^2}{2} + \frac{t^3}{12}$$

(iii) When $s_P = s_Q$, $\quad 9t + \frac{3t^2}{4} = \frac{t^2}{2} + \frac{t^3}{12}$

$$108t + 9t^2 = 6t^2 + t^3$$

$$\vdots$$

$$t(t-12)(t+9) = 0$$

Since $t > 0$, $\quad t = 12$

When $t = 12$, $\quad s_P = 9(12) + \frac{3(12)^2}{4} = 216$

P and Q collide at a point 216 m away from O.

Exercise

1. A particle travels in a straight line, so that, t seconds after passing a fixed point A on the line, its acceleration, a ms^{-2}, is given by $a = -2 - 2t$. It comes to rest at a point B when $t = 4$.
   (i) Find the velocity of the particle at A.
   (ii) Find the distance AB.

Solution

(i) $v = \int (-2 - 2t)dt$

$= -2t - t^2 + C$

When $t = 4$, $v = 0$, $\quad 0 = -8 - 16 + C$

$C = 24$

$\therefore v = -t^2 - 2t + 24$

When $t = 0$, $v = 24$

Velocity of particle at A = 24 ms^{-1}

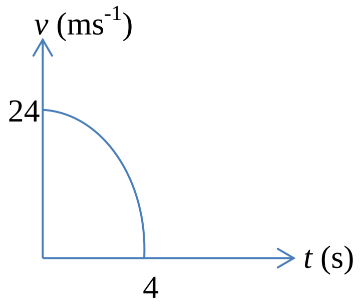
(ii) Distance $AB = \int_0^4 (-t^2 - 2t + 24)\mathrm{d}t$

$$= \left[-\frac{t^3}{3} - t^2 + 24t \right]_0^4$$

$$= \left[-\frac{64}{3} - 16 + 96 \right]_0^4$$

$$= 58\frac{2}{3} \text{ m}$$

(iii)



2. A particle moves in a straight line, so that, t seconds after leaving a fixed point O, its velocity, $v\ \mathrm{ms}^{-1}$, is given by

$$v = pt^2 + qt + 4,$$

where p and q are constants. When t = 1 the acceleration of the particle is $8\ \mathrm{ms}^{-2}$. When t = 2, the displacement of the particle from O is 22 m. Find the value of p and of q.

Solution

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$
$$= 2pt + q$$

When $t = 1$, $a = 8$, $\quad 2p + q = 8$

$$q = 8 - 2p \qquad \ldots\ldots(1)$$

$$s = \int (pt^2 + qt + 4)\mathrm{d}t$$
$$= \frac{pt^3}{3} + \frac{qt^2}{2} + 4t + C$$

When $t = 0$, s = 0, $\quad C = 0$

$$\therefore s = \frac{pt^3}{3} + \frac{qt^2}{2} + 4t$$

When $t = 2$, $s = 22$, $\quad \frac{8p}{3} + 2q + 8 = 22$

$$\frac{8p}{3} + 2q = 14 \qquad \ldots\ldots(2)$$

Sub (1) into (2) $\quad \frac{8p}{3} + 2(8 - 2p) = 14$

$$\frac{8p}{3} + 16 - 4p = 14$$
$$\frac{-4p}{3} = -2$$
$$p = \frac{3}{2}$$
$$q = 5$$

3. A particle, moving in a straight line, passes through a fixed point O with velocity $14\,\text{ms}^{-1}$. The acceleration, $a\,\text{ms}^{-2}$, of the particle, t seconds after passing through O, is given by $a = 2t - 9$. The particle subsequently comes to instantaneous rest, firstly at A and later at B. Find

(i) the acceleration of the particle at A and at B,
(ii) the greatest speed of the particle as it travels from A to B,
(iii) the distance AB.

June 2007

<u>Solution</u>

(i) $v = \int (2t-9)dt$

$= t^2 - 9t + C$

When $t=0$, $v=14$, $C=14$

$\therefore v = t^2 - 9t + 14$

When $v=0$, $t^2 - 9t + 14 = 0$

$(t-2)(t-7)=0$

$t = 2, 7$

At A, $t=2$, Acceleration $= 2(2) - 9$

$= -5\,\text{ms}^{-2}$

At B, $t=7$, Acceleration $= 2(7) - 9$

$= 5\,\text{ms}^{-2}$

(ii) $v = t^2 - 9t + \left(\frac{9}{2}\right)^2 + 14 - \left(\frac{9}{2}\right)^2$

$= \left(t - \frac{9}{2}\right)^2 - 6\frac{1}{4}$

Greatest speed = $6\frac{1}{4}$ ms^{-1}

(iii) $s = \int (t^2 - 9t + 14)dt$

$= \frac{t^3}{3} - \frac{9t^2}{2} + 14t + C$

When $t=0$, $s=0$, $C=0$

$\therefore s = \frac{t^3}{3} - \frac{9t^2}{2} + 14t$

When $t=2$, $s = \frac{8}{3} - \frac{36}{2} + 28 = 12\frac{2}{3}$

When $t=7$, $s = \frac{343}{3} - \frac{441}{2} + 98 = -8\frac{1}{6}$

Distance AB $= 12\frac{2}{3} + 8\frac{1}{6}$ or $-\int_2^7 (t^2 - 9t + 14)\mathrm{d}t$

$= 20\frac{5}{6}\,\text{m}$

4. A particle moves in a straight line such that its displacement, s m, from a fixed point O at a time t s, is given by

$$s = \ln(t+1) \quad \text{for } 0 \le t \le 3,$$

$$s = \frac{1}{2}\ln(t-2) - \ln(t+1) + \ln 16 \quad \text{for } t > 0.$$

Find
(i) the initial velocity of the particle,
(ii) the velocity of the particle when $t = 4$,
(iii) the acceleration of the particle when $t = 4$,
(iv) the value of t when the particle is instantaneously at rest,
(v) the distance travelled by the particle in the 4th second.

Solution

(i) For $0 \le t \le 3$, $v = \frac{\mathrm{d}}{\mathrm{d}t}[\ln(t+1)]$

$= \frac{1}{t+1}$

When $t = 0$, initial $v = 1$ ms^{-1}

(ii) For $t > 3$, $v = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\ln(t-2) - \ln(t+1) + \ln 16\right]$

$= \frac{1}{2(t-2)} - \frac{1}{t+1}$

When $t = 4$, $v = \frac{1}{4} - \frac{1}{5}$

$= \frac{1}{20}$ ms^{-1}

(iii) For $t > 3$, $a = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2(t-2)} - \frac{1}{t+1}\right]$

$= -\frac{1}{2(t-2)^2} + \frac{1}{(t+1)^2}$

When $t = 4$, $a = -\frac{1}{2(2)^2} + \frac{1}{5^2}$

$= -\frac{17}{200}$ ms^{-2}

(iv) When $v = 0$, $\frac{1}{2(t-2)} - \frac{1}{t+1} = 0$

$t + 1 - 2(t-2) = 0$

$t = 5$

(iv) When $t = 3$, $s = \ln 4 = 1.3862$

When $t = 4$, $s = \frac{1}{2}\ln 2 - \ln 5 + \ln 16 = 1.5097$

Distance travelled in the 4$^{\text{th}}$ seconds = 1.5097 – 1.3862

= 0.124 m

5. A particle moves in a straight line such that t s after passing through a fixed point O, its velocity, $v\ \text{ms}^{-1}$, is given by $v = k\cos 4t$, where k is a positive constant.
Find
(i) the value of t when the particle is first instantaneously at rest,
(ii) an expression for the acceleration of the particle t s after passing through O.

Given that the acceleration of the particle is $12\ \text{ms}^{-2}$ when $t = \dfrac{3\pi}{8}$,
(iii) find the value of k.

Using your value for k,
(iv) sketch the velocity-time curve for the particle for $0 \le t \le \pi$.
(v) find the displacement of the particle from O when $t = \dfrac{\pi}{24}$.

Solution

(i) When $v = 0$, $\cos 4t = 0$

$$4t = \frac{\pi}{2}$$

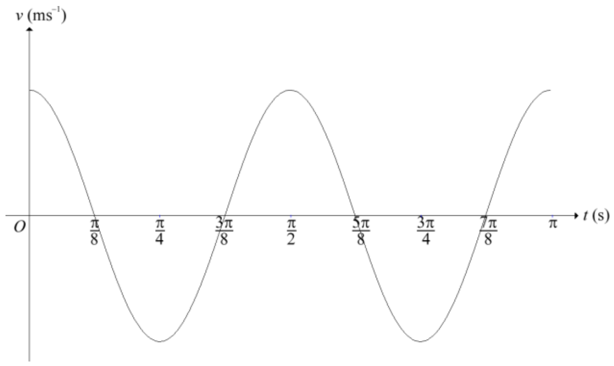
$$t = \frac{\pi}{8}$$

(ii) $$a = \frac{\mathrm{d}}{\mathrm{d}t}(k\cos 4t)$$
$$= -4k\sin 4t$$

(iii) When $t = \dfrac{3\pi}{8}$, $a = 12$, $\quad -4k\sin\dfrac{3\pi}{2} = 12$

$$-4k(-1) = 12$$
$$k = 3$$

(iv)



(v) $$s = \int 3\cos 4t\ \mathrm{d}t$$
$$= \frac{3}{4}\sin 4t + C$$

When $t = 0$, $s = 0$, $C = 0$

$$\therefore s = \frac{3}{4}\sin 4t$$

When $t = \dfrac{\pi}{24}$, $\quad s = \dfrac{3}{4}\sin\dfrac{\pi}{6}$

$$= \frac{3}{8}\ \text{m}$$

6. A motorcycle is driven along a straight horizontal road. As it passes a point A the brakes are applied and the motorcycle slows down, coming to rest at a point B. For the journey from A to B, the distance, s metres, of the motorcycle from A, t seconds after passing A, is given by

$$s = 400(1 - e^{-\frac{t}{10}}) - 16t.$$

(i) Find an expression, in terms of t, for the velocity of the motorcycle during the journey from A to B.
(ii) Find an expression, in terms of t, for the acceleration of the motorcycle during the journey from A to B.
(iii) Find the velocity of the motorcycle at A.
(iv) Show that the time taken for the journey from A to B is approximately 9.163 seconds.
(v) Find the average speed of the motorcycle for the journey from A to B

2009

<u>.Solution</u>

(i) Velocity of the motorcycle, $v = 400\left(-\frac{1}{10}\right)\left(-e^{-\frac{t}{10}}\right) - 16$

$$= 40e^{-\frac{t}{10}} - 16 \text{ ms}^{-1}$$

(ii) Acceleration of the motorcycle, $a = 40\left(-\frac{1}{10}\right)e^{-\frac{t}{10}}$

$$= -4e^{-\frac{t}{10}} \text{ ms}^{-2}$$

(iii) At A, $t = 0$, Velocity of the motorcycle at A $= 40 - 16 \text{ ms}^{-1}$

$$= 24 \text{ ms}^{-1}$$

(iv) At B, $v = 0$,

$$40e^{-\frac{t}{10}} - 16 = 0$$

$$e^{-\frac{t}{10}} = 0.4$$

$$-\frac{t}{10} = \ln 0.4$$

$$t = -10 \ln 0.4$$

$$= 9.163$$

The motorcycle took about 9.163 s to reach B.

(v) At B, $t = -10\ln 0.4$, $e^{-\frac{t}{10}} = 0.4$,

$$s = 400(1 - 0.4) - 16(-10\ln 0.4)$$

$$= 240 + 160\ln 0.4 \text{ m}$$

$$= 93.393 \text{ m}$$

Average speed $= \dfrac{240 + 160\ln 0.4}{-10\ln 0.4}$

$$= 10.2 \text{ ms}^{-1}$$

7. A particle P leaves a fixed point O and moves in a straight line so that, t s after leaving O, its displacement, s m, from O is given by

$$s = t\ln(t+1) - t.$$

Find, when $t = 20$,

(i) the displacement of P from O,

(ii) the velocity of P,

(iii) the acceleration of P.

2011

Solution

(i) When $t = 20$, $s = 20\ln 21 - 20$

$= 40.9$ m

(ii) $v = \dfrac{ds}{dt}$

$= \ln(t+1) + \dfrac{t}{t+1} - 1$

When $t = 20$, $v = \ln 21 - \dfrac{20}{21} - 1$

$= 3.00 \text{ ms}^{-1}$

(iii) $a = \dfrac{dv}{dt}$

$= \dfrac{1}{t+1} + \dfrac{t+1-t}{(t+1)^2}$

$= \dfrac{1}{t+1} + \dfrac{1}{(t+1)^2}$

When $t = 20$, $a = \dfrac{1}{21} + \dfrac{1}{21^2}$

$= \dfrac{22}{441}$ or 0.0499 ms^{-2}

8. A particle moves in a straight line, so that, t seconds after leaving a fixed point O, its velocity, $v\ \text{ms}^{-1}$, is given by $v = \dfrac{12}{(t+1)^2} - 3.$ Find

(i) an expression for the acceleration of the particle in terms of t,
(ii) the distance travelled by the particle before it comes to instantaneous rest.

2012

Solution

(i) Acceleration, $a = \dfrac{\mathrm{d}v}{\mathrm{d}t}$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[12(t+1)^{-2} - 3\right]$$

$$= 12(-2)(t+1)^{-3}(1)$$

$$= -\frac{24}{(t+1)^3}\ \text{ms}^{-2}$$

(ii) When $v = 0$, $\dfrac{12}{(t+1)^2} - 3 = 0$

$$12 = 3(t+1)^2$$

$$(t+1)^2 = 4$$

$$t+1 = \pm 2$$

$$\text{Since } t > 0, \therefore t = 1$$

Distance travelled $= \displaystyle\int_0^1 [12(t+1)^{-2} - 3]\mathrm{d}t$

$$= \left[\frac{12(t+1)^{-1}}{(-1)} - 3t\right]_0^1$$

$$= \left[-\frac{12}{t+1} - 3t\right]_0^1$$

$$= -6 - 3 - (-12)$$

$$= 3\ \text{m}$$

Alternatively

$$s = \int [12(t+1)^{-2} - 3]\mathrm{d}t$$

$$= \frac{12(t+1)^{-1}}{(-1)} - 3t + C$$

$$= -\frac{12}{t+1} - 3t + C$$

When $t = 0$, $s = 0$, $0 = -12 + C \Rightarrow C = 12$

$$\therefore s = \frac{12(t+1)^{-1}}{(-1)} - 3t + 12$$

When $t = 1$, $s = -\dfrac{12}{2} - 3 + 12$

$$= 3$$

9. A particle, moving in a straight line, passes through a fixed point O with a speed of $28\,\text{ms}^{-1}$. The acceleration, $a\,\text{ms}^{-2}$, of the particle, t seconds after passing through O, is given by $a = -16e^{-0.5t}$.

(i) Find the value of t when the particle is at instantaneous rest.

(ii) Find the distance of the particle from O when it is at instantaneous rest.

2013

Solution

(i) $v = \int -16e^{-0.5t}\,\mathrm{d}t$

$$= \frac{-16e^{-0.5t}}{-0.5} + C$$

$$= 32e^{-0.5t} + C$$

When $t = 0$, $v = 28$, $\quad 28 = 32 + C$

$$C = -4$$

$$v = 32e^{-0.5t} - 4$$

When $v = 0$, $\quad 32e^{-0.5t} - 4 = 0$

$$e^{-0.5t} = \frac{1}{8}$$

$$-0.5t = \ln\left(\frac{1}{8}\right)$$

$$t = \frac{\ln\left(\frac{1}{8}\right)}{-0.5}$$

$$= 4.16$$

(ii) $s = \int (32e^{-0.5t} - 4)\,\mathrm{d}t$

$$= \frac{32e^{-0.5t}}{-0.5} - 4t + C$$

$$= -64e^{-0.5t} - 4t + C$$

When $t = 0$, $s = 0$, $\quad 0 = -64 + C$

$$C = 64$$

$$\therefore s = -64e^{-0.5t} - 4t + 64$$

When $t = 4.1588$, $\quad s = -64e^{-0.5(4.1588)} - 4(4.1588) + 64$

$$= 39.4\text{ m}$$

10. A particle travels in a straight line, so that, t seconds after passing through a fixed point O, its acceleration, a ms^{-2}, is given by $a = \dfrac{8}{(t+2)^2}$. The particle comes to rest when $t = 2$.

Find

(i) an expression for the velocity of the particle in terms of t,

(ii) the distance from O at which the particle comes to rest.

Specimen

Solution

(i) $v = \int 8(t+2)^{-2}\,\mathrm{d}t$

$= -8(t+2)^{-1} + C$

$= -\dfrac{8}{t+2} + C$

When $t = 2$, $v = 0$, $\quad 0 = -2 + C$

$C = 2$

$\therefore v = -\dfrac{8}{t+2} + 2$

(ii) $s = \int [2 - 8(t+2)^{-1}]\,\mathrm{d}t$

$= 2t - 8\ln(t+2) + D$

When $t = 0$, $s = 0$, $\quad -8\ln 2 + D = 0$

$D = 8\ln 2$

$\therefore s = 2t - 8\ln(t+2) + 8\ln 2$

When $t = 2$, $\quad s = 4 - 8\ln 4 + 8\ln 2$

$= -1.55$

$\therefore$ Distance from O = 1.55 m

Alternatively

$s = \int_0^2 [2 - 8(t+2)^{-1}]\,\mathrm{d}t$

$= \left[2t - 8\ln(t+2)\right]_0^2$

$= 4 - 8\ln 4 + 8\ln 2$

$= -1.55$

$\therefore$ Distance from O = 1.55 m

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 2

Name: ________________________________ (____) Date: ________________

Class: Sec 4_____

Revision 2: Surds, Indices and Logarithms

Solving of Surd Equations

- Isolate the surd or unknown.
- Square both sides if necessary.
- Simplify and solve equation.
- Verify your answers.

Example

1. Without using a calculator, find the values of the integers a and b for which the solution of the equation $x\sqrt{24} = x\sqrt{3} + \sqrt{6}$ is $\dfrac{a+\sqrt{b}}{7}$. **2009**

$$x\sqrt{24} = x\sqrt{3} + \sqrt{6}$$
$$x\sqrt{24} - x\sqrt{3} = \sqrt{6}$$
$$x\sqrt{8} - x = \sqrt{2}$$
$$x(\sqrt{8} - 1) = \sqrt{2}$$

- Solve by making x the subject.
- Squaring both sides will introduce an additional solution which is invalid.

$$x = \frac{\sqrt{2}}{\sqrt{8}-1} \times \frac{\sqrt{8}+1}{\sqrt{8}+1}$$
$$= \frac{4+\sqrt{2}}{7}$$
$$a = 4$$
$$b = 2$$

Laws of Indices

The following rules are used for dividing and multiplying numbers written in index form.

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^n \times b^n = (ab)^n$
- $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

Example

1. Simplify $\dfrac{16^{x+1}+20(4^{2x})}{2^{x-3}8^{x+2}}$.

$$\frac{16^{x+1}+20(4^{2x})}{2^{x-3}8^{x+2}}=\frac{16(16^{x})+20(16^{x})}{\frac{64}{8}(16^{x})}$$
$$=\frac{36(16^{x})}{8(16^{x})}$$
$$=\frac{9}{2}$$

- Look for common factor between the numerator and denominator.
- In this case, the common factor is 4^{2x} or 16^{x}.

---

2. Without using a calculator, find the value of 6^{x}, given that $3^{x+2}=12^{2-x}$.

$$3^{x+2}=12^{2-x}$$
$$9(3^{x})=\frac{144}{12^{x}}$$
$$(12^{x})(3^{x})=\frac{144}{9}$$
$$36^{x}=16$$
$$(6^{x})^{2}=16$$
$$\text{Since, } 6^{x}>0,\ 6^{x}=4$$

- Make 6^{x} the subject.
- $6^{x}>0$

Solving Exponential Equations

Three types of exponential equations

1. $a^{x}=a^{y} \quad \Rightarrow x=y$
2. $a^{x}=b \quad \Rightarrow x\lg a=\lg b \ \Rightarrow x=\dfrac{\lg b}{\lg a}$
3. $ap^{2x}+bp^{x}+c=0$

- Compare indices
- Taking log on both sides
- Apply method of substitution
- $p^{x}>0$

Example

3. Solve the following equations.

(a) $2(16^{x})=2-3(4^{x})$

(b) $2e^{x}=7\sqrt{e^{x}}-3$

Solution

(a)
$$2(16^{x})=2-3(4^{x})$$
$$2(16^{x})+3(4^{x})-2=0$$
$$\text{Let } y=4^{x}$$
$$2y^{2}+3y-2=0$$
$$(2y-1)(y+2)=0$$
$$y=\frac{1}{2},\ -2$$

- This is a trinomial of degree 2.
- Solve by method of substitution
- $4^{x}>0$.

$$4^{x}=\frac{1}{2},\ -2\ \text{(NA)}$$
$$2^{2x}=2^{-1}$$
$$2x=-1$$
$$x=-\frac{1}{2}$$

Solving Logarithmic Equations

- Simplify the given equation by applying the appropriate Laws of Logarithm.
- Express the equation in the form $\log_a M = \log_a N$.
- Solve the equation $M = N$.
- Apply the method of substitution when none of the Laws of Logarithm are applicable.

Example

4. Solve the following equation.

(a) $\log_4(x+2)+\log_4(4-x)=\dfrac{3}{2}$

(b) $\log_3(x-8)=\log_9 81-\log_3 x$

(c) $\log_3 x = 4-\log_x 27$

(a) $\log_4(x+2)+\log_4(4-x)=\dfrac{3}{2}$

$$\log_4(x+2)(4-x)=\frac{3}{2}\log_4 4$$

$$\log_4(8+2x-x^2)=\log_4 4^{\frac{3}{2}}$$

$$8+2x-x^2=8$$

$$x^2-2x=0$$

$$x(x-2)=0$$

$$x=0,\ 2$$

- $\dfrac{3}{2}=\dfrac{3}{2}\log_4 4$
- Express the equation in the form $\log_a M = \log_a N$.
- Verify your answers.

(b) $\log_3(x-8)=\log_9 81-\log_3 x$

$$\log_3(x-8)+\log_3 x = 2\log_9 9$$

$$\log_3[x(x-8)]=2\log_3 3=\log_3 9$$

$$x^2-8x=9$$

$$x^2-8x-9=0$$

$$(x+1)(x-9)=0$$

$$x=-1\,(\text{NA}), 9$$

- Express all the logarithmic terms in a common base.
- Simplify the expression.
- $\log_3 x$ is undefined when $x = -1$
- Verify your answers.

(c) $\log_3 x = \log_x 27 - 2$

$$\log_3 x = \frac{3\log_3 3}{\log_3 x}-2$$

Let $t=\log_3 x$.

$$t=\frac{3}{t}-2$$

$$t^2+2t-3=0$$

$$(t-1)(t+3)=0$$

$$t=1,-3$$

$$\log_3 x = 1,-3$$

$$x=3^1,\ 3^{-3}$$

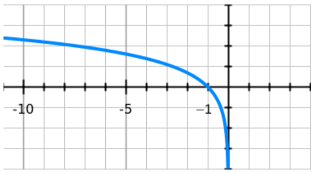
$$=3,\ \frac{1}{27}$$

- Express all the logarithmic terms in a common base.
- Simplify the expression.
- None of the laws of logarithms could be applied, use the method of substitution.
- Solve the equation.
- Verify your answers.

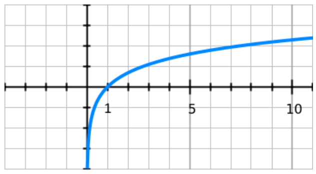
Graphs of Logarithmic Functions

Consider the graph of $y = \ln(ax+b)$,

- the asymptote by solving $ax+b=0$,
- the x-intercept by solving $ax+b=1$,
- the y-intercept (if any).



$y=\ln(-x)$



$y=\ln(x)$

Note

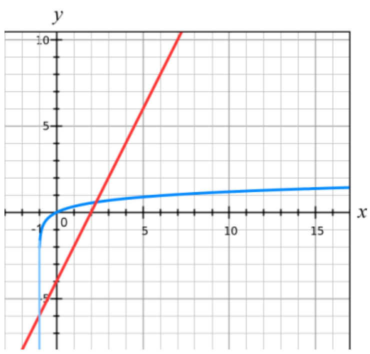
You might be provided with graph paper for rough work during the O Level Exam. However, you should sketch you graph on the writing paper provided.

Example

1. (i) Sketch the graph of $y=\ln(\sqrt{x+1})$, for $x>-1$.

   (ii) In order to solve the equation $x=e^{2x-4}-1$, a suitable straight line is drawn on the same set of axes as the graph of $y=\ln(\sqrt{x+1})$. Find the equation of this straight line.

(i)



- Solve $x+1=0$ to obtain the asymptote.
- Solve $x+1=1$ to obtain the x-intercept.

$x=e^{2x-4}-1$

$x+1=e^{2x-4}$

$\ln(x+1)=2x-4$

Insert $y=2x-4$.

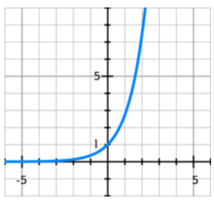
- Make $\ln(\sqrt{x+1})$ the subject of the equation $x=e^{2x-4}-1$.

Graphs of Logarithmic Exponential Functions

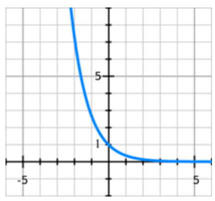
Note

Consider the graph of $y=e^{ax}$,

- the asymptote is the x-axis and y = 0,
- the y-intercept = 1.



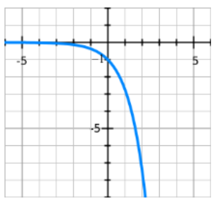
$y=e^{x}$



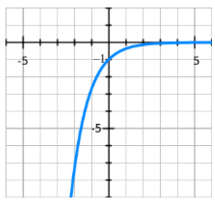
$y=e^{-x}$

Consider the graph of $y=e^{-ax}$,

- the asymptote is the x-axis and y = 0,
- the y-intercept $=-1$.



$y=-e^{x}$



$y=-e^{-x}$

Exercise

1. Without using a calculator, find the integers a and b such that $\dfrac{a+b\sqrt{3}}{5+2\sqrt{3}}=\dfrac{5+2\sqrt{3}}{2+\sqrt{3}}$. **2012**

$$\frac{a+b\sqrt{3}}{5+2\sqrt{3}}=\frac{5+2\sqrt{3}}{2+\sqrt{3}}$$

$$\begin{aligned} a+b\sqrt{3} &= \frac{(5+2\sqrt{3})^2}{2+\sqrt{3}} \\ &= \frac{(25+20\sqrt{3}+12)(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} \\ &= (37+20\sqrt{3})(2-\sqrt{3}) \\ &= 74+40\sqrt{3}-37\sqrt{3}-60 \\ &= 14+3\sqrt{3} \end{aligned}$$

$a=14$

$b=3$

---

2. Given that $\sqrt{p+q\sqrt{2}}=\dfrac{2}{3+2\sqrt{2}}$, where p and q are integers, without using a calculator, find the value of p and of q.

$$\begin{aligned} \sqrt{p+q\sqrt{2}} &= \frac{2}{3+2\sqrt{2}} \\ &= \frac{2(3-2\sqrt{2})}{9-8} \end{aligned}$$

$$\begin{aligned} p+q\sqrt{2} &= 4(3-2\sqrt{2})^2 \\ &= 4(9-12\sqrt{2}+8) \\ &= 68-48\sqrt{2} \end{aligned}$$

---

3. The curve $y=ax^n$, where a and n are constants, passes through the points (2, 40), (3, 135) and (4, k). Find the values of n, a and k.

At (2, 40), $40=a(2^n)$ ……(1)

At (3, 135), $135=a(3^n)$ ……(2)

At (4, k), $k=a(4^n)$ ……(3)

(2)÷(1) $\dfrac{135}{40}=\dfrac{a(3^n)}{a(2^n)}$

$$\frac{27}{8}=\left(\frac{3}{2}\right)^n$$

$$n=3$$

Sub $n=3$ into (1) $40=8a$

$a=5$

Sub $a=5$, $n=3$ into (3) $k=5(64)$

$=320$

4. Baby food is heated in a microwave to a temperature of 80°C. It subsequently cools in such a way that its temperature $T°C$, t minutes after removal from the microwave, is given by $T = 20 + Ae^{-kt}$, where A and k are constants.

(i) Explain why A = 60.

When t = 1, the temperature of the food is 65 °C.
(ii) Find the value of k correct to 3 significant figures.

A baby should only be given this food when the temperature of the food is less than 40 °C.
(iii) Determine, with working, whether it is safe to give the food 4 minutes after removal from the microwave. **2014**

(i) At t = 0, T = 80°C, $80 = 20 + A$

(ii) When t = 1,

$$65 = 20 + 60e^{-k}$$
$$60e^{-k} = 45$$
$$e^{-k} = \frac{3}{4}$$
$$-k = \ln\left(\frac{3}{4}\right)$$
$$k = 0.288$$

(iii) When t = 4,

$$T = 20 + 60e^{4\ln 0.75}$$
$$= 39.0 < 40$$

It is safe to give the food to the baby.

---

5. The curve $y = 5 - e^{2x}$ intersects the coordinate axes at the points A and B.
(i) Given that the line AB passes through the point with coordinates (ln 5, k), find the value of k.
(ii) In order to solve the equation $x = \ln\sqrt{9 - x}$, a graph of a suitable straight line is drawn on the same set of axes as the graph of $y = 5 - e^{2x}$. Find the equation of this straight line.

(i) When $x = 0$, $y = 4$

When $y = 0$,

$$5 - e^{2x} = 0$$
$$e^{2x} = 5$$
$$x = \frac{1}{2}\ln 5$$

$$m_{AB} = \frac{4}{-\frac{1}{2}\ln 5} = -\frac{8}{\ln 5}$$

Equation of AB,

$$y - 4 = -\frac{8}{\ln 5}x$$
$$y = 4 - \frac{8}{\ln 5}x$$

At (ln 5, k),

$$k = 4 - \frac{8}{\ln 5}(\ln 5)$$
$$= -4$$

(ii)

$$x = \ln\sqrt{9 - x}$$
$$= \frac{1}{2}\ln(9 - x)$$
$$2x = \ln(9 - x)$$
$$e^{2x} = 9 - x$$
$$-e^{2x} = x - 9$$
$$5 - e^{2x} = x - 4$$

Insert $y = x - 4$

6. Without using a calculator, find the fractions p and q, for which $\dfrac{\sqrt{2}+\sqrt{6}}{\sqrt{12}+\sqrt{3}}$ can be expressed as $p\sqrt{2}+q\sqrt{6}$.

$$\begin{aligned}
\frac{\sqrt{2}+\sqrt{6}}{\sqrt{12}+\sqrt{3}} &= \frac{\sqrt{2}+\sqrt{6}}{2\sqrt{3}+\sqrt{3}} \\
&= \frac{(\sqrt{2}+\sqrt{6})}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{\sqrt{6}+3\sqrt{2}}{9} \\
&= \frac{1}{9}\sqrt{6}+\frac{1}{3}\sqrt{2}
\end{aligned}$$

---

7. (a) A triangle has a base $(3\sqrt{5}+\sqrt{7})$ cm and an area $(5\sqrt{35}+37)$ cm^2.
Find the height of the triangle in the form $(a\sqrt{5}+b\sqrt{7})$ cm, where a and b are integers.

(b) Given that $\cos\theta = \dfrac{1}{5-2\sqrt{3}}$, where θ is acute, express $\sec^2\theta$ in the form $m+n\sqrt{3}$ where m and n are integers.

(a) Height
$$\begin{aligned}
&= \frac{2\text{Area}}{\text{Base}} \\
&= \frac{2(5\sqrt{35}+37)}{3\sqrt{5}+\sqrt{7}} \times \frac{3\sqrt{5}-\sqrt{7}}{3\sqrt{5}-\sqrt{7}} \\
&= \frac{2(75\sqrt{7}-35\sqrt{5}+111\sqrt{5}-37\sqrt{7})}{45-7} \\
&= \frac{2(38\sqrt{7}+76\sqrt{5})}{38} \\
&= (2\sqrt{7}+4\sqrt{5})\text{ cm}
\end{aligned}$$

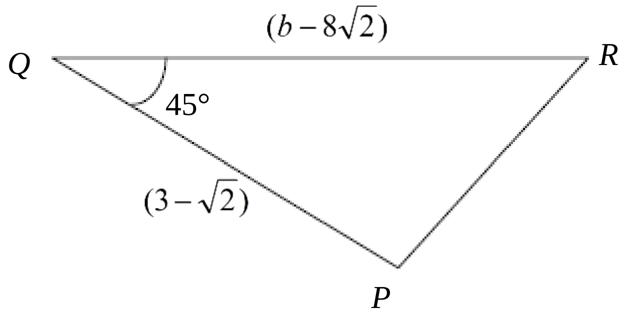
(b)
$$\begin{aligned}
\sec^2\theta &= \frac{1}{\cos^2\theta} \\
&= (5-2\sqrt{3})^2 \\
&= 25-20\sqrt{3}+12 \\
&= 37-20\sqrt{3}
\end{aligned}$$

8. In ΔPQR shown below, sides PQ and QR are $(3-\sqrt{2})$ cm and $(b-8\sqrt{2})$ cm respectively and $\angle PQR = 45°$.

Given that the area of ΔPQR is $(13\sqrt{2}-18)$ cm^2,

(i) show that $b = 12$,

(ii) find the perpendicular distance from R to PQ, leaving your answer in the form $p\sqrt{2}-q$.



(i) Area of $\Delta PQR = \frac{1}{2} \times (b-8\sqrt{2})(3-\sqrt{2})\sin 45°$

$$(13\sqrt{2}-18) = \frac{1}{2} \times (b-8\sqrt{2})(3-\sqrt{2})\left(\frac{\sqrt{2}}{2}\right)$$

$$b-8\sqrt{2} = \frac{4(13\sqrt{2}-18)}{3\sqrt{2}-2}$$

$$= \frac{4(13\sqrt{2}-18)}{3\sqrt{2}-2} \times \frac{3\sqrt{2}+2}{3\sqrt{2}+2}$$

$$= \frac{4(78+26\sqrt{2}-54\sqrt{2}-36)}{14}$$

$$= 12-8\sqrt{2}$$

$\therefore b = 12$

(ii) find the perpendicular distance from R to PQ, leaving your answer in the form $p\sqrt{2}-q$.

Perpendicular distance $= \frac{2(13\sqrt{2}-18)}{3-\sqrt{2}}$

$$= \frac{2(13\sqrt{2}-18)}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

$$= \frac{2(39\sqrt{2}+26-54-18\sqrt{2})}{7}$$

$$= \frac{2(21\sqrt{2}-28)}{7}$$

$$= (6\sqrt{2}-8)\text{ cm}$$

---

9. Simplify $\dfrac{x^{3n-1}y^2 \times 27^{\frac{2}{3}}}{3(\sqrt{x})^{2n}(y^{5n+1})}$.

$$\frac{x^{3n-1}y^2 \times 27^{\frac{2}{3}}}{3(\sqrt{x})^{2n}(y^{5n+1})} = \frac{x^{3n-1}y^2 \times 3^2}{3x^n y^{5n+1}}$$

$$= 3x^{2n-1}y^{1-5n}$$

10. Without using a calculator,

(i) find the value of r and of n, given that $\dfrac{3x^r}{r^2} \times \dfrac{2(r^{6-r})^2}{27x} = nx^2$,

(ii) simplify $\dfrac{3+\sqrt{2}}{2\sqrt{2}-1}$ in the form $a+b\sqrt{2}$.

(i) $\dfrac{3x^r}{r^2} \times \dfrac{2(r^{6-r})^2}{27x} = nx^2$

$\dfrac{2}{9}x^{r-1}r^{10-2r} = nx^2$

Comparing indices and coefficient,

$$
\begin{aligned}
r-1 &= 2 \\
r &= 3 \\
n &= \frac{2}{9}r^{10-2r} \\
&= \frac{2}{9}(3^4) \\
&= 18
\end{aligned}
$$

(ii) simplify $\dfrac{3+\sqrt{2}}{2\sqrt{2}-1}$ in the form $a+b\sqrt{2}$.

$$
\begin{aligned}
\frac{3+\sqrt{2}}{2\sqrt{2}-1} &= \frac{3+\sqrt{2}}{2\sqrt{2}-1} \times \frac{2\sqrt{2}+1}{2\sqrt{2}+1} \\
&= \frac{6\sqrt{2}+3+4+\sqrt{2}}{7} \\
&= \frac{7\sqrt{2}+7}{7} \\
&= \sqrt{2}+1
\end{aligned}
$$

---

11. Given that $(\sqrt{6}-2)x = (\sqrt{6}+2)$,

(i) find x in the form of $a+b\sqrt{6}$,

(ii) evaluate $x+\dfrac{1}{x}$ without using your calculator.

(i)
$$
\begin{aligned}
x &= \frac{\sqrt{6}+2}{\sqrt{6}-2} \\
&= \frac{\sqrt{6}+2}{\sqrt{6}-2} \times \frac{\sqrt{6}+2}{\sqrt{6}+2} \\
&= \frac{6+4\sqrt{6}+4}{2} \\
&= 5+2\sqrt{6}
\end{aligned}
$$

(ii)
$$
\begin{aligned}
x+\frac{1}{x} &= 5+2\sqrt{6}+\frac{1}{5+2\sqrt{6}} \\
&= 5+2\sqrt{6}+\frac{1}{5+2\sqrt{6}} \times \frac{5-2\sqrt{6}}{5-2\sqrt{6}} \\
&= 5+2\sqrt{6}+5-2\sqrt{6} \\
&= 10
\end{aligned}
$$

12. Simplify $\dfrac{a^n b^{n-1} - a^{2n+1}}{b^{2n-1} - b^n a^{n+1}}$.

$$\frac{a^n b^{n-1} - a^{2n+1}}{b^{2n-1} - b^n a^{n+1}} = \frac{a^n (b^{n-1} - a^{n+1})}{b^n (b^{n-1} - a^{n+1})}$$
$$= \frac{a^n}{b^n}$$

---

13. Simplify $\dfrac{81^{x+1} + 16(9^{2x})}{3^{x-3} \cdot 27^{x+2}}$.

$$\frac{81^{x+1} + 16(9^{2x})}{3^{x-3} \cdot 27^{x+2}} = \frac{81(3^{4x}) + 16(3^{4x})}{3^{x-3} \cdot 3^{3x+6}}$$
$$= \frac{97(3^{4x})}{3^{4x+3}}$$
$$= \frac{97}{27}$$

---

14. Find the solutions to the equation $(2\sqrt{3}+1)x^2 - (\sqrt{3}+2)x + 1 - \sqrt{3} = 0$, expressing your answers in the form $\dfrac{a\sqrt{3}+b}{11}$, where a and b are integers, when necessary.

$$x = \frac{(\sqrt{3}+2) \pm \sqrt{(\sqrt{3}+2)^2 - 4(2\sqrt{3}+1)(1-\sqrt{3})}}{2(2\sqrt{3}+1)}$$
$$= \frac{(\sqrt{3}+2) \pm \sqrt{3 + 4\sqrt{3} + 4 - 4(-6 + \sqrt{3} + 1)}}{2(2\sqrt{3}+1)}$$
$$= \frac{(\sqrt{3}+2) \pm \sqrt{27}}{2(2\sqrt{3}+1)}$$
$$= \frac{(\sqrt{3}+2) \pm 3\sqrt{3}}{2(2\sqrt{3}+1)}$$

$$x = \frac{4\sqrt{3}+2}{2(2\sqrt{3}+1)} \quad \text{or} \quad x = \frac{2-2\sqrt{3}}{2(2\sqrt{3}+1)}$$
$$= 1 \qquad\qquad = \frac{1-\sqrt{3}}{2\sqrt{3}+1} \times \frac{2\sqrt{3}-1}{2\sqrt{3}-1}$$
$$= \frac{2\sqrt{3} - 1 - 6 + \sqrt{3}}{11}$$
$$= \frac{3\sqrt{3} - 7}{11}$$

15. Show that if n is a positive integer, $7(9^{n+1})-6(3^{2n})+3^{2n+3}$ is exactly divisible by 14.

$7(9^{n+1})-6(3^{2n})+3^{2n+3}$

$=7(9)(3^{2n})-6(3^{2n})+(27)3^{2n}$

$=(3^{2n})(63-6+27)$

$=84(3^{2n})$

$=14(6)(3^{2n})$

Since $7(9^{n+1})-6(3^{2n})+3^{2n+3}$ is a multiple of 4, it is exactly divisible by 14.

---

16. The equation $2^{2a+b}-7(2^{a+b})=\frac{3}{2}(2^{a})-8$ has a solution $a=1$.

(i) Find the value of b.
(ii) Hence, find the other solution of a.

(i) $2^{2a+b}-7(2^{a+b})=\frac{3}{2}(2^{a})-8$

$2^{2+b}-7(2^{1+b})=\frac{3}{2}(2)-8$

$4(2^{b})-7(2)(2^{b})=-5$

$10(2^{b})=5$

$2^{b}=\frac{1}{2}$

$b=-1$

(ii) $2^{2a-1}-7(2^{a-1})=\frac{3}{2}(2^{a})-8$

$2^{2a}-7(2^{a})=3(2^{a})-16$

$(2^{a})^{2}-10(2^{a})+16=0$

Let $t=2^{a}$.

$t^{2}-10t+16=0$

$(t-8)(t-2)=0$

$t=8,\ 2$

$2^{a}=2^{3},\ 2^{1}$

$a=3,\ 1$

The other solution is $a=3$.

---

17. Solve the equation

(i) $\frac{x+5}{\sqrt{7x+29}}=1$

$(x+5)^{2}=7x+29$

$x^{2}+10x+25=7x+29$

$x^{2}+3x-4=0$

$(x+4)(x-1)=0$

$x=-4,\ 1$

(ii) $25^{x}-2(5^{x+1})+25=0$

$(5^{x})^{2}-10(5^{x})+25=0$

$(5^{x}-5)^{2}=0$

$5^{x}=5$

$x=1$

(iii) $2e^x = 7\sqrt{e^x} - 3$

$2e^x = 7e^{\frac{x}{2}} - 3$

Let $t = e^{\frac{x}{2}}$.

$2t^2 = 7t - 3$

$2t^2 - 7t + 3 = 0$

$(2t-1)(t-3) = 0$

$t = \frac{1}{2},\ 3$

$e^{\frac{x}{2}} = \frac{1}{2},\ 3$

$\frac{x}{2} = \ln\frac{1}{2},\ \ln 3$

$x = 2\ln\frac{1}{2},\ 2\ln 3$

$= -1.38, 2.18$

---

18. Solve the equation $\sqrt{13-2x} - x = 1$.

$\sqrt{13-2x} - x = 1$

$\sqrt{13-2x} = 1 + x$

$13 - 2x = (1+x)^2$

$13 - 2x = x^2 + 2x + 1$

$x^2 + 4x - 12 = 0$

$(x+6)(x-2) = 0$

$x = -6\ (\text{NA}),\ 2$

---

19. Solve the following equations.

(a) $2 + \sqrt{2x-1} = x$

$\sqrt{2x-1} = x - 2$

$2x - 1 = (x-2)^2$

$2x - 1 = x^2 - 4x + 4$

$x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0$

$x = 1\ (\text{NA}),\ 5$

(b) $(\sqrt{81})^x - 3^{x+1} = 3^x - 3$

$9^x - 3(3^x) - 3^x - 3 = 0$

$(3^x)^2 - 4(3^x) - 3 = 0$

Let $t = 3^x$.

$t^2 - 4t + 3 = 0$

$(t-1)(t-3) = 0$

$t = 1,\ 3$

$3^x = 1,\ 3$

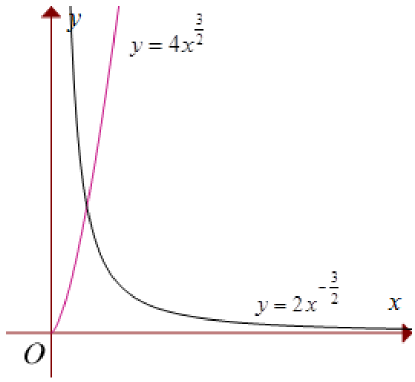
$x = 0,\ 1$

---

20. (i) On the same axes, sketch the graphs of $y = 4x^{\frac{3}{2}}$ and $y = 2x^{-\frac{3}{2}}$.

(ii) Determine, with explanation, the number of solutions of x for the equation $4x^{\frac{3}{2}} + 2x^{-\frac{3}{2}} = 0$.

(i) On the same axes, sketch the graphs of $y = 4x^{\frac{3}{2}}$ and $y = 2x^{-\frac{3}{2}}$.



(ii) Determine, with explanation, the number of solutions of x for the equation $4x^{\frac{3}{2}} + 2x^{-\frac{3}{2}} = 0$.

$4x^{\frac{3}{2}} + 2x^{-\frac{3}{2}} = 0 \Leftrightarrow 4x^{\frac{3}{2}} = -2x^{-\frac{3}{2}}$

No solution.

---

21. Show that $\log_9 xy = \frac{1}{2}\log_3 x + \frac{1}{2}\log_3 y$.

$$
\begin{aligned}
\log_9 xy &= \log_9 xy \\
&= \log_9 x + \log_9 y \\
&= \frac{\log_3 x}{\log_3 9} + \frac{\log_3 y}{\log_3 9} \\
&= \frac{\log_3 x}{2} + \frac{\log_3 y}{2} \\
&= \frac{1}{2}\log_3 x + \frac{1}{2}\log_3 y
\end{aligned}
$$

---

22. (a) Given that $\log_{13} x + \log_{13} y = \log_{13}(x - y)$, express x in terms of y. [3]

$$\log_{13} x + \log_{13} y = \log_{13}(x - y)$$
$$\log_{13} xy = \log_{13}(x - y)$$
$$xy = x - y$$
$$x - xy = y$$
$$x(1 - y) = y$$
$$x = \frac{y}{1 - y}$$

(b) Given that $u = \log_3 z$, find, in terms of u,

(i) $$\log_3 \frac{3}{z} = \log_3 3 - \log_3 z$$
$$= 1 - u$$

(ii) $$\log_3 27z = 3\log_3 3 + \log_3 z$$
$$= 3 + u$$

(iii) $$\log_z 9 = \frac{\log_3 9}{\log_3 z}$$
$$= \frac{2}{u}$$

---

23. (i) Evaluate $\log_2 x$ given that $\log_x 8 = \frac{2}{3}$.

$$\log_x 8 = \frac{2}{3}$$
$$\frac{\log_2 8}{\log_2 x} = \frac{2}{3}$$
$$\frac{3}{\log_2 x} = \frac{2}{3}$$
$$\log_2 x = \frac{9}{2}$$

(ii) Given that $t = \sqrt{5} + \sqrt{3}$, find the value of $t - \frac{2}{t}$.

$$t - \frac{2}{t} = \sqrt{5} + \sqrt{3} - \frac{2}{\sqrt{5} + \sqrt{3}} \times \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} - \sqrt{3}}$$
$$= \sqrt{5} + \sqrt{3} - (\sqrt{5} - \sqrt{3})$$
$$= 2\sqrt{3}$$

---

24. A certain virus grows from an initial population of 1000 to size S at the end of t days.

It is given that $S = 1000(2^{kt})$, where k is a constant, and that the population doubles at the end of 20 days.

(i) Show that the value of $k = \dfrac{1}{20}$.

(ii) Find the size of the population of the virus, to the nearest integer, at the end of 30 days.

(iii) Find the number of days taken for the virus to reach a population of 8000.

(i) When $t = 20$,
$$2000 = 1000(2^{20k})$$
$$2^{kt} = 2$$
$$20kt = 1$$
$$k = \frac{1}{20}$$

(ii) When $t = 30$,
$$S = 1000\,(2^{30\times\frac{1}{20}})$$
$$= 2820$$

(iii) When $S = 8000$,
$$8000 = 1000\,(2^{\frac{t}{20}})$$
$$2^{\frac{t}{20}} = 8 = 2^3$$
$$\frac{k}{20} = 3$$
$$k = 60$$

It takes 60 days.

---

25. Food ordered from a particular caterer is delivered at a temperature of 75°C. It subsequently cools in such a way that its temperature, T°C, t hours after delivery, is given by $T = 25 + Ae^{-kt}$, where A and t are constants.

(i) Find the value of A.

When $t = 2$, the temperature of the food is 40°C.

(ii) Find the value of k correct to 3 significant figures.

(iii) State, with explanations, the temperature of the room where the food is placed.

(i) When $t = 0$, $T = 75$,
$$75 = 25 + A$$
$$\therefore A = 50$$

(ii) When $t = 2$, $T = 40$,
$$40 = 25 + 50e^{-2k}$$
$$50e^{-2k} = 15$$
$$e^{-2k} = 0.3$$
$$-2k = \ln 0.3$$
$$k = -\frac{\ln 0.3}{2}$$
$$= 0.602$$

(iii) As $t \to \infty$, $T \to 25$,

$\therefore$ room temperature = 25°C.

26. A biologist conducted a research on a particular type of bacteria.
At the start of the experiment, there were 100 bacteria in the growth medium.
After 6 hours, there were 450 bacteria.
Assume that the growth of bacteria follows the equation $A = A_0 e^{kt}$, where A is the number of bacteria, A_0 is the initial number of bacteria, t is the time of growth in hours and k is the growth constant.

(i) Show that $A_0 = 100$ and $k = 0.251$.
(ii) Determine the number of hours for the number of bacteria to be 10 times the original number.
(iii) Determine the number of bacteria present after 1 day of growth.
(iv) Find the rate of change of A after 2 hours.

(i) When $t = 0$, $A = 100$, $A_0 = 100$

When $t = 6$, $A = 450$, $450 = 100e^{6k}$

$$e^k = 4.5$$

$$6k = \ln 4.5$$

$$k = \frac{\ln 4.5}{6}$$

$$= 0.25067$$

$$\approx 0.251$$

(ii) When $A = 10A_0$, $e^{0.25067t} = 10$

$$0.25067\,t = \ln 10$$

$$t = \frac{\ln 10}{0.25067}$$

$$= 9.19$$

(iii) When $t = 24$, $A = 100e^{24 \times 0.25067}$

$$= 41000$$

(iv) $\frac{dA}{dt} = A_0 k e^{kt}$,

When $t = 2$, $\frac{dA}{dt} = 100 \times 0.25067 e^{2(0.25067)}$

$= 41.4$ bacteria per hour

---

27. The population, x, of a certain micro-organism, present at t hours after the initial observation, is given by the formula $x = 200(1 + e^{-0.2t})$.

(i) Find the initial population of the micro-organism.
(ii) Find, to the nearest hundred, the population of the micro-organism, one day after the initial observation.
(iii) Estimate the population of the micro-organism several years after the initial observation.
(iv) Sketch the graph of $x = 200(1 + e^{-0.2t})$.

(i) When $t = 0$, $x = 200(1+1)$

$$= 400$$

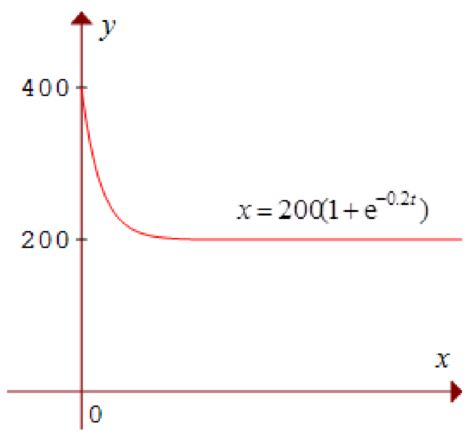
(ii) When $t = 24$, $x = 200(1 + e^{-4.8})$

$$\approx 200$$

(iii) As $t \to \infty$, $e^{-0.2t} \to 0$, $x \to 200$

Population of the micro-organism ≈ 200

(iv)



28. The temperature, $T\,°\mathrm{C}$, of a liquid after x minutes is given by formula $T = 100\ (0.9)^x$.

(i) Find the initial temperature of the liquid.
(ii) Find its temperature after 6 minutes.
(iii) The temperature decreases from its initial temperature by 19°C in m minutes. Find the value of m.
(iv) Find the temperature as x becomes very large.
(v) Sketch the graph of T against x.

(i) When $x = 0$, $T = 100$

(ii) When $x = 6$, $T = 100(0.9)$
≈ 53.1

(iii) When $T = 81$, $81 = 100(0.9^m)$

$0.9^m = 0.81$

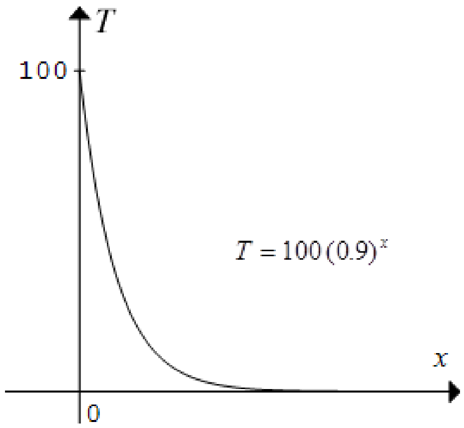
$m \lg 0.9 = \lg 0.81$

$m = \dfrac{\lg 0.81}{\lg 0.9}$

$= 2$

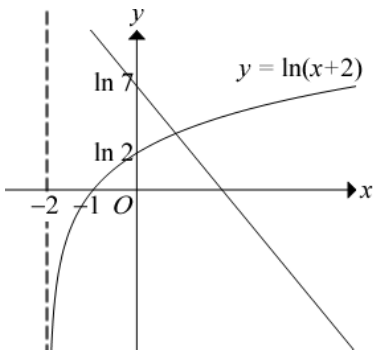
(iv) When $x \to \infty$, $T \to 0°\mathrm{C}$.

(v)



29. (i) Sketch the graph of $y = \ln(2 + x)$ for $x > -2$, showing clearly the asymptote and the axis intercepts.
(ii) By adding a suitable straight line to your sketch, determine the number of solution(s) of the equation $xe^x = 7 - 2e^x$.

(i)



(ii) $xe^x = 7 - 2e^x$

$xe^x + 2e^x = 7$

$e^x = \dfrac{7}{x+2}$

$x = \ln 7 - \ln(x+2)$

$\ln(x+2) = \ln 7 - x$

Insert $y = \ln 7 - x$

30. The equation of a curve is $\log_2 y = a\log_2 x + b$, where a and b are constants.

(a) (i) Given that the curve passes through the points (2, 8) and $\left(\frac{1}{2}, 32\right)$, find the value of a and of b,

(ii) Show that the relationship of x and y can be expressed in the form of $y = kx^n$ and state the value of k and n.

(b) Given that $\log_{27} x^3 = \log_9 u$, express u in terms of x.

(a) (i) At (2, 8), $\log_2 8 = a\log_2 2 + b$

$3 = a + b$ ....(1)

At $\left(\frac{1}{2}, 32\right)$, $\log_2 32 = a\log_2\left(\frac{1}{2}\right) + b$

$5 = -a + b$ ....(2)

(1) − (2) $-2 = 2a$

$a = -1$

$b = 4$

(ii) Show that the relationship of x and y can be expressed in the form of $y = kx^n$ and state the value of k and n.

$$\log_2 y = -\log_2 x + 4$$
$$\log_2 y = -\log_2 x + 4\log_2 2$$
$$\log_2 y = \log_2 \frac{16}{x}$$
$$y = \frac{16}{x}$$
$$y = 16x^{-1}$$
$$k = 16$$
$$n = -1$$

(b) Given that $\log_{27} x^3 = \log_9 u$, express u in terms of x.

$$\log_{27} x^3 = \log_9 u$$
$$\frac{\log_3 x^3}{\log_3 27} = \frac{\log_3 u}{\log_3 9}$$
$$\frac{3\log_3 x}{3\log_3 3} = \frac{\log_3 u}{2\log_3 3}$$
$$\log_3 x = \frac{\log_3 u}{2}$$
$$\log_3 u = 2\log_3 x$$
$$\log_3 u = \log_3 x^2$$
$$u = x^2$$

31. Solve the equation $\lg(4^x - 10) - x\lg 2 = \lg 3$.

$$\lg(4^x - 10) - x\lg 2 = \lg 3$$
$$\lg(4^x - 10) = \lg 3 + \lg 2^x$$
$$\lg(4^x - 10) = \lg[3(2^x)]$$
$$4^x - 10 = 3(2^x)$$
$$(2^x)^2 - 3(2^x) - 10 = 0$$

Let $t = 2^x$.

$$t^2 - 3t - 10 = 0$$
$$(t-5)(t+2) = 0$$
$$t = 5,\ -2$$
$$2^x = 5,\ -2 \text{ (NA)}$$
$$x = \frac{\lg 5}{\lg 2}$$
$$= 2.32$$

---

32. (i) Show that $\log_x p = \dfrac{1}{\log_p x}$ where $p > 0$. Hence solve the equation $4\log_x 2 = 4 - \log_2 x$.

(ii) Solve the equation $\log_3 x - \log_9 x = (\log_3 x)(\log_9 x)$.

(i) $$\log_x p = \frac{\log_p p}{\log_p x} = \frac{1}{\log_p x}$$

$$4\log_x 2 = 4 - \log_2 x$$
$$\frac{4}{\log_2 x} = 4 - \log_2 x$$

Let $t = \log_2 x$.

$$\frac{4}{t} = 4 - t$$
$$4 = 4t - t^2$$
$$t^2 - 4t + 4 = 0$$
$$(t-2)^2 = 0$$
$$t = 2$$
$$\log_2 x = 2$$
$$x = 2^2 = 4$$

(ii) $\log_3 x - \log_9 x = (\log_3 x)(\log_9 x)$.

$$\log_3 x - \frac{\log_3 x}{\log_3 9} = \log_3 x\left(\frac{\log_3 x}{\log_3 9}\right)$$
$$\log_3 x - \frac{\log_3 x}{2} = \log_3 x\left(\frac{\log_3 x}{2}\right)$$

Let $t = \log_3 x$.

$$t - \frac{t}{2} = t\left(\frac{t}{2}\right)$$
$$\frac{t}{2} = t\left(\frac{t}{2}\right)$$
$$\frac{t}{2} - t\left(\frac{t}{2}\right) = 0$$
$$\frac{t}{2}(1-t) = 0$$
$$t = 0,\ 1$$
$$\log_3 x = 0,\ 1$$
$$x = 1,\ 3$$

33. Solve the equation:

(i) $3(0.9)^x = 1.5$

$0.9^x = 0.5$

$x = \dfrac{\lg 0.5}{\lg 0.9}$

$\quad = 6.58$

(ii) $2\log_2 x - 6\log_x 2 = 1$

$2\log_2 x - \dfrac{6\log_2 2}{\log_2 x} = 1$

Let $t = \log_2 x$.

$2t - \dfrac{6}{t} = 1$

$2t^2 - t - 6 = 0$

$(2t+3)(t-2) = 0$

$t = -\dfrac{3}{2},\ 2$

$\log_2 x = -\dfrac{3}{2},\ 2$

$x = 2^{-\frac{3}{2}},\ 2^2$

$\quad = 0.354,\ 4$

---

34. Solve the following equations.

(i) $\sqrt{3x-9} - 3\sqrt{x-5} = 0$

$\sqrt{3x-9} = -3\sqrt{x-5}$

$3x - 9 = 9(x-5)$

$3x - 9 = 9x - 45$

$6x = 36$

$x = 6$

(ii) $e^x(2e^x - 39) = 20$

Let $t = e^x$.

$t(2t-39) = 20$

$2t^2 - 39t - 20 = 0$

$(2t+1)(t-20) = 0$

$t = -\dfrac{1}{2},\ 20$

$e^x = -\dfrac{1}{2}$ (NA), 20

$x = \ln 20$

$\quad = 3.00$

(iii) $\log_3 x = 16\ \log_x 3$

$\log_3 x = \dfrac{16\ \log_3 3}{\log_3 x}$

Let $t = \log_3 x$.

$t = \dfrac{16}{t}$

$t^2 = 16$

$t = -4,\ 4$

$\log_3 x = -4,\ 4$

$x = 3^{-4},\ 3^4$

$\quad = \dfrac{1}{81},\ 81$

(iv) $\log_2(2-3x) = 1 + 2\ \log_2 x$

$\log_2(2-3x) = \log_2 2 + 2\ \log_2 x$

$\qquad\qquad = \log_2(2x^2)$

$2 - 3x = 2x^2$

$2x^2 + 3x - 2 = 0$

$(2x-1)(x+2) = 0$

$x = \dfrac{1}{2},\ -2$ (NA)

Singapore Chinese Girls' School
Secondary 4(IP)
Integrated Mathematics II
Second Derivatives and Application

Name: ________________________________ (____)   Date: ________________

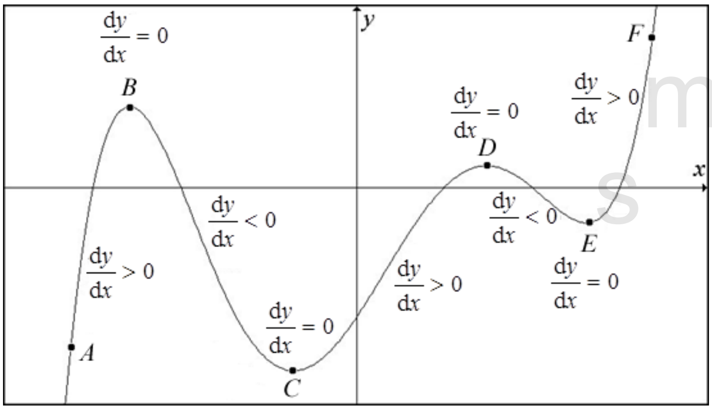
Class: Sec 4_____

Learning Objectives
At the end of the lesson, you will learn to • find the coordinates of a stationary or turning point of a given curves at using $\frac{\mathrm{d}y}{\mathrm{d}x}=0$. • observe the changes in gradients at a stationary point (or a turning point) to determine whether the point is a maximum point, minimum point or point of inflexion.

Worksheet 21: Nature of Stationary Points

Determination of Maximum and Minimum Points

Consider the graph of $y=\mathrm{f}(x)$ in the diagram below.



	Gradient of the curve		Gradient of the curve
Along AB	$\frac{\mathrm{d}y}{\mathrm{d}x}>0 \Rightarrow y$ increases as x increases	At D	$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow D$ is a stationary points
At B	$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow B$ is a stationary points	Along DE	$\frac{\mathrm{d}y}{\mathrm{d}x}<0 \Rightarrow y$ decreases as x increases
Along BC	$\frac{\mathrm{d}y}{\mathrm{d}x}<0 \Rightarrow y$ decreases as x increases	At E	$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow$ is a stationary points
At C	$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow C$ is a stationary points	Along EF	$\frac{\mathrm{d}y}{\mathrm{d}x}>0 \Rightarrow y$ increases as x increases
Along CD	$\frac{\mathrm{d}y}{\mathrm{d}x}>0 \Rightarrow y$ increases as x increases		

Stationary points

Points B, C, D and E where $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ are called **stationary points.**

Minimum points

Point C (or E) is called **a minimum point** because $y=\mathrm{f}(x)$ has **a minimum value** as compared to the points along BC and CD (along DE and EF).

Points C and E are also known as **local minimums** as they **do not represent the minimum value** of the whole curve.

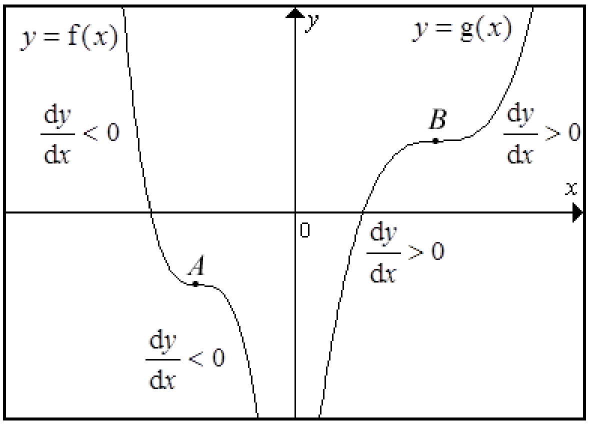
Maximum points

Point B (or D) is called **a maximum point** because $y=\mathrm{f}(x)$ has a **maximum value** as compared to the points along AB and BC (along CD and DE).

Points B and D are also known as **local maximums** as they **do not represent the maximum value** of the whole curve.

Points of Inflexion

Consider the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{g}(x)$ in the diagram below.



At points A and B, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, but they are both neither maximum nor minimum points.

We call these points the **stationary points of inflexion.**

Summary

Given a curve $y=\mathrm{f}(x)$,

(a) When $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ at $x=a$, then $(a,\mathrm{f}(a))$ is a **stationary point**. $(a,\mathrm{f}(a))$ is **a turning point** if it is either a **maximum point** or ***a* minimum point.**

(b) If $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes sign from **positive to negative** as it passes through $x=a$, then it a **maximum** point.

(c) If $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes sign from **negative to positive** as it passes through $x=a$, then it is a **minimum** point.

(d) If $\frac{\mathrm{d}y}{\mathrm{d}x}$ **does not change** sign as it passes through $x=a$, then it is a stationary **point of inflexion**.

Example 1

Find, by calculus, the stationary point of the curve $y = \frac{x^3}{3} - x$.

Observe the changes in the sign of the gradient of the curve and determine the nature of the point.

Hence, sketch the graph of $y = \frac{x^3}{3} - x$.

$$y = \frac{x^3}{3} - x$$

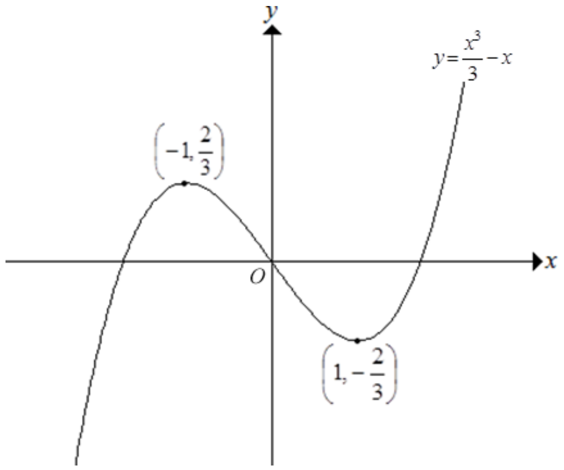
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - 1$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad x^2 - 1 = 0$

$$x = 1,\ -1$$

$$y = -\frac{2}{3}, \frac{2}{3}$$

The stationary point are $\left(1, -\frac{2}{3}\right)$ and $\left(-1, \frac{2}{3}\right)$.



At $\left(1, -\frac{2}{3}\right)$,

x	1^-	1	1^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	–	0	+
Slope	\	—	/

As x increases through <u>$x = 1$</u>, $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes sign from <u>negative</u> to <u>positive</u>, $\left(1, -\frac{2}{3}\right)$ is a <u>minimum</u> point.

At $\left(-1, \frac{2}{3}\right)$,

x	-1^-	1	-1^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	–	0	+
Slope	/	—	\

As x increases through x = <u>-1</u>, $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes sign <u>positive</u> to <u>negative</u>, $\left(-1, \frac{2}{3}\right)$ is a <u>maximum</u> point.

Example 2

Given that curve $y=(x-2)^3$, find the coordinates of the stationary point and determine its nature.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3(x-2)^2$$

When $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $\quad 3(x-2)^2=0$

$x=2$

$y=2$

x	2^-	2	2^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	+
Slope	/	—	/

$\therefore$ (2, 0) is a point of inflexion.

Example 3

A curve has the equation $y=\dfrac{2x+1}{x-1}$, where $x\neq 1$.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(b) State whether this curve has a stationary point. Justify your answer. **(Pg409Q2)**

(a) $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x-1)(2)-(2x+1)(1)}{(x-1)^2}$$

$$=\frac{2x-2-2x-1}{(x-1)^2}$$

$$=-\frac{3}{(x-1)^2}$$

(b) Since $(x-1)^2>0$, for $x\neq 1$, $-\dfrac{3}{(x-1)^2}<0$

$\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x}<0$,

Hence, y is a decreasing function without any stationary points.

Example 4

The equation of a curve is $y = x(x-1)^3$.

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ can be expressed in the form $(x-1)^2(ax+b)$ where a and b are integers.

(ii) Find the coordinates of the stationary points on the curve.

(iii) Observe the change in sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as x increases through each of the stationary points. Hence, deduce the nature of the points.

(iv) Sketch the graph of $y = x(x-1)^3$.

(i) $y = x(x-1)^3$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x-1)^3(1) + x(3)(x-1)^2 \\ &= (x-1)^2(x-1+3x) \\ &= (x-1)^2(4x-1)\end{aligned}$$

(ii) When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad (x-1)^2(4x-1) = 0$

$$x = 1,\ \frac{1}{4}$$

When $x = 0$, $y = 0$

When $x = \dfrac{1}{4}$, $y = \dfrac{1}{4}\left(-\dfrac{3}{4}\right)^3$

$$= -\frac{27}{256}$$

The stationary points are (1, 0) and $\left(\dfrac{1}{4}, -\dfrac{27}{256}\right)$.

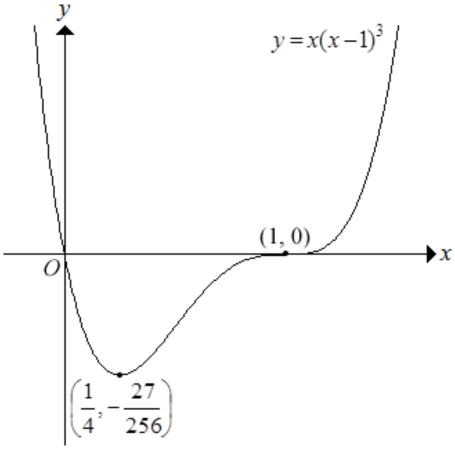
(iii)

x	1^-	1	1^+	$\frac{1}{4}^-$	$\frac{1}{4}$	$\frac{1}{4}^+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	+	–	0	+
Slope	/	—	/	\	—	/

(1, 0) is appoint of inflexion.

$\left(\dfrac{1}{4}, -\dfrac{27}{256}\right)$ is a minimum point.

(iv)



Example 5

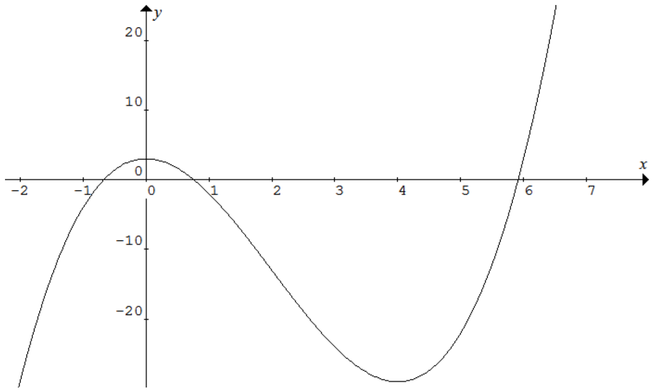
Given that $y = x^3 - 6x^2 + 3$, find

(a) an expression for $\frac{dy}{dx}$,

(b) the $x-$coordinates of the stationary points.

Show that the gradient of the curve between the stationary points is always negative. **(Pg409Q9)**

(a) $x = 1,\ x = 4$ $\qquad \frac{dy}{dx} = 3x^2 - 12x$



(b) When $\frac{dy}{dx} = 0$, $3x^2 - 12x = 0$

$$3x(x-4) = 0$$
$$x = 0,\ x = 4$$

$$\frac{dy}{dx} = 3x^2 - 12x$$
$$= 3x(x-4)$$

For $0 < x < 4$, $0 < 3x < 12$

$$-4 < x - 4 < 0$$
$$\therefore 3x(x-4) < 0$$
$$\frac{dy}{dx} < 0$$

Hence, the gradient of the curve between the stationary points is always negative.

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Assignment 22: Nature of Stationary Points

1. Given that $y = 2x(1-x)^3$,
   (i) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ can be expressed in the form $(1-x)^2(ax-b)$,
   (ii) find the coordinates of the stationary points,
   (iii) by considering the sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, determine the nature of the stationary points.

Solution

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2(1-x)^3 - 6x(1-x)^2$

$= 2(1-x)^2(1-x-3x)$

$= (1-x)^2(2-8x)$

(ii) When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $2(1-x)^2(1-4x) = 0$

$x = 1,\ \dfrac{1}{4}$

$y = 0,\ \dfrac{27}{128}$

The two points are (1, 0) and $\left(\dfrac{1}{4},\ \dfrac{27}{128}\right)$ are stationary point.

(iii)

x	$\frac{1}{4}^-$	$\frac{1}{4}$	$\frac{1}{4}^+$	1^-	1	1^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	–	–	0	–
Slope	/	—	\	\	—	\

$\left(\dfrac{1}{4},\ \dfrac{27}{128}\right)$ is a maximum point.

(1, 0) is a point of inflexion.

2. Given that $y=(x-5)\sqrt{7+x}$,

(i) find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$,

(ii) find the x-coordinate of the stationary point.

(iii) by considering the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$, determine the nature of the stationary points. **(Pg409Q5)**

Solution

(i) $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{7+x}+(x-5)\times\frac{1}{2\sqrt{7+x}}\\ &= \frac{2(7+x)+x-5}{2\sqrt{7+x}}\\ &= \frac{3x+9}{2\sqrt{7+x}}\end{aligned}$$

(ii) When $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $3x+9=0$

$$\begin{aligned}x &= -3\\ y &= (-3-5)\sqrt{7-3}\\ &= -16\end{aligned}$$

(–3, –16) is a stationary point.

(iii) At (–3, –16),

x	-3^-	3	-3^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	–	0	+
Slope	\	—	/

(–3, –16) is a minimum point.

3. Given that $y = \dfrac{2x-1}{x^2+2}$,

(i) find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$,

(ii) find the x-coordinate of the stationary point.

(iii) show that y increases as x increases between the stationary points. **(Pg409Q10)**

Solution

(i)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(x^2+2)-2x(2x-1)}{(x^2+2)^2}\\ &= \frac{2x^2+4-4x^2+2x}{(x^2+2)^2}\\ &= \frac{-2x^2+2x+4}{(x^2+2)^2}\\ &= \frac{-2(x-2)(x+1)}{(x^2+2)^2}\end{aligned}$$

(ii) When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad -2(x+1)(x-2) = 0$

$x = -1,\ x = 2$

(iii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{-2(x+1)(x-2)}{(x^2+2)^2}$

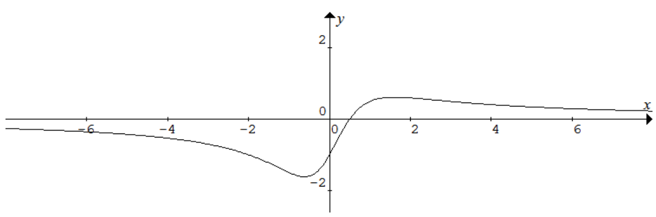
For $-1 < x < 2$, $\quad 0 < (x+1) < 3$

$-3 < (x-2) < 0$

$(x^2+2)^2 \ge 2$

$\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{2(x+1)(x-2)}{(x^2+2)^2} > 0$

Hence for $-1 < x < 2$, y increases as x increases between the stationary points.



4. Find the coordinates of the stationary point on the curve defined by $y = \dfrac{\sqrt{x-1}}{x}$. Determine the nature of this point.

$$\frac{dy}{dx} = \frac{x\left[\frac{1}{2}(x-1)^{-\frac{1}{2}}\right] - \sqrt{x-1}}{x^2}$$

$$= \frac{\frac{x}{2\sqrt{x-1}} - \sqrt{x-1}}{x^2}$$

$$= \frac{x - 2(x-1)}{2x^2\sqrt{x-1}}$$

$$= \frac{2-x}{2x^2\sqrt{x-1}}$$

When $\dfrac{dy}{dx} = 0$, $\quad 2 - x = 0$

$x = 2$

$y = \dfrac{1}{2}$

The stationary point is $\left(2, \dfrac{1}{2}\right)$.

x	2^-	2	2^+
$\dfrac{dy}{dx}$	$+$	0	$-$
Slope	/	—	\

Hence $\left(2, \dfrac{1}{2}\right)$ is a maximum point.

5. A curve has an equation of the form $y = ax + \frac{b}{x^2}$, where a and b are constants. Given that the curve has a stationary point at (3, 5). Find the value of a and of b. **(Pg409Q12)**

$\frac{\mathrm{d}y}{\mathrm{d}x} = a - \frac{2b}{x^3}$

At (3, 5), $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad a - \frac{2b}{27} = 0$

$a = \frac{2b}{27} \quad \ldots\ldots(1)$

$y = 5$, $\quad 5 = 3a + \frac{b}{9} \quad \ldots\ldots(2)$

Sub (1) into (2) $\quad 5 = 3\left(\frac{2b}{27}\right) + \frac{b}{9}$

$\frac{b}{3} = 5b = 15$

$a = \frac{10}{9}$

6. The curve $y = x^2(x-k)^2$, where k is a non-zero constant, has three stationary points.

(i) Find $\frac{dy}{dx}$ in terms of k.

(ii) Given that the curve passes through the point (1, 1), find the coordinates of each stationary point.

(iii) Determine the nature of the points.

(iv) Sketch the curve for $-1 \le x \le 3$. **(Pg410Q16)**

(i) $$\begin{aligned} \frac{dy}{dx} &= 2x(x-k)^2 + 2x^2(x-k) \\ &= 2x(x-k)(x-k+x) \\ &= 2x(x-k)(2x-k) \end{aligned}$$

(ii) At (1, 1), $$\begin{aligned} 1 &= (1-k)^2 \\ 1-k &= 1, -1 \\ k &= 0\ (\text{NA}), 2 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 2x(x-2)(2x-2) \\ &= 4x(x-2)(x-1) \end{aligned}$$

When $\frac{dy}{dx} = 0$, $\quad 4x(x-2)(x-1) = 0$

$$x = 0, 2, 1$$
$$y = 0, 0, 1$$

The three points are (0, 0), (1, 1) and (2, 0).

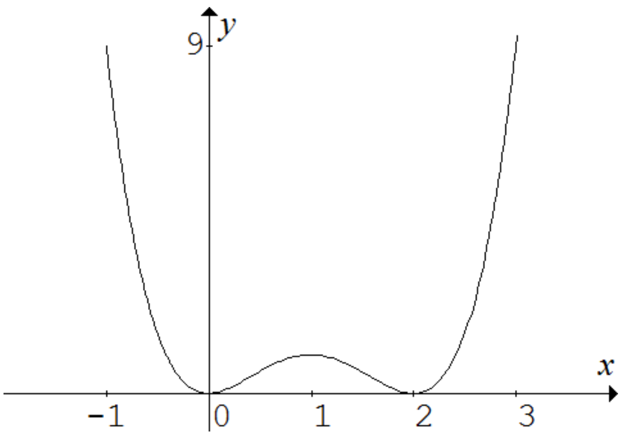
(iii)

x	0^-	0	0^+	1^-	1	1^+	2^-	2	2^+
$\frac{dy}{dx}$	–	0	+	+	0	–	–	0	+
Slope	\	—	/	/	—	\	\	—	/

(0, 0) and (2, 0) are minimum points

(1, 1) is a maximum point.

(iv)



7*. The curve $y = x^3 + ax^2 + bx + c$, where a, b and c are constants, touches the x-axis at $x = 1$ and crosses the x -axis at $x = 4$. Find

(i) the values of a, b and c,

(ii) the coordinates of all the stationary points on the curve, and state the nature of each point,

(iii) the equation of the tangent to the curve at $x = 0$. **(Pg410Q18)**

(i) At (1, 0), $0 = 1 + a + b + c$ ……(1)

At (4, 0), $0 = 64 + 16a + 4b + c$ ……(2)

(2)–(1) $0 = 63 + 15a + 3b$

$0 = 21 + 5a + b$

$b = -21 - 5a$ ……(3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2ax + b$$

At (1, 0), $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $3 + 2a + b = 0$ ……(4)

Sub (3) into (4) $3 + 2a - 21 - 5a = 0$

$3a = -18$

$a = -6$

$b = 9$

$c = -4$

(ii) $y = x^3 - 6x^2 + 9x - 4$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + 9$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3, 1$

$y = -4, 0$

The stationary points are (3, –4) and (1, 0).

(iii)

x	1^-	1	1^+	3^-	3	3^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	–	–	0	+
Slope	/	—	\	\	—	/

(1, 0) is a maximum point.

(3, –4) is a minimum point.

(iv) When $x = 0$, $y = -4$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 9$$

The equation of tangent is $y = 9x - 4$

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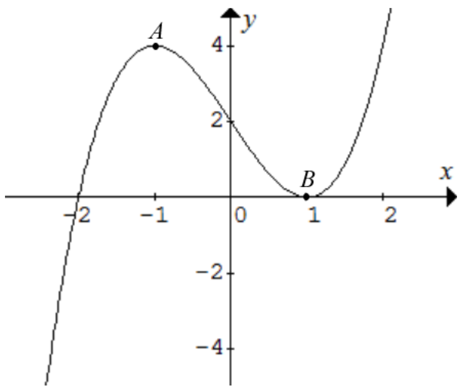
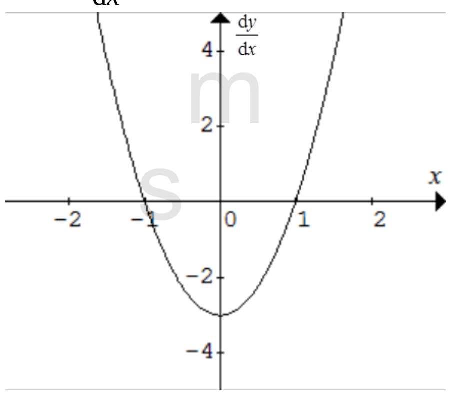
Learning Objectives
At the end of the lesson, you will learn to • use the value of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at a stationary point $A(a, \mathrm{f}(a))$ to determine whether A is a maximum or a minimum point.

Worksheet 23: Second Derivative

Determining Maximum and Minimum Points Using the Second Derivative of *y*

The diagrams below shows the curve $y = \mathrm{f}(x)$ and the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ against x.





From the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ against x, it is observed that

(a) $\frac{\mathrm{d}y}{\mathrm{d}x}$ decreases as x increases through A,

(b) the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at A, i.e. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0$ at A.

Thus a turning point is **a maximum** when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0$.

(c) $\frac{\mathrm{d}y}{\mathrm{d}x}$ increases as x increases through B,

(d) the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at B, i.e. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$ at B.

Thus a turning point is **a minimum** when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$.

Summary

Given a curve $y = \mathrm{f}(x)$,

(a) When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq 0$ at $x = a$, then $(a,\ \mathrm{f}(a))$ is a turning point.

(b) If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, then $(a,\ \mathrm{f}(a))$ is a minimum point.

(c) If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, then $(a,\ \mathrm{f}(a))$ is a maximum point.

Example 1

Given that $y = (x-1)^2(x+2)$, find

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$,

(b) the stationary values of y and determine the nature of these values.

(a)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2(x-1)(x+2)+(x-1)^2\\ &= 2x^2+2x-4+x^2-2x+1\\ &= 3x^2-3\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(3x^2-3)\\ &= 6x\end{aligned}$$

(b) When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad x = -1,\ 1$

When $x = -1$, $\quad y = 4$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$$

$\therefore$ y is a maximum value.

When $x = 1$, $\quad y = 0$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$$

$\therefore$ y is a minimum value.

Example 2

Find the coordinates of the turning point of the curve $y = 8x + \dfrac{1}{2x^2}$ and determine whether this point is a maximum or minimum point. **(Pg409Q8)**

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(8x + \frac{1}{2}x^{-2}\right)\\ &= 8 - x^{-3}\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(8 - x^{-3})\\ &= 3x^{-4}\end{aligned}$$

When $\dfrac{dy}{dx} = 0$, $8 - x^{-3} = 0$

$$x = \frac{1}{2}$$

$$\begin{aligned}y &= 8\left(\frac{1}{2}\right) + \frac{1}{2\left(\frac{1}{2}\right)^2}\\ &= 6\end{aligned}$$

When $x = \dfrac{1}{2}$, $\dfrac{d^2y}{dx^2} = 3\left(\dfrac{1}{2}\right)^{-4}$

$$= 48 > 0$$

$\left(\dfrac{1}{2},\ 6\right)$ is a minimum point.

Example 3

Show that the curve $y=\dfrac{x+2}{x-3}$ where $x\neq 3$ has neither a maximum nor a minimum point.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x-3)-(x+2)}{(x-3)^2}\\ &= -\frac{5}{(x-3)^2}\end{aligned}$$

Since $x\neq 3$, $(x-3)^2>0$, $-\dfrac{5}{(x-3)^2}<0$.

Hence, $\dfrac{\mathrm{d}y}{\mathrm{d}x}\neq 0$.

$\therefore\ y=\dfrac{x+2}{x-3}$ does not have any turning points.

Example 4

Show that the curve $y=\dfrac{2x-1}{1-x}$ where $x\neq 1$ has neither a maximum nor a minimum point.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(1-x)+(2x-1)}{(1-x)^2}\\ &= \frac{1}{(1-x)^2}\end{aligned}$$

Since $(1-x)^2>0$, $\dfrac{1}{(1-x)^2}>0$.

Hence, $\dfrac{\mathrm{d}y}{\mathrm{d}x}\neq 0$.

$\therefore\ y=\dfrac{2x-1}{1-x}$ does not have any turning points.

Example 5

Given that curve $y = (x+1)^4$, find

(a) an expression for $\dfrac{dy}{dx}$,

(b) the coordinates of the stationary point and determine its nature.

(a) $\dfrac{dy}{dx} = 4(x+1)^3$

(b) When $\dfrac{dy}{dx} = 0$, $\quad 4(x+1)^3 = 0$

$x = -1$

$y = 0$

$\dfrac{d^2y}{dx^2} = 12(x+1)^2$

When $x = -1$, $\quad \dfrac{d^2y}{dx^2} = 0$ (inconclusive).

x	-1^-	-1	-1^+
$\dfrac{dy}{dx}$	< 0	0	> 0
Slope	\	—	/

Hence $(-1,\ 0)$ is a minimum point.

Example 6

Given that curve $y = (x-1)^3 + 2$, find

(a) an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$,

(b) the coordinates of the stationary point and determine its nature.

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 3(x-1)^2$

(b) When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\quad 3(x-1)^2 = 0$

$x = 1$

$y = 2$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6(x-1)$

When $x = 1$, $\quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ (inconclusive).

x	1^-	1	1^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	> 0	0	> 0
Slope	/	—	/

Hence $(1,\ 2)$ is a point of inflexion.

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Assignment 24: Second Derivative

1. Find the coordinates of the stationary point(s) of the following curves and determine the nature of each point.
   (a) $y = x(x-6)^2$
   (b) $y = x^2 + \dfrac{16}{x}$

(a)
$$\begin{aligned}\frac{dy}{dx} &= (x-6)^2 + 2x(x-6)\\ &= (x-6)(x-6+2x)\\ &= (x-6)(3x-6)\\ &= 3(x-6)(x-2)\end{aligned}$$

When $\dfrac{dy}{dx} = 0$, $\quad 3(x-6)(x-2) = 0$

$x = 6,\ 2$

$y = 0,\ 32$

The stationary points are (6, 0) and (2, 32).

$$\begin{aligned}\frac{d^2y}{dx^2} &= 3(x-2) + 3(x-6)\\ &= 6x - 24\end{aligned}$$

When $x = 6$, $\quad \dfrac{d^2y}{dx^2} = 12 > 0$

(6, 0) is a minimum point.

When $x = 2$, $\quad \dfrac{d^2y}{dx^2} = -12 < 0$

(2, 32) is a maximum point.

(b)
$$\begin{aligned}\frac{dy}{dx} &= 2x - \frac{16}{x^2}\\ &= \frac{2(x^3-8)}{x^2}\end{aligned}$$

When $\dfrac{dy}{dx} = 0$, $\quad 2(x^3 - 8) = 0$

$x^3 = 8$

$x = 2$

$y = 12$

The stationary points are (2, 12).

$$\frac{d^2y}{dx^2} = 2 + \frac{32}{x^3}$$

When $x = 2$, $\quad \dfrac{d^2y}{dx^2} = 6 > 0$

(2, 12) is a minimum point.

2. The graph of $y = 2x^3 + ax^2 + b$ has a stationary point $(-3, 19)$. Find the value of a and of b. Determine whether this stationary point is a maximum or a minimum. **(Pg410Q14)**

$$\frac{dy}{dx} = 6x^2 + 2ax$$

At $(-3, 19)$, $\frac{dy}{dx} = 0$,

$$6(-3)^2 + 2a(-3) = 0$$
$$54 - 6a = 0$$
$$a = 9$$

At $(-3, 19)$, $y = 19$,

$$19 = 2(-3)^3 + 9(-3)^2 + b$$
$$19 = -54 + 81 + b$$
$$b = -8$$

$$\frac{d^2y}{dx^2} = 12x + 18$$

When $x = -3$, $\frac{d^2y}{dx^2} = 12(-3) + 18 < 0$

$(-3, 19)$ is a maximum point.

---

3. Prove that the curve $y = 3x^4 + kx$ has only one turning point. Find the turning point where $k = 12$ and determine the nature of this point. **(Pg410Q11)**

$$\frac{dy}{dx} = 12x^3 + k$$

When $\frac{dy}{dx} = 0$,

$$12x^3 + k = 0$$
$$x^3 = -\frac{k}{12}$$
$$x = -\sqrt[3]{\frac{k}{12}}$$

Hence, $y = 3x^4 + kx$ has only one turning point at $x = -\sqrt[3]{\frac{k}{12}}$.

$$\frac{d^2y}{dx^2} = 36x^2$$

When $k = 12$,

$$x = -1$$
$$y = -9$$
$$\frac{d^2y}{dx^2} = 36 > 0$$

$(-1, -9)$ is a minimum point.

4. The curve $y = ax + \dfrac{b}{2x-1}$ has a stationary point at $A(2, 7)$.

(i) Find the value of a and of b.

(ii) Find the other stationary point.

(iii) Determine whether the point A is a maximum or a minimum point. **(Pg410Q15)**

(i) $$\frac{dy}{dx} = a - 2b(2x-1)^{-2}$$
$$= a - \frac{2b}{(2x-1)^2}$$

At $A(2, 7)$, $\dfrac{dy}{dx} = 0$, $\quad a - \dfrac{2b}{9} = 0$

$$a = \frac{2b}{9} \qquad \ldots\ldots(1)$$

At $A(2, 7)$, $y = 7$, $\quad 7 = 2a + \dfrac{b}{3}$

$$2a = 7 - \frac{b}{3} \qquad \ldots\ldots(2)$$
$$a = \frac{1}{2}\left(7 - \frac{b}{3}\right)$$

Sub (1) into (2) $\quad 2\left(\dfrac{2b}{9}\right) = 7 - \dfrac{b}{3}$

$$4b = 63 - 3b$$
$$7b = 63$$
$$b = 9$$
$$a = 2$$

(ii) $$\frac{dy}{dx} = 2 - \frac{18}{(2x-1)^2}$$

When $\dfrac{dy}{dx} = 0$, $\quad 2 - \dfrac{18}{(2x-1)^2} = 0$

$$2(2x-1)^2 - 18 = 0$$
$$(2x-1)^2 = 9$$
$$2x - 1 = \pm 3$$
$$x = -1,\ 2$$
$$y = -5,\ 2$$

The other stationary point is $(-1, -5)$.

(iii) $$\frac{d^2y}{dx^2} = \frac{72}{(2x-1)^3}$$

When $x = 2$, $\quad \dfrac{d^2y}{dx^2} > 0$

$A(2, 7)$ is a minimum point.

5. A curve has the equation $y = 2 - (3 - x)^4$. The point (p, q) is the stationary point on the curve.

(i) Determine the value of p and of q.

(ii) Determine whether y is increasing or decreasing

(a) for values of x less than p,

(b) for values of x greater than p.

(iii) What do the results of part (ii) imply about the stationary point? **(2014)**

(i) $\frac{dy}{dx} = -4(3-x)^3(-1)$

$= 4(3-x)^3$

At (p, q), $\frac{dy}{dx} = 0$, $\quad 4(3-p)^3 = 0$

$p = 3$

At (p, q), $y = q$, $q = 2$

(ii) (a) When $x < 3$, $\quad 4(3-x)^3 > 0$

$\therefore \frac{dy}{dx} > 0$

Hence, y is increasing when $x < 3$.

(b) When $x > 3$, $\quad 4(3-x)^3 < 0$

$\therefore \frac{dy}{dx} < 0$

Hence, y is decreasing when $x > 3$.

(iii) The stationary point is a maximum point.

(iv) $\frac{d^2y}{dx^2} = 4(3)(3-x)^2(-1)$

$= -12(3-x)^2$

At (3, 2), $\frac{d^2y}{dx^2} = 0$

Singapore Chinese Girls' School
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Integrated Mathematics II
Second Derivatives and Application

Name: ______________________________ ( ) Date: ________________

Class: Sec 4_____

Learning Objectives
At the end of the lesson, you will learn to • apply the second derivative test to solve word problems involving real life situations.

Worksheet 25: Maxima and Minima Problems

1. A rectangle has sides x cm and y cm. If the area of the rectangle is 16 cm^2, show that its perimeter, P cm, is given by $P = 2x + \frac{32}{x}$. Hence, calculate the value of x which gives P a stationary value and show that this value of P is a minimum.

$$\text{Area of rectangle} = 16\text{ cm}^2$$
$$xy = 16$$
$$y = \frac{16}{x}$$

$$\text{Perimeter of rectangle} = 2x + 2y\text{ cm}$$
$$\begin{aligned} P &= 2x + 2\left(\frac{16}{x}\right) \\ &= 2x + \frac{32}{x} \end{aligned}$$

$$\frac{\mathrm{d}P}{\mathrm{d}x} = 2 - \frac{32}{x^2}$$
$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{64}{x^3}$$

When $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$, $\quad 2 - \frac{32}{x^2} = 0$

$x^2 - 16 = 0$

$x = \pm 4$

Since $x > 0$, $x = 4$

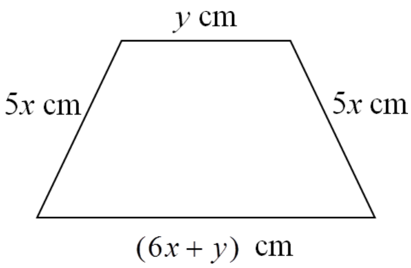
When $x = 4$, $\quad \frac{\mathrm{d}^2P}{\mathrm{d}x^2} > 0$

Hence, P is a minimum.

2. A piece of wire of length 104 cm, is bent to form a trapezium as shown in the diagram.
   (i) Express y in terms of x and show that the area, A cm, enclosed by the wire is given by

$$A = 208x - 20x^2.$$

   (ii) Find the value of x for which the area enclosed by the wire is stationary.
   (iii) Explain why this value of x gives the largest area possible. **(Pg416Q5 modified)**

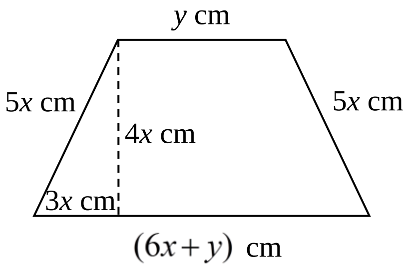


(i) $\text{Perimeter of trapezium} = 104\text{ cm}$

$16x + 2y = 104$

$2y = 104 - 16x$

$y = 52 - 8x$



$\text{Area of trapezium} = \frac{1}{2}(6x + 2y) \times 4x\text{ cm}^2$

$A = 2x(6x + 2y)$

$\quad = 12x^2 + 4x(52 - 8x)$

$\quad = 12x^2 + 208x - 32x^2$

$\quad = 208x - 20x^2$

(ii) $\frac{dA}{dx} = 208 - 40x$

When $\frac{dA}{dx} = 0$, $\quad 208 - 40x = 0$

$x = 5.2$

(iii) When $x = 5.2$, $\frac{d^2A}{dx^2} < 0$

Hence, A is a maximum.

<u>Alternatively</u>

As x increases through $x = 5.2$, $\frac{dA}{dx}$ changes sign from positive to negative, A is a maximum.

3. An open rectangular box of height h cm has a horizontal rectangular base of sides x cm and $2x$ cm. The volume of the box is 36 cm^3.

(i) Express h in terms of x, and show that the total external surface area, A cm^2, of the box is given by $A = 2x^2 + \dfrac{108}{x}$.

(ii) Calculate the value of x and of h, which would make the total surface area a minimum.

(P416Q9)

(i) $\text{Volume} = 36 \text{ cm}^3$

$$36 = x \times 2x \times h$$

$$h = \frac{18}{x^2}$$

$$A = 6xh + 2x^2$$

$$= \frac{108}{x} + 2x^2$$

(ii) $\dfrac{dA}{dx} = 4x - \dfrac{108}{x^2}$

When $\dfrac{dA}{dx} = 0$, $\quad 4x - \dfrac{108}{x^2} = 0$

$$4x^3 = 108$$

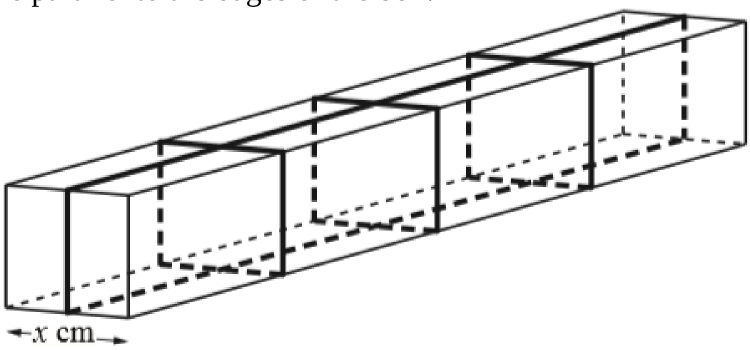
$$x^3 = 27$$

$$x = 3$$

$$h = 2$$

The box has a minimum area when $x = 3$ and $h = 2$.

4. The diagram shows a box in the shape of a cuboid with a square cross-section of side x cm. The volume of the box is 3500 cm^3. Four pieces of tape are fastened round the box as shown. The pieces of tape are parallel to the edges of the box.



(i) Given that the total length of the four pieces of tape is L cm, show that

$$L = 14x + \frac{7000}{x^2}.$$

(ii) Given that x can vary, find the stationary value of L and determine the nature of this stationary value. **(June2003)**

(i) Let the length of the box be l cm.

$x^2 = 3500$

$l = \frac{3500}{x^2}$

$$L = 14(x) + 2l$$
$$= 14(x) + 2\left(\frac{3500}{x^2}\right)$$
$$= 14x + \frac{7000}{x^2}$$

(ii) $$\frac{dL}{dx} = 14 - 2\left(\frac{7000}{x^3}\right)$$
$$= 14 - \frac{14000}{x^3}$$

When $\frac{dL}{dx} = 0$, $\quad 14 - \frac{14000}{x^3} = 0$

$x^3 = 1000$

$x = 10$

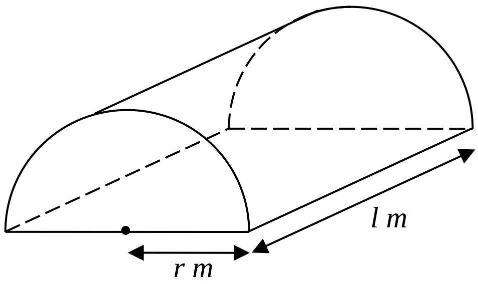
When $x = 10$ $\quad L = 14(10) + \frac{7000}{10^2}$

$= 210$

(ii) $$\frac{d^2L}{dx^2} = \frac{42000}{x^4}$$

When $x = 10$, $\frac{d^2L}{dx^2} > 0$, $\therefore$ L is a minimum value.

5. The diagram shows a greenhouse standing on a horizontal rectangular base. The vertical semicircular ends and the curved roof are made from polythene sheeting. The radius of each semicircle is r m and the length of the greenhouse is l m. Given that 120 m^2 of polythene sheeting is used for the greenhouse, express l in terms of r and show that the volume, V m^3 of the greenhouse is given by $V = 60r - \dfrac{\pi r^3}{2}$. Given that r can vary, find, to 2 decimal places, the value of r for which V has a stationary value. Find this value of V and determine whether it is a maximum or a minimum. **(N02)**



$$\pi r^2 + \pi r l = 120$$

$$l = \frac{120 - \pi r^2}{\pi r}$$

$$\text{Volume of the greenhouse} = \frac{1}{2}\pi r^2 l \ \text{cm}^2$$

$$V = \frac{1}{2}\pi r^2\left(\frac{120 - \pi r^2}{\pi r}\right)$$

$$= 60r - \frac{\pi r^3}{2}$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 60 - \frac{3\pi r^2}{2}$$

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -3\pi r$$

When $\dfrac{\mathrm{d}S}{\mathrm{d}x} = 0$, $\quad 60 - \dfrac{3\pi r^2}{2} = 0$

$$40 - \pi r^2 = 0$$

$$r = \sqrt{\frac{40}{\pi}}$$

$$= 3.57$$

$$V = 143$$

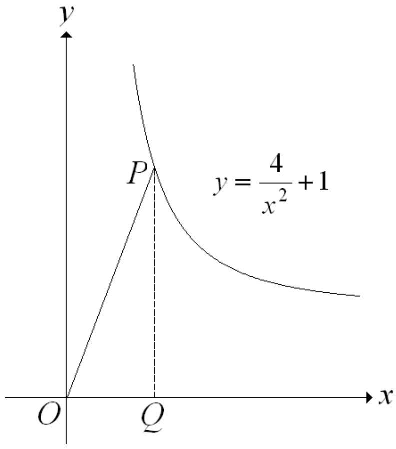
When $r = 3.57$, $\dfrac{d^2V}{dr^2} < 0$

Hence, V has a maximum value when $r = 3.57$.

6. The diagram below shows the curve $y = \dfrac{4}{x^2} + 1$. A variable point P, moves on the curve such that PQ is always parallel to the y-axis and Q is on the x-axis. Let the x-coordinate of P be p, $p > 0$, and the area of triangle OPQ be A square units, where O is the origin.

(i) Obtain an expression for A in terms of p.

(ii) Calculate the smallest possible area of triangle OPQ.



(i) Coordinates of $P = \left(p, \quad \dfrac{4}{p^2} + 1 \right)$

Area of triangle OPQ, $A = \dfrac{1}{2} \times p \times \left(\dfrac{4}{p^2} + 1 \right)$ unit^2

$$= \frac{p}{2}\left(\frac{4}{p^2} + 1 \right) \text{ unit}^2$$

$$= \frac{2}{p} + \frac{p}{2} \text{ unit}^2$$

(ii) $\dfrac{dA}{dp} = -\dfrac{2}{p^2} + \dfrac{1}{2}$

$\dfrac{d^2A}{dp^2} = \dfrac{4}{p^3} > 0 \Rightarrow A$ is a minimum value.

When $\dfrac{dA}{dp} = 0$, $-\dfrac{2}{p^2} + \dfrac{1}{2} \Rightarrow p = 2$

$$\Rightarrow \text{Minimum } A = \frac{2}{2} + \frac{2}{2} \text{ unit}^2$$

$$= 2 \text{ unit}^2$$

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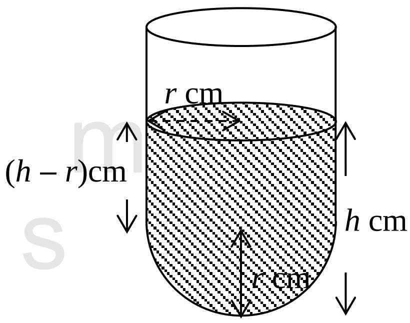
<u>Assignment 26: Maxima and Minima Problems</u>

1. The diagram shows a container made by fixing a hollow hemisphere of radius r cm to a hollow right circular cylinder of the same radius. Water is filled to a height of h cm.
If the volume of water in the container is 144π cm^3, show that

$$h = \frac{144}{r^2} + \frac{r}{3}.$$

Hence, find the value of r and of h such that the surface area of the container that is in contact with the water is a minimum.



Volume of water = 144π cm^3

$$\frac{2\pi r^3}{3} + \pi r^2 (h - r) = 144\pi$$

$$\frac{2r^3}{3} + r^2 h - r^3 = 144$$

$$r^2 h = 144 + \frac{r^3}{3}$$

$$h = \frac{144}{r^2} + \frac{r}{3}$$

Let surface area be A cm^2.

$$A = 2\pi r^2 + 2\pi r(h - r)$$

$$= 2\pi r^2 + 2\pi r\left(\frac{144}{r^2} + \frac{r}{3} - r\right)$$

$$= \frac{288\pi}{r} + \frac{2\pi r^2}{3}$$

$$\frac{dA}{dr} = -\frac{288\pi}{r^2} + \frac{4\pi r}{3} \Rightarrow \frac{d^2A}{dr^2} = \frac{576\pi}{r^3} + \frac{4\pi}{3}$$

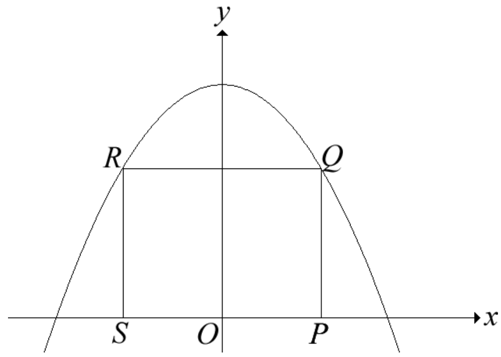
When $\frac{dA}{dr} = 0$, $\quad -\frac{288\pi}{r^2} + \frac{4\pi r}{3} = 0$

$r = 6$
$h = 6$

When $r = 6$, $\frac{d^2A}{dr^2} = 4\pi > 0$

Surface area in contact with the water is a minimum when $r = h = 6$ cm.

2. The diagram shows part of a curve $y = 9 - \dfrac{x^2}{2}$. $PQRS$ is a rectangle with $P(p, 0)$ is inscribed in the region bounded by the curve and the x-axis. Find the dimensions of the rectangle if its area is to be a maximum.



$P(0,\ p) \Rightarrow \left(p,\ 9 - \dfrac{p^2}{2} \right)$.

Area of $PQRS$, $A = 2p\left(9 - \dfrac{p^2}{2}\right)$ unit^2

$= 18p - p^3$ unit^2

$\dfrac{dA}{dp} = 18 - 3p^2$

$\dfrac{d^2A}{dp^2} = -6p$

When $\dfrac{dA}{dp} = 0$, $\quad 18 - 3p^2 = 0$

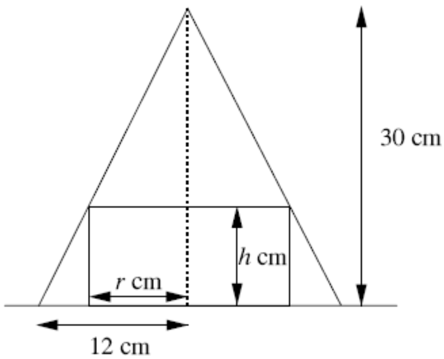
$p = \sqrt{6}\ \ (p > 0)$

$y = 9 - \dfrac{p^2}{2} = 6$

When $p = \sqrt{6}$, $\quad \dfrac{d^2A}{dp^2} = -6\sqrt{6} < 0$

The dimensions of the rectangle with a maximum area are $2\sqrt{6}$ units by 6 units.

3. The diagram shows the cross-section of a hollow cone of height 30 cm and base radius 12 cm and a solid cylinder of radius *r* cm and height *h* cm. Both stand on a horizontal surface with the cylinder inside the cone. The upper circular edge of the cylinder is in contact with the cone.



(i) Express *h* in terms of *r* and hence show that the volume of the cylinder, V cm^3, is given by

$$V = \pi\left(30r^2 - \frac{5}{2}r^3\right).$$

Given that r can vary,

(i) find the volume of the largest cylinder which can stand inside the cone and show that, in this case, the cylinder occupies $\frac{4}{9}$ of the volume of the cone.

(i) $\frac{h}{30} = \frac{12-r}{12}$

$$h = \frac{30(12-r)}{12}$$

$$= 30 - \frac{5}{2}$$

(ii) $V = \pi r^2 h$

$$= \pi r^2\left(30 - \frac{5}{2}\right)$$

$$= \pi\left(30r^2 - \frac{5}{2}r^3\right)$$

$$\frac{dV}{dr} = 60\pi r - \frac{15\pi r^2}{2}$$

$$\frac{d^2V}{dr^2} = 60\pi - 15\pi r$$

When $\frac{dV}{dr} = 0$, $60\pi r - \frac{15\pi r^2}{2} = 0$

$8\pi r - \pi r^2 = 0$

$\pi r(8 - r) = 0$

$r = 8 \ (r > 0)$

$V = 640\pi$

$= 2010$

$\frac{d^2V}{dr^2} = 60\pi - 15\pi(8) < 0$

$$\frac{\text{Volume of the largest cylinder}}{\text{Volume of the cone}}$$

$$= \frac{640\pi}{\frac{1}{3}\pi(12^2)(30)}$$

$$= \frac{4}{9}$$

4. In the diagram, the cross-section of the prism is an isosceles triangle ABC in which $AB = AC = 5x$ cm and $BC = 6x$ cm.

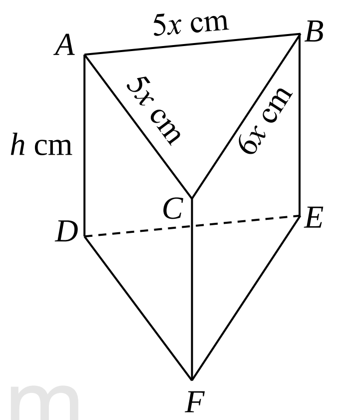
The length of each of the parallel edges AD, BE and CF is h cm.
Given that the volume of the prism is 4500 cm^3,

(i) write an expression for h in terms of x,

(ii) show that the total surface area, A cm^2, is given by $24x^2 + \dfrac{6000}{x}$,

(iii) express $\dfrac{\mathrm{d}A}{\mathrm{d}x}$ in terms of x,

(iv) find the stationary value of A. **(Pg417Q11)**



(i) Volume = 4500 cm^3

$$\frac{1}{2}(6x)(4x)h = 4500$$

$$h = \frac{4500}{x^2}$$

$$= \frac{375}{x^2}$$

(ii)
$$A = 2 \times \frac{1}{2}(6x)(4x) + 5xh + 5xh + 6xh$$
$$= 24x^2 + 16xh$$
$$= 24x^2 + 16x\left(\frac{375}{x^2}\right)$$
$$= 24x^2 + \frac{6000}{x}$$

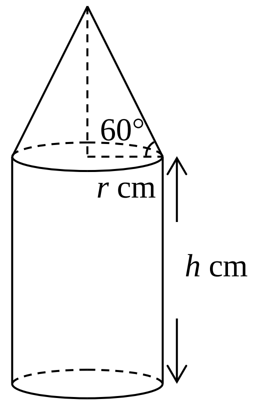
(iii) $$\frac{\mathrm{d}A}{\mathrm{d}x} = 48x - \frac{6000}{x^2}$$

(iv) When $\dfrac{\mathrm{d}A}{\mathrm{d}x} = 0$, $\quad 48x - \dfrac{6000}{x^2} = 0$

$$48x^3 = 6000$$
$$x^3 = 125$$
$$x = 5$$
$$A = 24(5)^2 + \frac{6000}{5}$$
$$= 1800$$

Stationary value of A = 1800

5. The diagram shows a toy which is made up of a right circular cone fixed to a closed cylinder with height h cm. The slant height of the cone makes an angle of 60° with the horizontal axis and the radius of the cone is r cm.



Given that the volume of the toy is 50π cm^3, show that

$$h = \frac{50}{r^2} - \frac{r\sqrt{3}}{3}.$$

If the total surface area of the toy is A, show that

$$A = \pi r^2\left(3 - \frac{2\sqrt{3}}{3}\right) + \frac{100\pi}{r}.$$

Hence, find the value of r for which A has a stationary value.
Determine if A is a maximum or a minimum.

Height of cone $= r\tan 60°$ cm
$= r\sqrt{3}$ cm

Volume of toy $= \frac{1}{3}\pi r^2(r\sqrt{3}) + \pi r^2 h$ cm^3

$$50\pi = \frac{\sqrt{3}}{3}\pi r^3 + \pi r^2 h$$

$$\pi r^2 h = 50\pi - \frac{\sqrt{3}}{3}\pi r^3$$

$$h = \frac{50\pi - \frac{\sqrt{3}}{3}\pi r^3}{\pi r^2}$$

$$h = \frac{50}{r^2} - \frac{r\sqrt{3}}{3}$$

$$\frac{r}{l} = \cos 60° \Rightarrow l = 2r$$

Total surface area of the toy is A

$= \pi r^2 + 2\pi r h + \pi r l$ cm^2

$= \pi r^2 + 2\pi r\left(\frac{50}{r^2} - \frac{r\sqrt{3}}{3}\right) + \pi r(2r)$ cm^2

$= 3\pi r^2 + \frac{100\pi}{r} - \frac{2\pi r^2\sqrt{3}}{3}$ cm^2

$= \pi r^2\left(3 - \frac{2\sqrt{3}}{3}\right) + \frac{100\pi}{r}$ cm^2

$$\frac{dA}{dr} = 2\pi r\left(3 - \frac{2\sqrt{3}}{3}\right) - \frac{100\pi}{r^2}$$

When $\frac{dA}{dr} = 0$, $\quad 2\pi r\left(3 - \frac{2\sqrt{3}}{3}\right) - \frac{100\pi}{r^2} = 0$

$$2\pi r\left(3 - \frac{2\sqrt{3}}{3}\right) = \frac{100\pi}{r^2}$$

$$r^3 = \frac{100\pi}{2\pi\left(3 - \frac{2\sqrt{3}}{3}\right)}$$

$$r = \sqrt[3]{\frac{100\pi}{2\pi\left(3 - \frac{2\sqrt{3}}{3}\right)}}$$

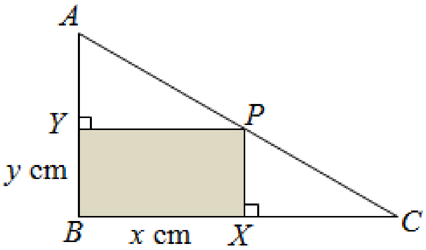
$= 3.0035$ cm
≈ 3.00 cm

$$\frac{d^2A}{dr^2} = 2\pi\left(3 - \frac{2\sqrt{3}}{3}\right) + \frac{200\pi}{r^3}$$

$= 34.8 > 0$

A is a minimum when $r = 3.00$ cm

6. The right-angled triangle ABC has $AB = 3$ cm, $BC = 4$ cm and $AC = 5$ cm. The rectangle $BXPY$ is such that $BX = x$ cm and $BY = y$ cm.



(i) Express y in terms of x.

(ii) Show that the area, A cm^2, of $A = \frac{3}{4}x(4 - x)$.

(iii) Find the value of x for which the A is stationary.

(iv) Explain why this value of x gives the largest area possible. **(Pg420Q5 modified)**

(i) $\frac{y}{3} = \frac{4 - x}{4}$

$y = \frac{3(4 - x)}{4}$

(ii) $A = xy$

$= \frac{3}{4}x(4 - x)$

(iii) $\frac{dA}{dx} = \frac{d}{dx}\left(3x - \frac{3}{4}x^2\right)$

$= 3 - \frac{3}{2}x$

When $\frac{dA}{dx} = 0$, $3 - \frac{3}{2}x = 0$

$x = 2$

When $x = 2$, $A = \frac{3}{4}(2)(4 - 2)$

$= 3$

(iii) When $x = 2$, $\frac{d^2A}{dx^2} = -\frac{3}{2} < 0$

Hence, A is a maximum.

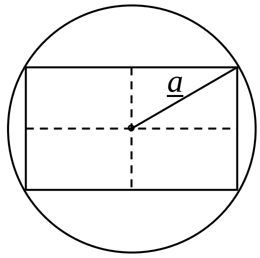
Alternatively

As x increases through $x = 2$, $\frac{dA}{dx}$ changes sign from positive to negative, A is a maximum.

7*. Show that the largest rectangle inscribed in a circle of radius a cm is a square of side $a\sqrt{2}$ cm.

(Pg417Q15)

Let the length, width and area of the rectangle be x cm, y cm and A cm^2.



$$x^2 + y^2 = (2a)^2$$

$$y = \sqrt{4a^2 - x^2}$$

$$\begin{aligned} A &= xy \\ &= x\sqrt{4a^2 - x^2} \\ &= x(4a^2 - x^2)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \frac{dA}{dx} &= (4a^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}x(4a^2 - x^2)^{-\frac{1}{2}}(-2x) \\ &= (4a^2 - x^2)^{-\frac{1}{2}}(4a^2 - x^2 - x^2) \\ &= \frac{4a^2 - 2x^2}{\sqrt{4a^2 - x^2}} \end{aligned}$$

When $\frac{dA}{dx} = 0$, $\quad \frac{4a^2 - 2x^2}{\sqrt{4a^2 - x^2}} = 0$

$$\begin{aligned} 4a^2 - 2x^2 &= 0 \\ 2x^2 &= 4a^2 \\ x^2 &= 2a^2 \\ x &= a\sqrt{2} \\ y &= a\sqrt{2} \end{aligned}$$

x	$a\sqrt{2}^-$	$a\sqrt{2}$	$a\sqrt{2}^+$
$\frac{dy}{dx}$	+	0	–
Slope	/	—	\

$\therefore$ Largest rectangle inscribed in a circle of radius a cm is a square of side $a\sqrt{2}$ cm.

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 14

Name: ______________________________ (____)   Date: ______________

Class: Sec 4______

Revision 14 Maxima and Minima

Given a curve $y = \mathrm{f}(x)$, when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = a$, $(a, \mathrm{f}(a))$ is a stationary point.

First Derivative Test

- If $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes sign from **positive to negative** as it passes through $x = a$, then it a **maximum point.**
- If $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes sign from **negative to positive** as it passes through $x = a$, then it is **a minimum point.**
- If $\frac{\mathrm{d}y}{\mathrm{d}x}$ **does not change** sign as it passes through $x = a$, then it is a stationary **point of inflexion**.

Second Derivative Test

- If $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$, then $(a,\ \mathrm{f}(a))$ is **a minimum point**.
- If $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0$, then $(a,\ \mathrm{f}(a))$ is **a maximum point**.
- If $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$, the nature of the stationary point **could not be determined**. Apply the **First derivative test.**

Example

1. The curve with equation $y = e^{-x}\sin x$ has a stationary point for which $0 \le x \le \pi$.
   (i) Find the x-coordinate of this point.
   (ii) Determine the nature of this this point.

Solution

(i) $\dfrac{dy}{dx} = -e^{-x}\sin x + e^{-x}\cos x$

$= -e^{-x}(\sin x - \cos x)$

When $\dfrac{dy}{dx} = 0$, $\quad -e^{-x}\sin x + e^{-x}\cos x = 0$

Since $e^{-x} > 0$, $\quad \sin x = \cos x$

$\tan x = 1$

$x = \dfrac{\pi}{4}$

- Apply the Product Rule.
- Solve $\dfrac{dy}{dx} = 0$ to find the value of x which gives a stationary value of y.

(ii) $\dfrac{d^2y}{dx^2} = e^{-x}(\sin x - \cos x) - e^{-x}(\cos x + \sin x)$

$= -2e^{-x}\cos x$

When $x = \dfrac{\pi}{4}$, $\dfrac{d^2y}{dx^2} < 0$

Hence the stationary point is a maximum point.

- Perform the **Second Derivative Test**.
- Check the sign of the value of $\dfrac{d^2y}{dx^2}$ to determine nature of the stationary value.

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2. The equation of a curve is $y = e^{x^3}$.
   (i) Find expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.
   (ii) Find the coordinate of the stationary point and determine the nature of this point.

Solution

(i) $\dfrac{dy}{dx} = 3x^2e^{x^3}$

$\dfrac{d^2y}{dx^2} = 6xe^{x^3} + 3x^2(3x^2e^{x^3})$

$= 3xe^{x^3}(2 + 3x^3)$

- Apply the Product Rule
- Solve $\dfrac{dy}{dx} = 0$ to find the value of x which gives a stationary value of y.

(ii) When $\dfrac{dy}{dx} = 0$, $\quad 3x^2e^{x^3} = 0$

Since $e^{x^3} > 0$, $\quad x = 0$

$y = 1$

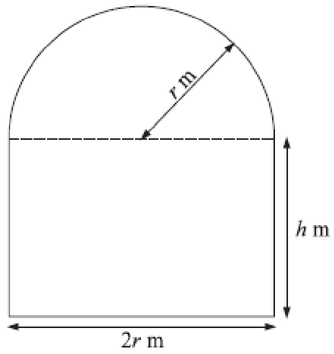
When $x = 0$, $\quad \dfrac{d^2y}{dx^2} = 0$ (inconclusive)

x	0^-	0	0^+
$\dfrac{dy}{dx}$	$+$	0	$-$

(0, 1) is a point of inflexion.

- $a^x > 0$ or $e^{x^3} > 0$
- When the **Second Derivative Test** fails, performs the **First Derivative Test**.

3. The diagram shows a glass window consisting of a recatangle of height h m and and with $2r$ m and a semicircle of radius r m. The perimeter of the window is 8 m.



(i) Express h in terms of r.

(ii) Show that the area of the window, A m^2, is given by

$$A = 8r - 2r^2 - \frac{1}{2}\pi r^2.$$

Given that r can vary,

(iii) find the value of r for which A has a stationary value,

(iii) determine whether this stationary value is a maximum or a minimum.

Solution

(i) $\pi r + 2r + 2h = 8$

$2h = 8 - \pi r - 2r$

$h = 4 - \frac{1}{2}\pi r - r$

(ii) $A = \frac{1}{2}\pi r^2 + 2rh$

$= \frac{1}{2}\pi r^2 + r(8 - \pi r - 2r)$

$= 8r - 2r^2 - \frac{1}{2}\pi r^2$

- $a^x > 0$ or $e^{x^3} > 0$
- When the **Second Derivative Test** fails, performs the **First Derivative Test**.

(iii) $\frac{\mathrm{d}A}{\mathrm{d}r} = 8 - 4r - \pi r$

When $\frac{\mathrm{d}A}{\mathrm{d}r} = 0,\quad 8 - 4r - \pi r = 0$

$r = \frac{8}{4+\pi}$

$= 1.12$

- Solve $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$ to find the value of r which gives a stationary value of A.

(iv) $\frac{\mathrm{d}^2A}{\mathrm{d}r^2} = -4 - \pi < 0$

A is a maximum value.

- Check the sign of the value of $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}$ to determine nature of the stationary value.

Exercise

1. The curve $y=\sqrt{x}+\dfrac{9}{\sqrt{x}}$.

   (i) Find expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

   (ii) Show that the curve has a stationary value when $x=9$.

   (iii) Find the nature of this stationary value.

Solution

(i) $$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}+9\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$=\frac{1}{2}x^{-\frac{1}{2}}-\frac{9}{2}x^{-\frac{3}{2}} \text{ or } \frac{1}{2\sqrt{x}}-\frac{9}{2x\sqrt{x}}$$

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}-\frac{9}{2}\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$$

$$=-\frac{1}{4}x^{-\frac{3}{2}}+\frac{27}{4}x^{-\frac{5}{2}}$$

(ii) When $x=9$, $$\frac{dy}{dx}=\frac{1}{2}(9^{-\frac{1}{2}})-\frac{9}{2}(9^{-\frac{3}{2}})$$

$$=0$$

Hence, the curve has a stationary value when $x=9$.

(iii) When $x=9$, $$\frac{d^2y}{dx^2}=\frac{1}{2}\left(-\frac{1}{2}\right)(9^{-\frac{3}{2}})-\frac{9}{2}\left(-\frac{3}{2}\right)(9^{-\frac{5}{2}})$$

$$=-\frac{1}{4(27)}+\frac{27}{4(243)}$$

$$=\frac{1}{54}>0$$

The stationary value is a minimum.

2. A piece of wire, of length 2 m, is divided into two pieces. One piece is bent to form a square of side x m and the other is bent to form a circle of radius r m.

   (i) Express r in terms of x and show that the total area, A m^2, of the two shapes is given by

$$A = \frac{(\pi+4)x^2 - 4x + 1}{\pi}.$$

   Given that x can vary, find

   (ii) the stationary value of A,

   (iii) the nature of this stationary value.

Solution

(i) $4x + 2\pi r = 2$

$$2\pi r = 2 - 4x$$

$$r = \frac{2-4x}{2\pi}$$

$$= \frac{1-2x}{\pi}$$

$$A = \pi r^2 + x^2$$

$$= \pi\left(\frac{1-2x}{\pi}\right)^2 + x^2$$

$$= \frac{1-4x+4x^2+\pi x^2}{\pi}$$

$$= \frac{(\pi+4)x^2 - 4x + 1}{\pi}$$

(ii) $\frac{dA}{dx} = \frac{2x(\pi+4)-4}{\pi}$

When $\frac{dA}{dx} = 0$, $\quad 2x(\pi+4) - 4 = 0$

$$2x = \frac{4}{\pi+4}$$

$$x = \frac{2}{\pi+4}$$

$$A = \frac{(\pi+4)\left(\frac{2}{\pi+4}\right)^2 - 4\left(\frac{2}{\pi+4}\right) + 1}{\pi}$$

$$= 0.140$$

(iii) $\frac{d^2A}{dx^2} = \frac{2(\pi+4)}{\pi} > 0$

$A = 0.140$ is a minimum value.

3. A cuboid has a total surface area of 120 cm^2. Its base measures x cm by $2x$ cm and its height is h cm.

(i) Obtain an expression for h in terms of x.

Given that the volume of the cuboid is V cm^3,

(ii) show that $V = 40x - \dfrac{4x^3}{3}$.

Given that x can vary, find

(iii) the stationary value of V and the nature of this stationary value.

Solution

(i) $2(2x^2 + hx + 2hx) = 120$

$$2x^2 + 3hx = 60$$

$$3hx = 60 - 2x^2$$

$$h = \frac{60 - 2x^2}{3x}$$

(ii)
$$\begin{aligned} V &= 2x^2 h \\ &= 2x^2\left(\frac{60 - 2x^2}{3x}\right) \\ &= 40x - \frac{4x^3}{3} \end{aligned}$$

(iii) $\dfrac{\mathrm{d}V}{\mathrm{d}x} = 40 - 4x^2$

When $\dfrac{\mathrm{d}V}{\mathrm{d}x} = 0$, $\quad 40 - 4x^2 = 0$

$$x^2 = 10$$

Since $x > 0$, $\quad x = \sqrt{10}$

$$V = 20\left(\frac{60 - 20}{3\sqrt{10}}\right)$$

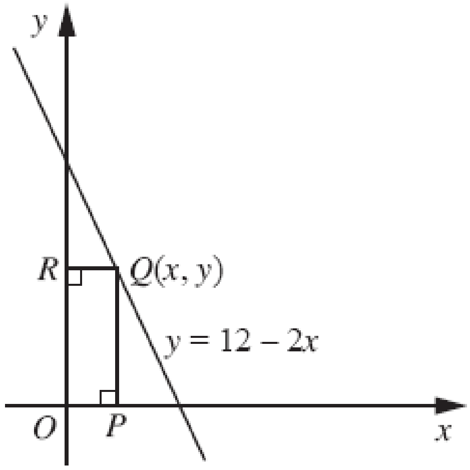
$$= 84.3$$

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -8x$$

When $x = \sqrt{10}$, $\dfrac{\mathrm{d}^2V}{\mathrm{d}x^2} = -8\sqrt{10} < 0$

Hence $V = 84.3$ is a maximum value.

4. The diagram shows part of the line $y = 12 - 2x$. The point $Q(x, y)$ lies on this line and the points P and R lie on the coordinates axes such that $OPQR$ is a rectangle.



(i) Write down an expression, in terms of x, for the area A of the rectangle $OPQR$.
(ii) Given that x can vary, find the value of x for which A has a stationary value.
(iii) Find this stationary value of A and determine its nature.

<u>Solution</u>

(i) $A = x(12 - 2x)$
$= 12x - 2x^2$

(ii) $\dfrac{dA}{dx} = 12 - 4x$

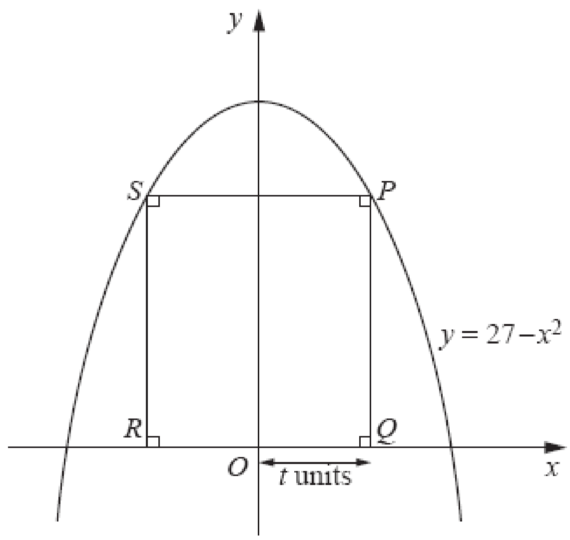
When $\dfrac{dA}{dx} = 0$, $\quad 12 - 4x = 0$
$x = 3$

(iii) When $x = 3$, $\quad A = 3(12 - 6)$
$= 18$

$\dfrac{d^2A}{dx^2} = -4 < 0$

$\therefore A = 18$ is a maximum value.

5. The diagram shows part of the curve $y = 27 - x^2$. The points P and S lie on this curve, The points Q and R lie on the x-axis and $PQRS$ is a rectangle. The length of OQ is t units.



(i) Find the length of PQ in terms of t and hence show that the area, A square units, of $PQRS$ is given by

$$A = 54t - 2t^3.$$

(ii) Given that t can vary, find the value of t for which A has a stationary value.

(iii) Find this stationary value of A and determine its nature.

<u>Solution</u>

(i) Coordinates of $P = (t, 27 - t^2)$

$PQ = 27 - t^2$ units

$A = 2t(27 - t^2)$

$\quad = 54t - 2t^3$

(ii) $\dfrac{\mathrm{d}A}{\mathrm{d}t} = 54 - 6t^2$

When $\dfrac{\mathrm{d}A}{\mathrm{d}t} = 0$, $\qquad 54 - 6t^2 = 0$

$\qquad t^2 = 9$

Since $t > 0$, $\qquad t = 3$

(iii) When $t = 3$, $\qquad A = 54(3) - 2(3)^3$

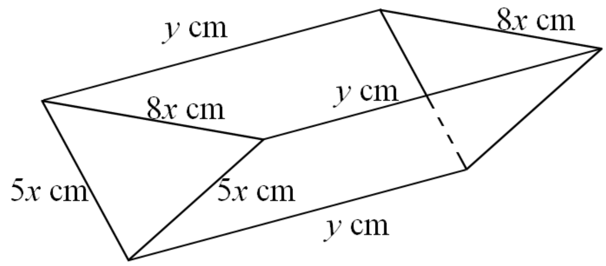
$\qquad = 108$

$\qquad \dfrac{\mathrm{d}^2A}{\mathrm{d}x^2} = -12t$

$\qquad = -36 < 0$

A has a maximum value when $t = 3$.

6. The diagram shows an open container constructed out of 200 cm^2 of cardboard. The two vertical end pieces are isosceles triangles with sides $5x$, $5x$ and $8x$ cm, and the two side pieces are rectangles of length y cm and width $5x$, cm, as shown. The open top is a horizontal rectangle.



(i) Express y in terms of x.

(ii) Show that the volume V cm^3, of the container is given by $V = 240x - 28.8x^3$.

Given that x can vary,

(iii) find the value of x for which V has a stationary value,

(iv) determine whether this stationary value is a maximum or a minimum.

Solution

(i) $2\left(\frac{1}{2} \times 8x \times 3x\right) + 2(5xy) = 200y$

$$24x^2 + 10xy = 200$$

$$10xy = 200 - 24x^2$$

$$y = \frac{200 - 24x^2}{10}$$

$$= 20 - 2.4x^2$$

(ii) $V = \frac{1}{2} \times 8x \times 3x \times y$

$$= 12x(20 - 2.4x^2)$$

$$= 240x - 28.8x^3$$

(iii) $\frac{\mathrm{d}V}{\mathrm{d}x} = 240 - 86.4x^2$

When $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$, $\quad 240 - 86.4x^2 = 0$

$$x^2 = \frac{240}{86.4}$$

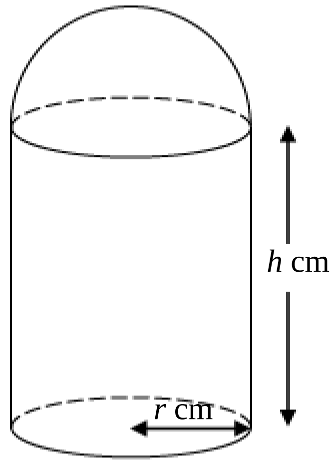
Since $x > 0$, $\quad x = \frac{5}{3}$

(iv) $\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -172.8x$

When $x = \frac{5}{3}$, $\frac{\mathrm{d}^2V}{\mathrm{d}x^2} < 0$

Hence, V is a maximum value.

7. The diagram shows a container which consists of a cylinder and a solid hemispherical cap which fits exactly on top of the cylinder.
The cylinder has a radius r cm, height h cm and it can hold a volume of 500 cm^3.



(i) Show that the total external surface area, A cm^2, of the container is given by $A = 3\pi r^2 + \dfrac{1000}{r}$.

Given that r can vary,
(ii) find the stationary value of A,
(iii) determine the nature of this stationary value of A.

Solution

(i) $\pi r^2 h = 500$

$$h = \frac{500}{\pi r^2}$$

$$A = \pi r^2 + 2\pi r^2 + 2\pi r h$$

$$= 3\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$$

$$= 3\pi r^2 + \frac{1000}{r}$$

(ii) $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 6\pi r - \dfrac{1000}{r^2}$

When $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 0$, $\quad 6\pi r - \dfrac{1000}{r^2} = 0$

$$6\pi r^3 = 1000$$

$$r = \sqrt[3]{\frac{1000}{6\pi}}$$

$$= 3.7575$$

$$A = 3\pi(3.7575)^2 + \frac{1000}{3.7575}$$

$$= 399$$

(iii) $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2} = 6\pi + \dfrac{2000}{r^3}$

When $r = 3.7575$, $\quad \dfrac{\mathrm{d}^2A}{\mathrm{d}r^2} > 0$

Hence, A is a minimum value.

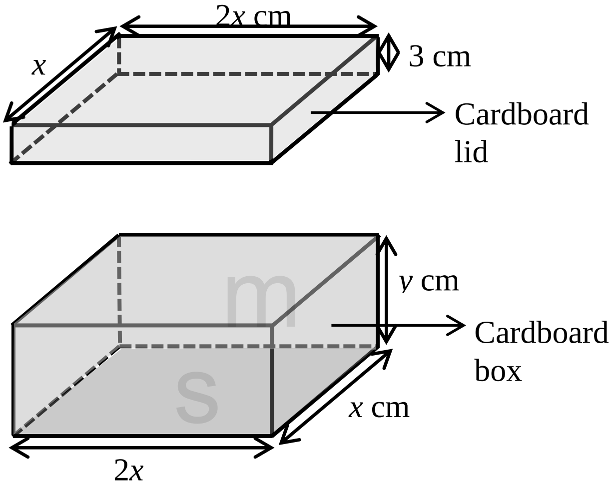
8. The diagram shows an open cardboard box with a rectangular base and a close fitting cardboard lid which slips over the top of the box.

The dimensions of the lid are $2x$ cm, x cm and 3 cm.
The total area of cardboard used in making the box and the lid is 2400 cm^2.

(i) Obtain an expression for y in terms of x, and hence show that the volume, V cm^3 of the box is given by $V = 800x - \frac{4x^3}{3} - 6x^2$.

(ii) Given that x can vary, find the value of x for which volume of the box is stationary. Calculate this stationary value of V.

(iii) Explain why this value of x gives the largest volume of the box.



Solution

(i) $2x^2 + 18x + 6xy + 2x^2 = 2400$

$6xy = 2400 - 4x^2 - 18x$

$$y = \frac{2400 - 4x^2 - 18x}{6x}$$

$$= \frac{1200 - 2x^2 - 9x}{3x}$$

$$V = 2x^2 y$$

$$= 2x^2\left(\frac{1200 - 2x^2 - 9x}{3x}\right)$$

$$= 800x - \frac{4x^3}{3} - 6x^2$$

(ii) $\frac{dV}{dx} = 800 - 4x^2 - 12x$

When $\frac{dV}{dx} = 0$, $\quad 800 - 4x^2 - 12x = 0$

$$x^2 + 3x - 200 = 0$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-200)}}{2}$$

$$= 12.7, \ -15.7 \text{ (NA)}$$

(iii) $\frac{d^2V}{dx^2} = -8x - 12$

When $x = 12.7$, $\quad \frac{d^2V}{dx^2} < 0$

Hence, V is a maximum value.

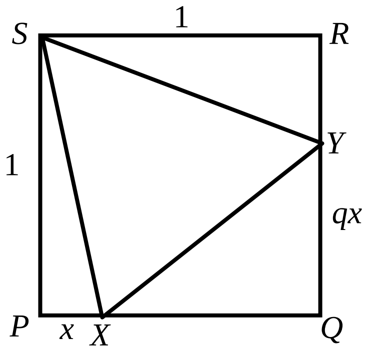
9. The diagram shows a square $PQRS$ of side 1 m.

The length of $PX = x$ m and $QY = qx$ m, where q is a constant such that $q > 1$.

(i) Given that the area of triangle SXY is A m^2, show that $A = \frac{1}{2}(1 - x + qx^2)$.

(ii) Given that x can vary, show that $QY = YR$ when A is a minimum. Hence, express the minimum value of A in terms of q.

Solution



(i) $$A = 1 - \frac{1}{2}(1)(1-qx) - \frac{1}{2}(1)(x) - \frac{1}{2}(qx)(1-x)$$

$$= 1 - \frac{1}{2} + \frac{qx}{2} - \frac{x}{2} - \frac{qx}{2} + \frac{qx^2}{2}$$

$$= \frac{1}{2} - \frac{x}{2} - + \frac{qx^2}{2}$$

$$= \frac{1}{2}(1 - x + qx^2)$$

(ii) $$\frac{dA}{dx} = \frac{1}{2}(-1 + 2qx)$$

When $\frac{dA}{dx} = 0$, $\quad \frac{1}{2}(-1+2qx) = 0$

$$2qx = 1$$

$$x = \frac{1}{2q}$$

$$\frac{d^2A}{dx^2} = 2q > 0$$

A is a minimum when $x = \frac{1}{2q}$.

When $x = \frac{1}{2q}$, $\quad QY = qx$ m

$$= q\left(\frac{1}{2q}\right) \text{ m}$$

$$= \frac{1}{2} \text{ m}$$

$$= YR$$

Minimum A $$= \frac{1}{2}\left[1 - \frac{1}{2q} + q\left(\frac{1}{2q}\right)^2\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{2q} + \frac{1}{4q}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{4q}\right)$$

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 6

Name: ______________________________ (____) Date: ______________

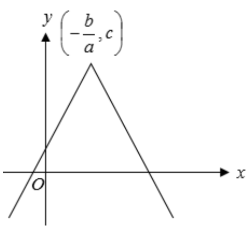
Class: Sec 4______

Revision 6: Modulus Functions

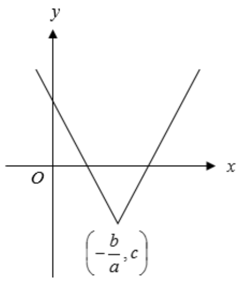
Equations Involving Absolute Valued Functions

- $|x| = p \Rightarrow x = p$ or $x = -p$
- $|a| = |b| \Rightarrow a = b$ or $a = -b$

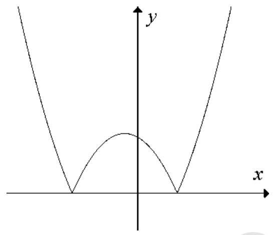
Graphs of Absolute Valued Functions



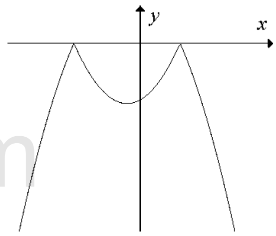
$y = c - |ax + b|$



$y = |ax + b| + c$



$y = |ax^2 + bx + c|$



$y = -|ax^2 + bx + c|$

Example

1. Solve the equation $|x^2 - 6| = 5x$.

$x^2 - 6 = 5x$	$x^2 - 6 = -5x$	1. Consider the two cases.
$x^2 - 5x - 6 = 0$	$x^2 + 5x - 6 = 0$	2. As the LHS > 0, $5x$ > 0.
$(x-6)(x+1) = 0$	$(x+6)(x-1) = 0$	3. Verify the answers.
$x = 6,\ -1$ (NA)	$x = -6$ (NA), 1	

---

2. Solve the equation $|2x+4| + 1 = |3x+6|$.

Method 1

$|2x+4| + 1 = |3x+6|$

$2|x+2| + 1 = 3|x+2|$

$|x+2| = 1$

$x + 2 = 1$ $\qquad$ $x + 2 = -1$

$x = -1$ $\qquad$ $x = -3$

Alternative method

$|2x+4| + 1 = |3x+6|$

$2x + 4 + 1 = 3x + 6$ $\qquad$ $-(2x+4) + 1 = -(3x+6)$

$x = -1$ $\qquad$ $-2x - 4 + 1 = -3x - 6$

$x = -3$

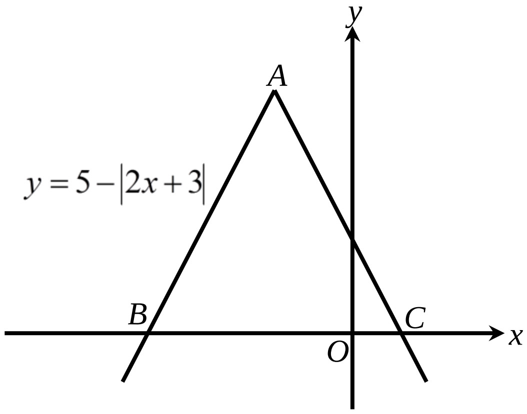
Since $2x+4$ and $3x+6$ are multiples of $x+2$, only two following cases are considered.

Exercise

1. The diagram shows part of the graph of $y = 5 - |2x+3|$.

(i) Find the coordinates of the points A, B and C.

(ii) Solve the equation $5 - |2x+3| = x$.



In each of the following cases, find the set of values of x that satisfies the given inequality.

(iii) $y \geq 0$

(iv) $y < x$

(i) At A, $2x + 3 = 0$

$$x = -\frac{3}{2}$$

$\therefore A\left(-\frac{3}{2},\ 5\right)$.

At B and C, $y = 0$,

$$5 - |2x+3| = 0$$
$$|2x+3| = 5$$
$$2x + 3 = \pm 5$$
$$2x = 2, -8$$
$$x = 1, -4$$

$\therefore B(-4, 0)$ and $C(1, 0)$.

(ii) $5 - |2x+3| = x$

$|2x+3| = 5 - x$

$2x + 3 = 5 - x$ or $2x + 3 = x - 5$

$3x = 2$ $\quad x = -8$

$x = \frac{2}{3}$

(iii) $y \geq 0 \Rightarrow -4 \leq x \leq 1$

(iv) $y < x \Rightarrow x \leq -8$ or $x \geq \frac{2}{3}$

2. (a) Solve the equation $|x+4|+3=2x$.

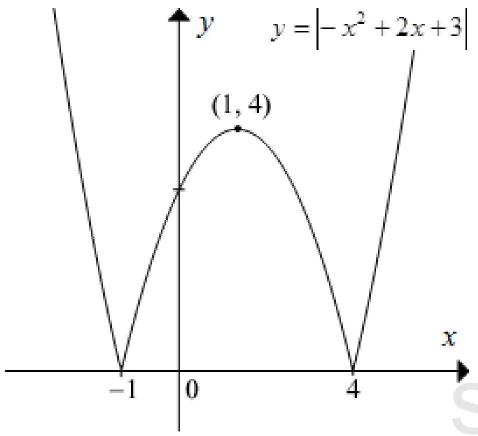
$$|x+4|+3=2x$$
$$|x+4|=2x-3$$

$x+4=2x-3$ or $x+4=3-2x$

$x=7$ $\qquad$ $3x=1$

$x=\frac{1}{3}$ (NA)

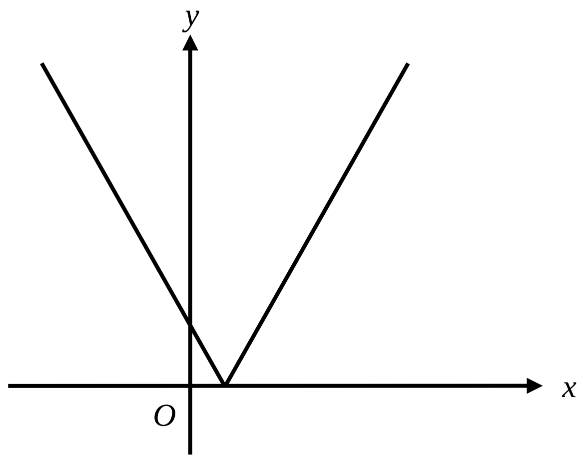
(b) Sketch the curve $y=|-x^2+2x+3|$, showing clearly the coordinates of the maximum point and the points where the curve meets the axes.



(c) The diagram below shows part of the graph of $y=|2x-1|$.

(i) Given that the line $y=mx+3$ intersects the graph $y=|2x-1|$ at only one point. Write down the possible values of m.

(ii) Determine the number of points of intersection between the graph of $y=|2x-1|$ and the line $y=x+c$ where $c > 0$. Explain your answer.



(i) $y=mx+3$ must be parallel to the right hand arm or left hand arm of $y=|2x-1|$.
$m=\pm 2$

(ii) 2 solutions.

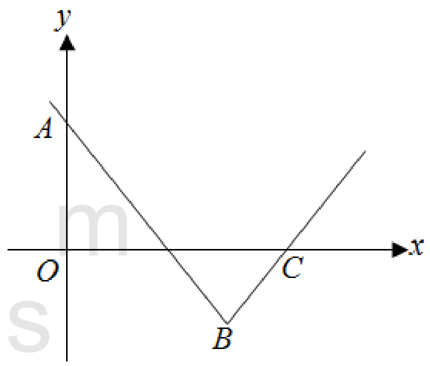
3. (a) Solve the equation $\left|\dfrac{3x^2-16}{2x}\right|=1$.

$$\left|\frac{3x^2-16}{2x}\right|=1$$

$$|3x^2-16|=|2x|$$

$3x^2-16=2x$	**or**	$3x^2-16=-2x$
$3x^2-2x-16=0$		$3x^2+2x-16=0$
$(3x-8)(x+2)=0$		$(3x+8)(x-2)=0$
$x=\dfrac{8}{3},\ -2$		$x=-\dfrac{8}{3},\ 2$

(b) The diagram shows part of the graph of $y=|5-2x|-2$.
Find the coordinates of the points A, B and C.



At A, $x=0$, $y=3$

$\therefore A(0,\ 3)$.

At B, $5-2x=0$

$$x=\frac{5}{2}$$

$\therefore B\left(\dfrac{5}{2},\ -2\right)$.

At C, $y=0$,

$$|5-2x|-2=0$$

$$|5-2x|=2$$

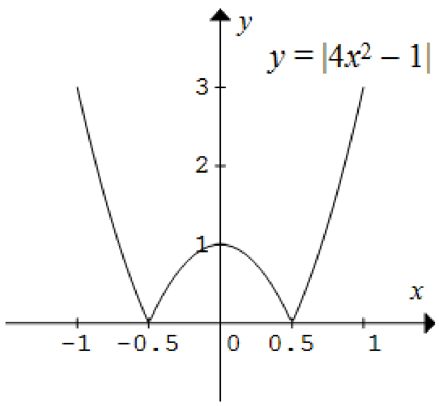
$$2x-5=2$$

$$2x=7$$

$$x=\frac{7}{2}$$

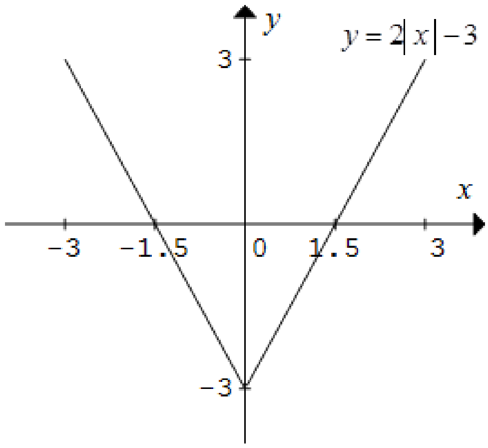
$\therefore C\left(\dfrac{7}{2},\ 0\right)$.

(c) Sketch the graph of $y=|4x^2-1|$ for $-1\le x\le 1$. Hence state the range of y.



Range of y: $0\le y\le 3$

4. (a) Draw the graph of $y = 2|x| - 3$ for $-3 \le x \le 3$. State the range of y.



Range of y: $-3 \le y \le 3$

(b) Solve the equation $|x^2 - 16| = 6x$.

$|x^2 - 16| = 6x$

$x^2 - 16 = 6x$	or	$x^2 - 16 = -6x$
$x^2 - 6x - 16 = 0$		$x^2 + 6x - 16 = 0$
$(x-8)(x+2) = 0$		$(x+8)(x-2) = 0$
$x = 8,\ -2$ (NA)		$x = -8$ (NA), 2

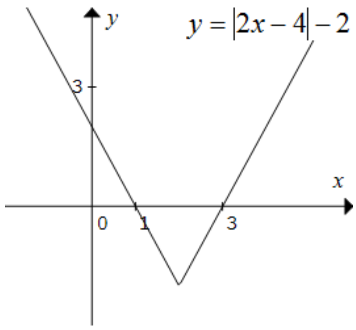
$\therefore x = 2,\ 8$

---

5. (i) Sketch the graph of $y = |2x - 4| - 2$.

(ii) Solve the equation of $|2x - 4| - 2 = x$.

(i)



(ii) $|2x - 4| - 2 = x$

$|2x - 4| = x + 2$

$2x - 4 = x + 2$	or	$2x - 4 = -x - 2$
$x = 6$		$3x = 2$
		$x = \dfrac{2}{3}$

6. (i) Find the coordinates of all points at which the graph of $y = |2x^2 - 5|$ meets the coordinate axes.

(ii) Solve the equation $|2x^2 - 5| = 3x$.

(iii) Sketch, on the same diagram, the graphs of $y = |2x^2 - 5|$ and $y = 3x$.

Hence find the range of values of $|2x^2 - 5| < 3x$.

(i) When $y = 0$, $|2x^2 - 5| = 0$

$$x^2 = \frac{5}{2}$$

$$x = \pm\sqrt{\frac{5}{2}}$$

When $x = 0$, $y = 5$

The three points are (0, 5), $\left(\sqrt{\frac{5}{2}}, 0\right)$ and $\left(-\sqrt{\frac{5}{2}}, 0\right)$.

(ii) $|2x^2 - 5| = 3x$

$2x^2 - 5 = 3x$ or $2x^2 - 5 = -3x$

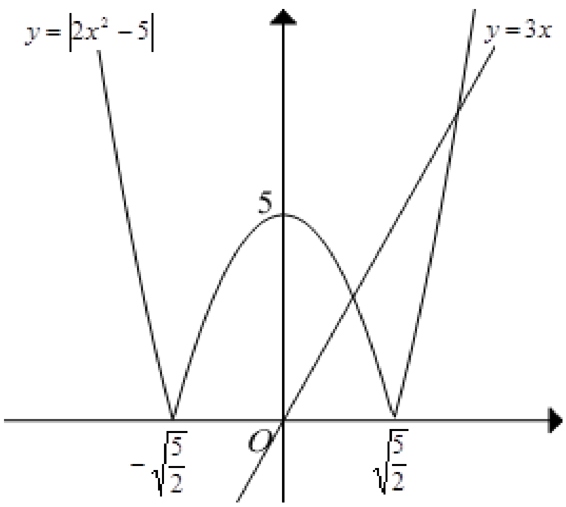
$2x^2 - 3x - 5 = 0$ $\quad$ $2x^2 + 3x - 5 = 0$

$(2x - 5)(x + 1) = 0$ $\quad$ $(2x + 5)(x - 1) = 0$

$x = \frac{5}{2}, -1$ (NA) $\quad$ $x = -\frac{5}{2}$ (NA), 1

$\therefore x = 1, \frac{5}{2}$

(iii)



$$|2x^2 - 5| < 3x \Rightarrow 1 < x < \frac{5}{2}$$

7. Solve the equation $\left|x^2-4x-33\right|=12$.

$$\left|x^2-4x-33\right|=12$$

$x^2-4x-33=12$	or	$x^2-4x-33=-12$
$x^2-4x-45=0$		$x^2-4x-21=0$
$(x-9)(x+5)=0$		$(x-7)(x+3)=0$
$x=9,\ -5$		$x=7,\ -3$

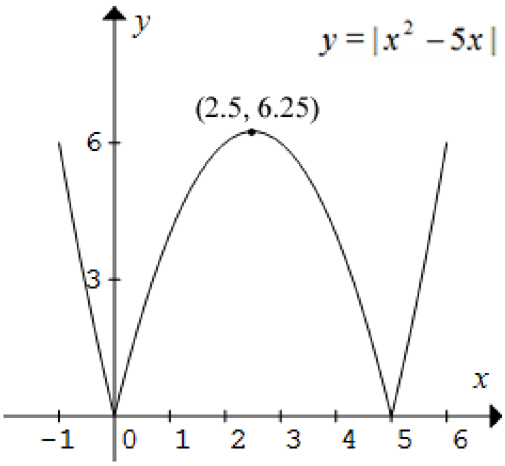
$\therefore x=-5,\ -3,\ 7,\ 9$

---

8. (i) Sketch the graph of $y=|x^2-5x|$ for $-1\le x\le 6$, indicating clearly the points where the curve cuts the x- axis.

(ii) Solve the equation $6-|x^2-5x|=0$.

(i)



(ii) $6-|x^2-5x|=0$

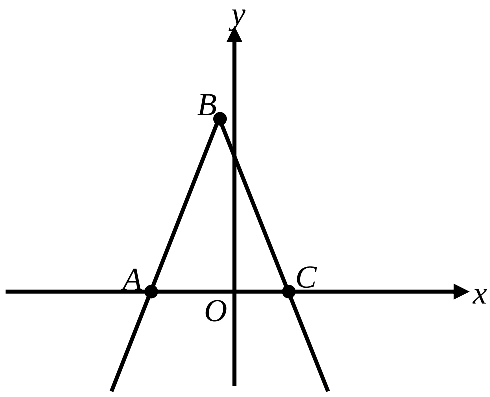
$|x^2-5x|=6$

$x^2-5x=6$	or	$x^2-5x=-6$
$x^2-5x-6=0$		$x^2-5x+6=0$
$(x-6)(x+1)=0$		$(x-2)(x-3)=0$
$x=6,\ -1$		$x=2,\ 3$

$\therefore x=-1,\ 2,\ 3,\ 6$

---

9. The diagram shows part of the graph of $y = 5 - |3x+1|$.



(i) Find the coordinates of A, of B and of C.
(ii) Determine the area of the triangle ABC.
(iii) Solve the equation $5 - |3x+1| = 2x$.

(i) At B,

$$3x + 1 = 0$$
$$x = -\frac{1}{3}$$
$$\therefore B\left(-\frac{1}{3},\ 5\right).$$

At A and C, $y = 0$,

$$5 - |3x+1| = 0$$
$$|3x+1| = 5$$
$$3x + 1 = \pm 5$$
$$3x = 4, -6$$
$$x = \frac{4}{3}, -2$$

$\therefore A(-2, 0)$ and $C\left(\frac{4}{3}, 0\right)$.

(ii) Area of triangle ABC $= \frac{1}{2} \times \left(\frac{4}{3} + 2\right) \times 5$

$$= \frac{25}{3} \text{ unit}^2$$

(iii) $5 - |3x+1| = 2x$

$|3x+1| = 5 - 2x$

$3x + 1 = 5 - 2x$ or $3x + 1 = 2x - 5$

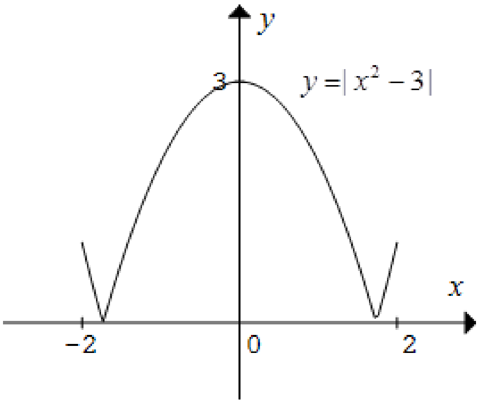
$5x = 4$ $\qquad$ $x = -6$

$x = \frac{4}{5}$

10. (i) Sketch the graph of $y = |x^2 - 3|$ for $-2 \le x \le 2$.

(ii) Solve the equation $x = |3x - 5| - 2$.

(i)



(ii) $x = |3x - 5| - 2$

$|3x - 5| = x + 2$

$3x - 5 = x + 2$ or $3x - 5 = -2 - x$

$2x = 7$ $\quad$ $4x = 3$

$x = \dfrac{7}{2}$ $\quad$ $x = \dfrac{3}{4}$

$\therefore x = \dfrac{7}{2}, \dfrac{3}{4}$

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 5

Name: ______________________________ (   )   Date: ________________

Class: Sec 4______

Revision 5: Partial Fractions

The following three cases will be assessed.

1. $\dfrac{\mathrm{f}(x)}{(ax+b)(cx+d)}=\dfrac{A}{(ax+b)}+\dfrac{B}{(cx+d)}$

2. $\dfrac{\mathrm{f}(x)}{(ax+b)(cx+d)^2}=\dfrac{A}{(ax+b)}+\dfrac{B}{(cx+d)}+\dfrac{C}{(cx+d)^2}$

3. $\dfrac{\mathrm{f}(x)}{(ax+b)(x^2+c^2)}=\dfrac{A}{(ax+b)}+\dfrac{Bx+C}{(x^2+c^2)}$

Exercise

1. Express the following in partial fractions.

(a) $\dfrac{3x^2-9x-10}{(x^2-4)(x+2)}=\dfrac{3x^2-9x-10}{(x-2)(x+2)^2}$

Let $\dfrac{3x^2-9x-10}{(x-2)(x+2)^2}=\dfrac{A}{x-2}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$

$$=\frac{A(x+2)^2+B(x-2)(x+2)+C(x-2)}{(x-2)(x+2)^2}$$

$3x^2-9x-10=A(x+2)^2+B(x-2)(x+2)+C(x-2)$

Let $x=2$, $\quad 12-18-10=16A$
$16A=-16$
$A=-1$

Let $x=-2$, $\quad 12+18-10=-4C$
$-4C=20$
$C=-5$

Comparing coefficient of x^2, $\quad A+B=3$
$B=3-A$
$=4$

$\therefore \dfrac{3x^2-9x-10}{(x-2)(x+2)^2}=-\dfrac{1}{x-2}+\dfrac{4}{x+2}-\dfrac{5}{(x+2)^2}$

(b) $\dfrac{2x}{x^2+6x+5}=\dfrac{2x}{(x+1)(x+5)}$

Let $\dfrac{2x}{(x+1)(x+5)}=\dfrac{A}{x+1}+\dfrac{B}{x+5}$

$$=\frac{A(x+5)+B(x+1)}{(x+1)(x+5)}$$

$2x=A(x+5)+B(x+1)$

Let $x=-1$, $\quad -2=4A$

$A=-\dfrac{1}{2}$

Let $x=-5$, $\quad -10=-4B$

$B=\dfrac{5}{2}$

$\therefore \dfrac{2x}{(x+1)(x+5)}=-\dfrac{1}{2(x+1)}+\dfrac{5}{2(x+5)}$

(c) $\dfrac{2x^2-5x+9}{(x+2)(x-1)}$

Let $\dfrac{2x^2-5x+9}{(x+2)(x-1)}=2+\dfrac{A}{x+2}+\dfrac{B}{x-1}$

$$=\frac{2(x+2)(x-1)+A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$2x^2-5x+9=2(x+2)(x-1)+A(x-1)+B(x+2)$

Let $x=1$, $\quad 6=3B$

$B=2$

Let $x=-2$, $\quad 8+10+9=-3A$

$27=-3A$

$A=-9$

$\therefore \dfrac{2x^2-5x+9}{(x+2)(x-1)}=2-\dfrac{9}{x+2}+\dfrac{2}{x-1}$

(d) $\dfrac{7x^2-21x+8}{(x-5)(x^2+1)}$

$$\text{Let } \frac{7x^2-21x+8}{(x-5)(x^2+1)} = \frac{A}{x-5}+\frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1)+(Bx+C)(x-5)}{(x-5)(x^2+1)}$$

$$7x^2-21x+8 = A(x^2+1)+(Bx+C)(x-5)$$

Let $x=5$, $\quad 175-105+8=26A$

$78=26A$

$A=3$

Comparing coefficient of x^2, $\quad A+B=7$

$B=4$

Let $x=0$, $\quad 8=A-5C$

$8=3-5C$

$C=-1$

$$\therefore \frac{7x^2-21x+8}{(x-5)(x^2+1)} = \frac{3}{x-5}+\frac{4x-1}{x^2+1}$$

(e) $\dfrac{3x^2+4x-20}{(2x+1)(x^2+4)}$

$$\text{Let } \frac{3x^2+4x-20}{(2x+1)(x^2+4)} = \frac{A}{2x+1}+\frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4)+(Bx+C)(2x+1)}{(2x+1)(x^2+4)}$$

$$3x^2+4x-20 = A(x^2+4)+(Bx+C)(2x+1)$$

Let $x=-\frac{1}{2}$, $\quad 3\left(-\frac{1}{2}\right)^2+4\left(-\frac{1}{2}\right)-20 = A\left[\left(-\frac{1}{2}\right)^2+4\right]$

$-\frac{85}{4}=\frac{17A}{4}$

$A=-5$

Comparing coefficient of x^2, $\quad A+2B=3$

$2B=8$

$B=4$

Let $x=0$, $\quad -20=4A+C$

$-20=-20+C$

$C=0$

$$\therefore \frac{3x^2+4x-20}{(2x+1)(x^2+4)} = -\frac{5}{2x+1}+\frac{4x}{x^2+4}$$

(f) $\dfrac{x^2+4x+5}{x^2+5x+6}=1+\dfrac{-x-1}{(x+2)(x+3)}$

Let $\dfrac{-x-1}{(x+2)(x+3)}=\dfrac{A}{x+2}+\dfrac{B}{x+3}$

$$=\frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$$

$-x-1=A(x+3)+B(x+2)$

Let $x=-3$, $\quad 3-1=-B$

$B=-2$

Let $x=-2$, $\quad 2-1=A$

$A=1$

$\therefore \dfrac{x^2+4x+5}{x^2+5x+6}=1+\dfrac{1}{x+2}-\dfrac{2}{x+3}$

(g) $\dfrac{1}{(2x+1)(x^2+1)}$

Let $\dfrac{1}{(2x+1)(x^2+1)}=\dfrac{A}{2x+1}+\dfrac{Bx+C}{x^2+1}$

$$=\frac{A(x^2+1)+(Bx+C)(2x+1)}{(2x+1)(x^2+1)}$$

$1=A(x^2+1)+(Bx+C)(2x+1)$

Let $x=-\dfrac{1}{2}$, $\quad 1=\dfrac{5A}{4}$

$A=\dfrac{4}{5}$

Comparing coefficient of x^2, $\quad A+2B=0$

$B=-\dfrac{A}{2}$

$=-\dfrac{2}{5}$

Let $x=0$, $\quad 1=A+C$

$C=1-\dfrac{4}{5}$

$=\dfrac{1}{5}$

$\therefore \dfrac{1}{(2x+1)(x^2+1)}=\dfrac{4}{5(2x+1)}+\dfrac{1-2x}{5(x^2+1)}$

(h) $\dfrac{8+3x-2x^2}{(x-1)(x+2)^2}$

Let $\dfrac{8+3x-2x^2}{(x-1)(x+2)^2}=\dfrac{A}{x-1}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$

$$=\frac{A(x+2)^2+B(x-1)(x+2)+C(x-1)}{(x-1)(x+2)^2}$$

$8+3x-2x^2=A(x+2)^2+B(x-1)(x+2)+C(x-1)$

Let $x=1$, $\quad 8+3-2=9A$

$A=1$

Let $x=-2$, $\quad 8-6-8=-3C$

$C=2$

Comparing coefficient of x^2, $\quad A+B=-2$

$B=-2-A$

$=-3$

$$\therefore \frac{8+3x-2x^2}{(x-1)(x+2)^2}=\frac{1}{x-1}-\frac{3}{x+2}+\frac{2}{(x+2)^2}$$

---

2. Factorise x^3+3x^2-4 completely.
Hence, express $\dfrac{6}{x^3+3x^2-4}$ in partial fractions.

Let $\mathrm{f}(x)=x^3+3x^2-4$

$\mathrm{f}(1)=1+3-4=0$

$(x-1)$ is afctor of $\mathrm{f}(x)$.

Let $\mathrm{f}(x)=(x-1)(x^2+bx+4)$.

Comparing coefficient of x, $\quad -b+4=0$

$b=4$

$x^3+3x^2-4=(x-1)(x^2+4x+4)$

$=(x-1)(x+2)^2$

Let $\dfrac{6}{(x-1)(x+2)^2}=\dfrac{A}{x-1}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$

$$=\frac{A(x+2)^2+B(x-1)(x+2)+C(x-1)}{(x-1)(x+2)^2}$$

$6=A(x+2)^2+B(x-1)(x+2)+C(x-1)$

Let $x=1$, $\quad 6=9A$

$A=\dfrac{2}{3}$

Let $x=-2$, $\quad 6=-3C$

$C=-2$

Comparing coefficient of x^2, $\quad A+B=0$

$$B=-A$$

$$=-\frac{2}{3}$$

$$\therefore \frac{6}{x^3+3x^2-4}=\frac{2}{3(x-1)}-\frac{2}{3(x+2)}-\frac{2}{(x+2)^2}$$

---

3. Express $\dfrac{x-5}{x^3-x^2+x-1}$ in partial fractions.

$$x^3-x^2+x-1=x^2(x-1)+(x-1)$$

$$=(x-1)(x^2+1)$$

$$\frac{x-5}{x^3-x^2+x-1}=\frac{x-5}{(x-1)(x^2+1)}$$

Let $\dfrac{x-5}{(x-1)(x^2+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$

$$=\frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x+2)^2}$$

$$x-5=A(x^2+1)+(Bx+C)(x-1)$$

Let $x=1$, $\quad -4=2A$

$A=-2$

Let $x=0$, $\quad -5=A-C$

$C=A+5$

$=3$

Comparing coefficient of x^2, $\quad A+B=0$

$$B=-A$$

$$=2$$

$$\therefore \frac{x-5}{(x-1)(x^2+1)}=-\frac{2}{x-1}+\frac{2x+3}{x^2+1}$$

---

4. Express $\dfrac{x^4+9}{x^3+3x}$ in the form $x+\dfrac{A}{x}+\dfrac{Bx+C}{x^2+3}$, where A, B and C are constants to be determined.

$$\frac{x^4+9}{x^3+3x}=\frac{x^4+3x^2-3x^2+9}{x^3+3x}$$

$$=x+\frac{9-3x^2}{x(x^2+3)}$$

Let $\dfrac{9-3x^2}{x(x^2+3)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+3}$

$$=\frac{A(x^2+3)+x(Bx+C)}{x^3+3x}$$

$$9-3x^2=A(x^2+3)+x(Bx+C)$$

Let $x=0$, $\quad 9=3A$

$A=3$

Comparing coefficient of x^2, $\quad A+B=-3$

$B=-3-A$

$=-6$

Let $x=1$, $\quad 9-3=4A+B+C$

$C=6-4A-B$

$=6-12+6$

$=0$

$$\therefore \frac{x^4+9}{x^3+3x}=x+\frac{3}{x}-\frac{6x}{x^2+3}$$

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Plane Geometry

Name: ________________________________ (____) Date: ______________

Class: Sec 4_____

Worksheet 1: Congruent and Similar Triangles

1. **Parallel Lines**

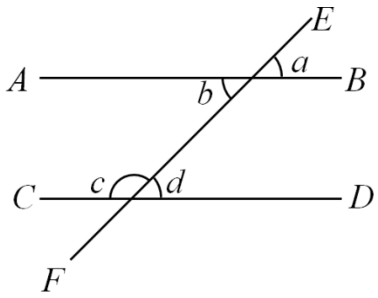
In the diagram, AB and CD are two parallel lines.
If a transversal EF cuts the lines AB and CD,

- the alternates angles are equal ,

  $\angle b = \angle d$ (alt $\angle$s, $AB//CD$),
- the corresponding angles are equal,

  $\angle a = \angle d$ (corr $\angle$s, $AB//CD$),
- the interior angles are supplementary,

  $\angle b + \angle c = 180°$ (int $\angle$s, $AB//CD$).

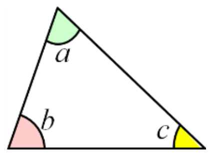


2. **Triangles**

- **Sum of Interior Angles of a Triangle**

  The interior angles of a triangle add up to 180°.

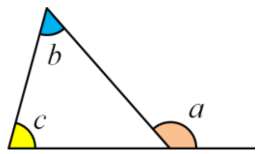
  $\angle a + \angle b + \angle c = 180°$ ($\angle$ sum of Δ)



- **Exterior Angle of a Triangle**

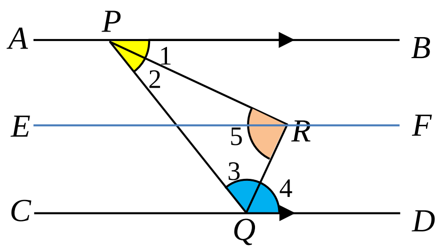
  The measure of an exterior angle of a triangle is equal to the sum of the measures of the two non-adjacent interior angles,

  $\angle a = \angle b + \angle c$ (Ext $\angle$ sum of Δ)



Example 1

In the diagram, the lines AB and CD are parallel. The lines PR bisects angle BPQ and the line RQ bisects angle PQD.
Prove that the line PR is perpendicular to the line QR.

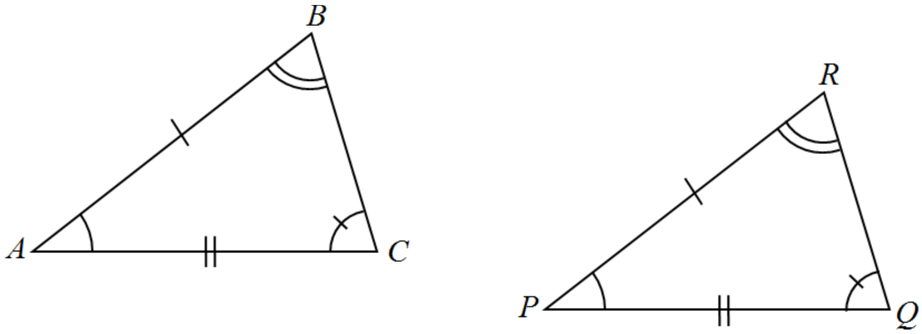


Proof

Statement	Reasons
Insert a line ERF such that $ERF//AB//CD$.	
$\angle 1 = \angle 2$	(PR bisects $\angle BPQ$)
$\angle 3 = \angle 4$	(RQ bisects $\angle PQD$)
$\angle 5 = \angle 2 + \angle 3$	(alt $\angle$s, $ERF//CD$)
$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 90°$	($\angle$ sum of Δ)
$2\angle 5 = 180°$	($\angle$ sum of Δ)
$\angle 5 = 90°$	
Hence $PR \perp QR$.	

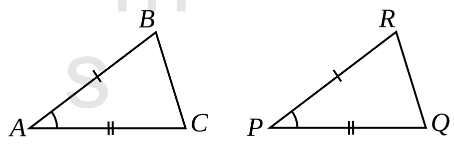
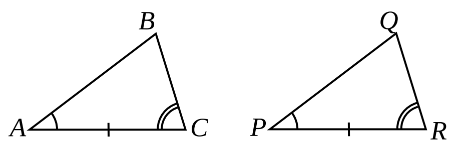
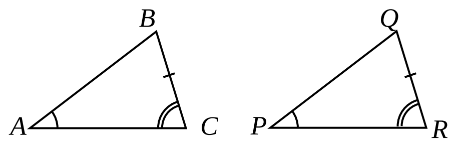
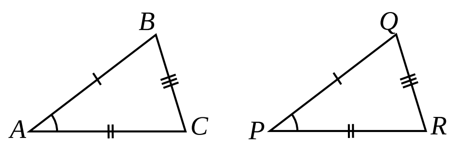
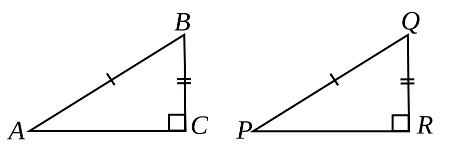
3. **Congruent Triangles**

Two triangles *ABC* and *PQR* are said to be congruent if they have the same shape and size.
i.e. their corresponding sides and angles must be congruent.



Triangles	Corresponding congruent angles	Corresponding congruent sides
$\Delta ABC \equiv \Delta PQR$	$\angle A = \angle P$ $\angle B = \angle Q$ $\angle C = \angle R$	$AB = PQ$ $BC = QR$ $AC = PR$

4. **Tests for Congruent Triangles**

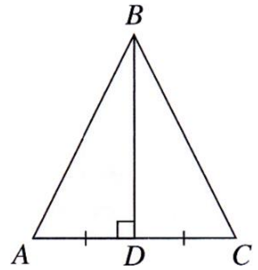
Test			
1	**SAS**	If **two sides** and **the included angle** on one triangle are congruent to the corresponding parts of another, then the triangles are congruent.	
2	**ASA**	If **two angles** and **the included side** of one triangle are congruent to the corresponding parts of another, then the triangles are congruent.	
3	**AAS**	If **two adjacent angles** and **a side** of one triangle are congruent to the corresponding parts of another, then the triangles are congruent.	
4	**SSS**	If the **three sides** of one triangle are congruent to the three sides of another, then the triangles are congruent.	
5	**RHS**	If **the hypotenuse** and **one side** of a **right-angled triangle** are congruent to the corresponding parts of the second right-angled triangle, the two triangles are congruent.	

Example 2

In the diagram, D is the mid-point of AC and the line BD is perpendicular to the line AC.
Prove that

(i) ΔABD is congruent to ΔCBD,

(ii) BD bisects $\angle ABC$.



Proof

(i) BD is common.

$AD = DC$ (Given)

$\angle ADB = \angle CDB$ ($BD \perp AC$)

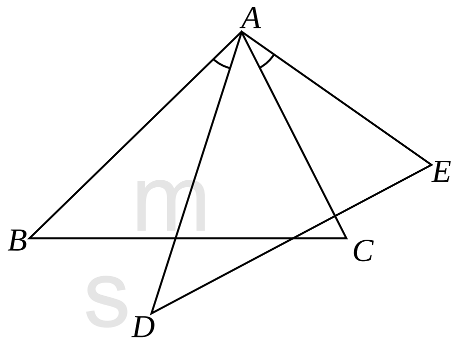
$\therefore \Delta ABD$ is congruent to ΔCBD. (SAS)

(ii) $\angle ABD = \angle CBD$ ($\Delta ABD \equiv \Delta CBD$)

Hence, BD bisects $\angle ABC$.

Example 3

In the figure, $\angle BAD = \angle CAE$, $AB = AD$ and $AC = AE$.
Prove that ABC and ADE are congruent triangles.



Proof

$AB = AD$

$\angle BAD = \angle CAE$

$\angle BAD + \angle DAC = \angle DAC + \angle CAE$

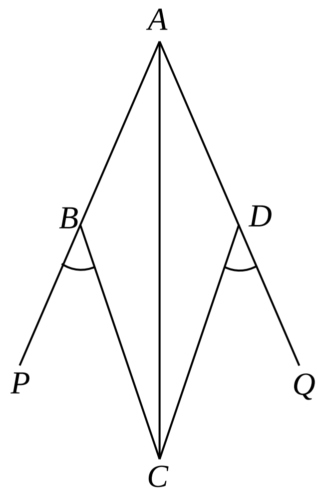
$\therefore \angle BAC = \angle DAE$

$AC = AE$

$\therefore ABC$ and ADE are congruent triangles. (SAS)

Example 4

In the diagram, AC is the angle bisector of $\angle BAD$ and $\angle PBC = \angle CDQ$.
Prove that $AB = AD$.



Proof

$\angle BAC = \angle CAD$ (AC bisects $\angle BAD$)

$\angle ABC = 180° - \angle PBC$

$= 180° - \angle CDQ$

$= \angle ADC$

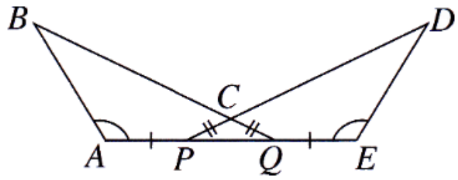
AC is common.

$\therefore$ ΔABC and ΔADC are congruent. (AAS)

Hence, $AB = AD$.

Example 5

In the diagram, $\angle BAQ = \angle DEP$, $AP = EQ$ and $CP = CQ$.



Prove that

(i) $AQ = EP$,

(ii) ΔABQ is congruent to ΔEDP,

(iii) $\angle QBA = \angle PDE$. **(Pg246 Q2)**

Proof

(i) $AQ = AP + PQ$

$= EQ + QP$

$= EP$

(ii) $\angle BAQ = \angle DEP$ (given)

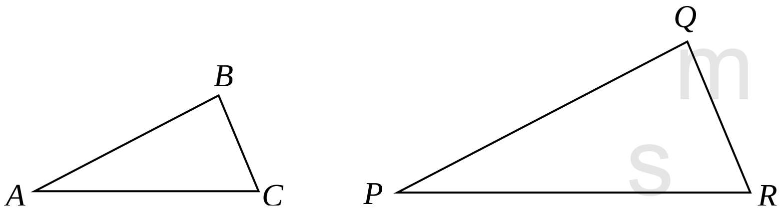
$AQ = EP$ (from (i))

$\angle AQB = \angle EPD$ (base $\angle$s of an isos Δ)

ΔABQ is congruent to ΔEDP. (AAS)

(iii) $\angle QBA = \angle PDE$ ($\Delta ABQ \equiv \Delta EDP$)

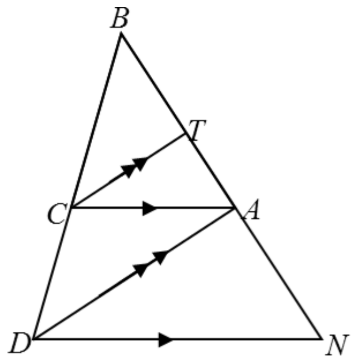
Similar Triangles



Two triangles *ABC* and *PQR* are similar if:

- their corresponding angles are equal $\left(\text{i.e. } \angle A = \angle P, \angle B = \angle Q, \angle C = \angle R\right)$,
- their corresponding sides are proportional with the same ratio. $\left(\text{i.e. } \dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{AC}{PR}\right)$.

Example 6



In the figure, BCD and $BTAN$ are straight lines. $CT \mathbin{//} DA$ and $CA \mathbin{//} DN$.

Prove that

(i) $\angle TCA = \angle ADN$.

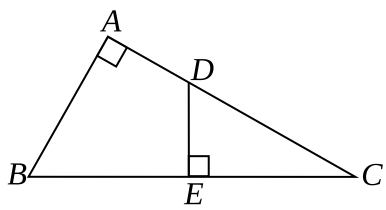
(ii) triangle CAT and triangle DNA are similar.

Complete the following proof.

Proof

	Statement	Reason
(i)	$\angle TCA = \angle CAD$	(Alt $\angle$s, $CT//DA$)
	$\angle CAD = \angle ADN$	(Alt $\angle$s, $CT//DA$)
	$\therefore \angle TCA = \angle ADN$	
(ii)	In ΔCAT and ΔDNA,	
	$\angle TCA = \angle ADN$	(from (i))
	$\angle TAC = \angle AND$	(Corr $\angle$s, $CA//DN$)
	ΔCAT and ΔDNA are similar.	

Example 7



In the diagram, $\angle BAC = \angle DEC = 90°$.

Prove that

(i) $\angle ABC = \angle EDC$,

(ii) $AB \times CD = DE \times BC$.

Proof

(i) $\angle DCE = \angle ACB$ (Common $\angle$)

$\angle BAC = \angle DEC = 90°$ (Given)

$\angle ABC = 90° - \angle DCE$ ($\angle$ sum of Δ)

$= 90° - \angle ABC$

$= \angle BAC$ ($\angle$ sum of Δ)

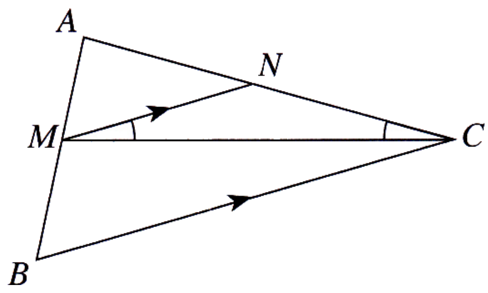
(ii) ΔABC is similar to ΔDEC (AA)

$$\frac{AB}{BC} = \frac{DE}{CD}$$

$\therefore AB \times DE = CD \times BC$

Example 8

In the diagram, MN and BC are parallel lines, and $\angle NMC = \angle NCM$.
Prove that
(i) ΔABC is similar to ΔAMN,
(ii) $NC \times AC = AN \times BC$. **(Pg246 Q4)**



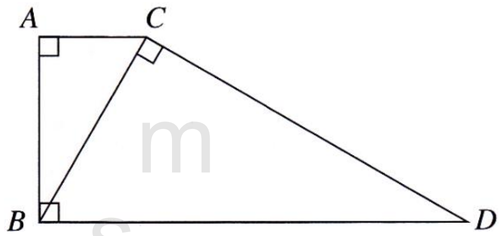
Proof

(i) $\angle MAN = \angle BAC$ (Common $\angle$)
$\angle AMN = \angle ABC$ (Corr $\angle$s, $MN//BC$)
$\therefore$ ΔABC is similar to ΔAMN. (AA)

(ii) $\angle NMC = \angle NCM$ (Given)
$\therefore$ $MN = NC$
$\dfrac{MN}{BC} = \dfrac{AN}{AC}$ (Corr sides of similar Δs)
$MN \times AC = AN \times BC$
$\therefore$ $NC \times AC = AN \times BC$

Example 9

In the diagram, $\angle ABD$, $\angle BAC$ and $\angle DCB$ are right angles.
Prove that
(i) ΔABC is similar to ΔCDB,
(ii) $AC \times BD = CB^2$. **(Pg246Q5)**

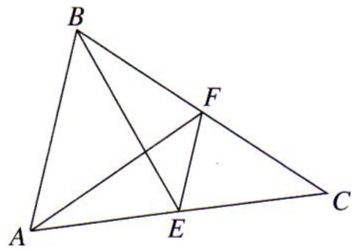


Proof

(i) $\angle CAB + \angle ABD = 180°$
$\therefore AC//BD$ (Int $\angle$s are supplementary)
$\angle BCD = \angle BAC = 90°$ (Given)
$\angle ACB = \angle CBD$ (Alt $\angle$s, $AC//BD$)
ΔABC is similar to ΔCDB. (AA)

(ii) $\dfrac{AC}{BC} = \dfrac{BC}{BD}$ (Corr sides of similar Δs)
$\therefore AC \times BD = CB^2$

Example 10



In the diagram, $\angle EAF = \angle EBF$, and E and F are points on AC and BC respectively.
Prove that
(i) $\angle AFC = \angle BEC$,
(ii) ΔAFC is similar to ΔBEC ,
(ii) $CF \times CB = CE \times CA$. **(Pg248Q14)**

Complete the following proof.

	Statement	**Reason**
(i)	$\angle ACF = \angle BCE$	(Common $\angle$)
	$\angle ACF = \angle CBE$	(Given)
	$\therefore\ \angle AFC = \angle BEC$	($\angle$ sum of Δ)
(ii)	$\angle ACF = \angle BCE$	(Common $\angle$)
	$\angle ACF = \angle CBE$	(Given)
	$\therefore \Delta AFC$ is similar to ΔBEC.	(AA)
(iii)	$\dfrac{CF}{CE} = \dfrac{CA}{CB}$	(Corr sides of similar Δs)
	$\therefore CF \times CB = CE \times CA$	

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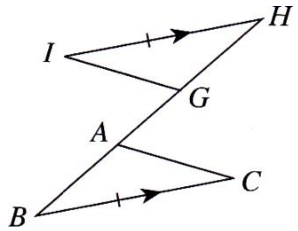
Assignment 2: Congruent and Similar Triangles

1. In the diagram, $CB = HI$, $GB = AH$ and the line BC is parallel to the line IH.
Prove that
(i) ΔGHI is congruent to ΔABC,
(ii) $CA//GI$. **(Pg268Q1)**



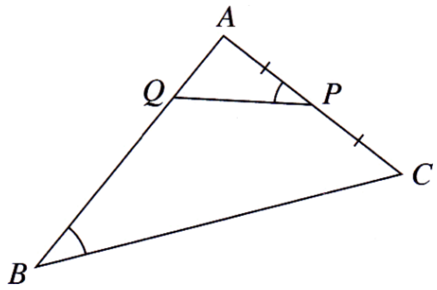
Proof

(i) $CB = HI$ (Given)
$\angle IHG = \angle CBA$ (Alt $\angle$s, $BC//HI$)
$GB = AH$ (Given)
$BA + AG = HG + GA$
$\therefore BA = HG$
ΔGHI is congruent to ΔABC. (SAS)

(ii) $\angle HIG = \angle BAC$ (Corr $\angle$s of congruent Δs)
$\therefore \angle IGA = \angle CAG$ (Adj $\angle$s on a st line)
Since the alternate angles are equal, $CA//GI$.

---

2. In the diagram, P is the mid-point of AC and $\angle APQ = \angle ABC$.
Prove that
(i) ΔAPQ is similar to ΔABC,
(ii) $AQ = \dfrac{AC^2}{2AB}$. **(Pg268Q2)**



Proof

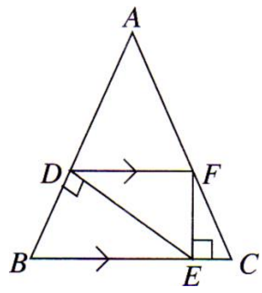
(i) $\angle QAP = \angle CAB$ (Common $\angle$)
$\angle APQ = \angle ABC$ (Given)
ΔAPQ is congruent to ΔABC. (AA)

(ii) $\dfrac{AQ}{AC} = \dfrac{AP}{AB}$ (Corr sides of similar Δs)

$\dfrac{AQ}{AC} = \dfrac{\frac{1}{2}AC}{AB}$ ($2AP = AC$)

$\therefore AQ = \dfrac{AC^2}{2AB}$

3. The diagram shows an isosceles triangle ABC with $AB = AC$. The points D, E and F lie on AB, BC and CA respectively, such that DF is parallel to BC, ED is perpendicular to BC.
Prove that.
(i) ΔBDE is similar to ΔEFD,
(ii) ΔEFD is similar to ΔCEF. **(Pg247Q12)**



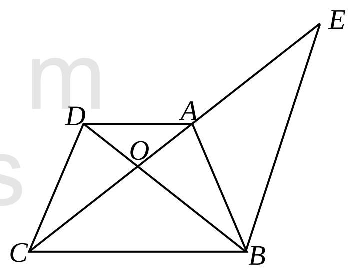
Proof

(i) $\angle EDF = \angle BED$ (Alt $\angle$s, DF//BE)
$\angle EFD = \angle FEC$ (Alt $\angle$s, DF//BE)
$= 90°$
$= \angle BDE$
$\therefore \Delta BDE$ is similar to ΔEFD. (AA)

(ii) $\angle EFD = \angle FEC$ (Alt $\angle$s, DF//BE)
$\angle DBE = \angle ECF$ (Base $\angle$s of an isos Δ)
$= \angle FED$ (Corr$\angle$s of similar ΔBDE and ΔEFD)
$\therefore \Delta EFD$ is similar to ΔCEF. (AA)

---

4. In the diagram $ABCD$ is an isosceles trapezium where AD is parallel to BC. CA is produced to a point E such that and BE is parallel to CD.
Prove that $OC^2 = OA \times OE$



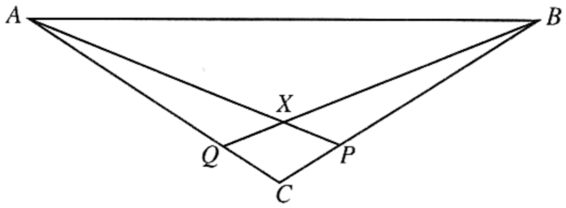
$\angle ODA = \angle OBC$ (Alt $\angle$s, $CD//BE$)
$\angle OAD = \angle OCB$ (Alt $\angle$s, $CD//BE$)
ΔODA is similar to ΔOBC. (AA)
$\therefore \dfrac{OD}{OB} = \dfrac{OA}{OC}$ ......(1)

$\angle DCO = \angle OEB$ (Alt $\angle$s, $CD//BE$)
$\angle DOC = \angle EOB$ (Vert opp $\angle$s)
ΔDOC is similar to ΔBOE. (AA)
$\therefore \dfrac{OC}{OE} = \dfrac{OD}{OB}$ ......(1)

Comparing (1) and (2)
$\dfrac{OC}{OE} = \dfrac{OA}{OE}$
$\therefore OC^2 = OA \times OE$

5. The diagram shows an isosceles triangle ABC in which $AC = BC$. Lines are draw from A and B to meet BC and AC at P and Q respectively. The lines AP and BQ intersect at X.

Given that $PC = QC$, show that
(i) AXB is an isosceles triangle,
(ii) $PX = QX$. **(2009)**



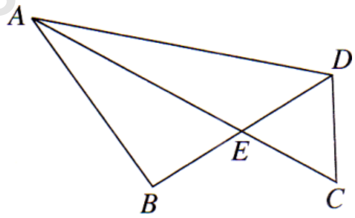
Proof

(i) $AC = BC$ (Given)
$AQ + QC = BP + PC$
$\therefore AQ = BP$
$\angle QAB = \angle PBA$ (Base $\angle$s of an isos Δ)
AB is common.
ΔQAB and ΔPBA are congruent. (SAS)
$\angle XAB = \angle XBA$ (Corr $\angle$s of congruent Δs)
Hence, AXB is an isosceles Δ.

(ii) $\angle BXP = \angle AXQ$ (vert opp $\angle$s)
$BX = AX$ (AXB is an isosceles Δ)
$\angle PBX = \angle QAX$ (ΔACP and ΔBCQ are congruent.)
ΔBXP and ΔAXQ are congruent. (AAS)
Hence, $PX = QX$.

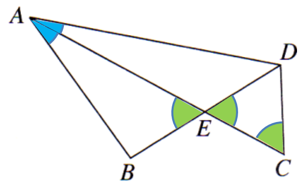
---

6. In the diagram, triangle CDE is isosceles with $DE = DC$, and the line AE bisects the angle BAD. Prove that
(i) ΔABE is similar to ΔADC,
(ii) $AB \times AC = AD \times AE$. **(Pg247Q11)**



Proof

(i) $\angle DCE = \angle DEC$ (Base $\angle$s of isos Δ)
$= \angle AEC$ (Vert opp $\angle$s)
$\angle DAE = \angle EAB$ (AE bisects $\angle BAD$)
$\therefore \Delta ABE$ is similar to ΔADC. (AA)



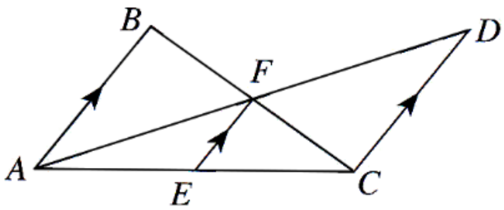
(ii) $\dfrac{AB}{AD} = \dfrac{AE}{AC}$ (ΔABE is similar to ΔADC)
$\therefore AB \times AC = AD \times AE$

---

7. In the diagram, the lines AB, EF and CD are parallel to one another. Given that $AE : EC = 1 : 2$, prove that

(i) $BF : BC = 1 : 2$,

(ii) $AB : CD = 1 : 1$. **(Pg248Q15)**



Proof

(i) $\angle ECF = \angle ACB$ (Common $\angle$)

$\angle CFE = \angle CBA$ (Corr $\angle$s, $AB//EF$)

$\therefore \Delta ACB$ and ΔECF are similar. (AA)

$$\frac{CF}{CB} = \frac{CE}{CA} = \frac{1}{2} \qquad \text{(Corr sides of similar } \Delta\text{s)}$$

$$CF = \frac{1}{2} BC$$

$$\therefore BF : BC = 2 : 1$$

(ii) $\angle FAE = \angle DAC$ (Common $\angle$)

$\angle AEF = \angle ACD$ (Corr $\angle$s, $CD//EF$)

$\therefore \Delta ACB$ and ΔECF are similar. (AA)

$$\frac{EF}{CD} = \frac{AE}{AC} = \frac{1}{2} \qquad \text{(Corr sides of similar } \Delta\text{s)}$$

$$CD = 2EF$$

$$\frac{EF}{AB} = \frac{1}{2} \qquad \text{(Corr sides of similar } \Delta\text{s, from (i))}$$

$$AB = 2EF$$

$$AB = 2EF = CD$$

Hence, $AB : CD = 1 : 1$

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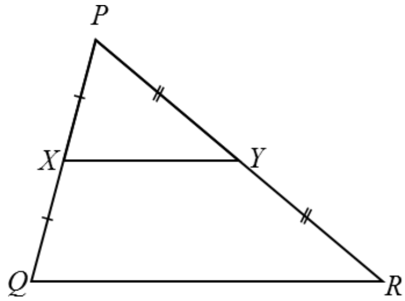
Worksheet 3: The Midpoint Theorem

The Midpoint Theorem

If PQR is a triangle with X and Y as midpoints of PQ and PR respectively, then

- the straight line XY is parallel to QR
- $XY = \frac{1}{2}QR$



Proof

$PX = XQ$ and $PY = YR$ (Given)

$\therefore \frac{PX}{PQ} = \frac{1}{2}, \frac{PY}{PR} = \frac{1}{2}$

Hence, $\frac{PX}{PQ} = \frac{PY}{PR}$

Also $\angle XPY = \angle QPR$ (Common $\angle$s)

Hence, ΔXPY is similar to ΔQPR. (2 corr sides are proportional & the included $\angle$s are equal)

$\therefore \angle PXY = \angle PQR$ (Corr $\angle$s)

Hence, $XY // QR$.

$\frac{XY}{QR} = \frac{1}{2} \Rightarrow XY = \frac{1}{2}QR$ (Similar Δs)

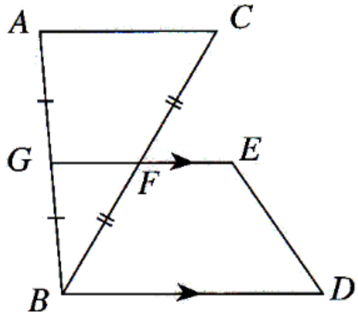
Example 1

In the diagram, G and F are the midpoints of AB and BC respectively. GE and BC intersect at the point F, GE is parallel to BD. Prove that AC is parallel to BD. **(Pg246Q6)**

Proof

$AC // GFE$ (Midpoint Theorem)

$\therefore AC // GE // BD$

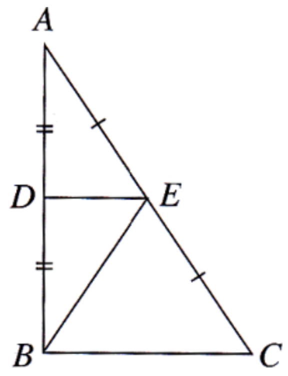


Example 2

In the diagram, D and E are the midpoints of AB and AC respectively.

(i) Prove that $DE//BC$.

(ii) If $BC = 2x$ cm, explain why $DE = x$ cm. **(Pg247Q7)**



Proof

(i) $DE//BC$ (Midpoint Theorem)

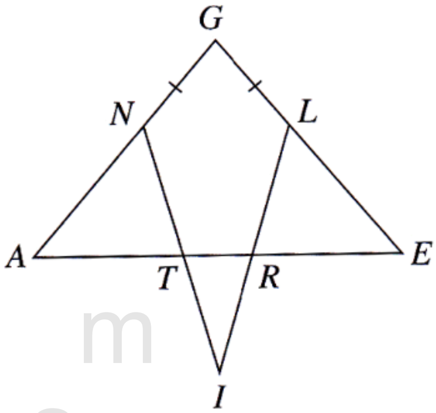
(ii) By Midpoint Theorem, if $BC = 2x$ cm then $DE = \frac{1}{2} BC$

$= x$ cm

Example 3

In the diagram, $AG = EG$, $NG = LG$, $AR = ET$ and $NI = LI$. The points T and R are midpoints of NI and LI respectively.

Prove that ΔANT is congruent to ΔELR.



Proof

$TR//NL$	(Midpoint Theorem)
$\angle LNT = \angle NLR$	(Base $\angle$s of an isos Δ)
$\angle LNT = \angle NTA$	(Alt $\angle$s, $TR//NL$)
$\angle NLR = \angle LRE$	(Alt $\angle$s, $TR//NL$)
$\therefore\ \angle NTA = = \angle LRE$	
$\angle NAT = \angle LER$	(Base $\angle$s of an isos Δ)

$AR = ET$

$AT + TR = TR + RE$

$\therefore\ AT = RE$

$\therefore \Delta ANT$ is congruent to ΔELR. (AAS)

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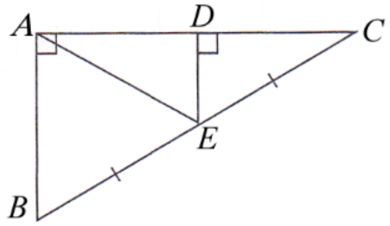
<u>Assignment 4: The Midpoint Theorem</u>

1. In the diagram, ABC and DEC are right-angled triangles with $\angle BAC = \angle EDC = 90°$. E is the midpoint of BC.



Prove that

(i) D is the midpoint of AC,

(ii) AEC is an isosceles triangle. **(Pg248Q14)**

<u>Proof</u>

(i) $\angle BAC = \angle EDC = 90°$ (Given)

$\angle DCE = \angle ACB$ (Common $\angle$)

$\therefore \Delta ACB$ and ΔDCE are similar. (AA)

$$\frac{CE}{CB} = \frac{CD}{CA} = \frac{1}{2}$$ (Corr sides of similar Δs)

$CA = 2CD$

Hence, D is the midpoint of AC.

(ii) $AD = CD$ (Given)

$\angle ADE = \angle CDE = 90°$ (adj $\angle$s on a st line)

DE is common.

$\therefore \Delta CDE \equiv \Delta ADE.$ (SAS)

$EC = EA$ (Corr sides of congruent Δs)

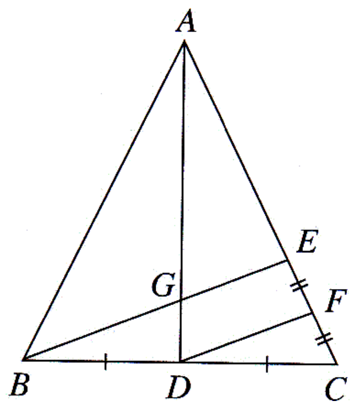
Hence, AEC is an isosceles triangle.

2. In the diagram, D and F are the midpoints of BC and EC respectively. The points E is on AC such that $AC = 3EC$. BE and AD intersect at the point G.

Show that

(i) $AE = 4EF$,

(ii) $\dfrac{AG}{GD} = 4$. **(Pg248Q18)**



<u>Proof</u>

(i) $AE = AC - EC$

$= 3EC - EC$

$= 2EC$

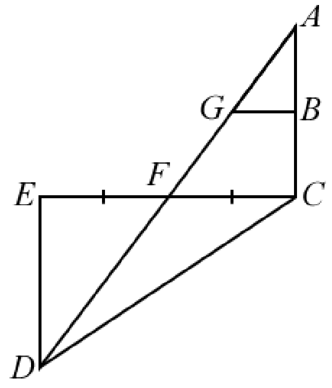
$= 2(2EF)$

$= 4EF$

(ii) $DE//EB$	(Midpoint Theorem)
$\angle EAG = \angle FAD$	(Common $\angle$)
$\angle AEG = \angle AFD$	(Corr $\angle$s, $DE//EB$)
$\therefore \Delta AEG$ and ΔAFD are similar.	(AA)
$\dfrac{AG}{GD} = \dfrac{AE}{EF} = \dfrac{4}{1}$	(Corr sides of similar Δs)

$\therefore \dfrac{AG}{GD} = 4$

3. In the diagram, B and G are the midpoints of AC and AF respectively. F is the midpoint of both AD and EC. AC is parallel to DE.



Prove that

(i) $GB = \dfrac{EC}{4}$,

(ii) area of ΔAGB = $\dfrac{1}{8}$ of the area of ΔACD.

Proof

(i) $FC = 2GB$ (Midpoint Theorem)

$EC = 2FC$

$= 2(2GB)$

$\therefore GB = \dfrac{EC}{4}$

(ii) $\angle GAB = \angle FAC$ (Common $\angle$)

$\angle GBA = \angle FCA$ (Corr $\angle$s, $GB//FC$)

$\therefore \Delta AGB$ and ΔAFC are similar. (AA)

$$\frac{\text{Area of } \Delta AGB}{\text{Area of } \Delta AFC} = \left(\frac{1}{2}\right)^2$$

$$\text{Area of } \Delta AGB = \frac{1}{4} \text{Area of } \Delta AFC$$

$$\text{Area of } \Delta AFC = \frac{1}{2} \text{Area of } \Delta ACD$$

$$\begin{aligned}\text{Area of } \Delta AGB &= \frac{1}{4} \text{Area of } \Delta AFC \\ &= \frac{1}{4} \times \frac{1}{2} \text{Area of } \Delta ACD \\ &= \frac{1}{8} \text{Area of } \Delta ACD\end{aligned}$$

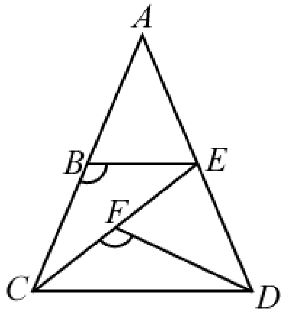
4. In the diagram, B and E are the midpoints of AC and AD respectively. FD bisects CE and $\angle CBE = \angle DFC$.

Prove that

(i) BE is parallel to CD,

(ii) triangle BEC is similar to triangle FCD,

(iii) $\dfrac{BE}{FC} = \dfrac{EC}{CD}$.



<u>Proof</u>

(i) $BE // CD$	(Midpoint Theorem)
(ii) $\angle BEC = \angle CED$	(Given)
$\angle CBE = \angle FCD$	(Alt $\angle$s, $BE//CD$)
ΔBEC is similar to ΔFCD.	(AA)
(iii) $\dfrac{BE}{FC} = \dfrac{EC}{CD}$	(Corr sides of similar Δs)

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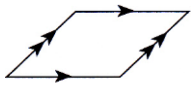
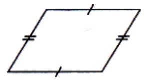
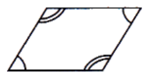
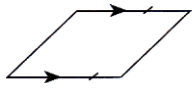
Worksheet 5: Angles Properties of Quadrilaterals

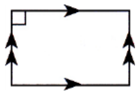
The table below shows some properties commonly used for proving special quadrilaterals.

(TBPg251)

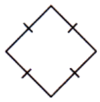
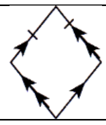
1. Parallelogram

• quadrilateral with 2 pairs of parallel sides	
• quadrilateral with 2 pairs of equal sides	
• quadrilateral with 2 pairs of equal opposite angles	
• quadrilateral with 1 pair of equal and parallel opposite sides	

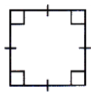
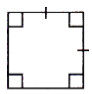
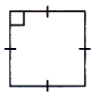
2. Rectangle

• quadrilateral with 4 right angles	
• parallelogram with 1 right angle	

3. Rhombus

• quadrilateral with 4 equal sides	
• parallelogram with equal adjacent sides	

4. Square

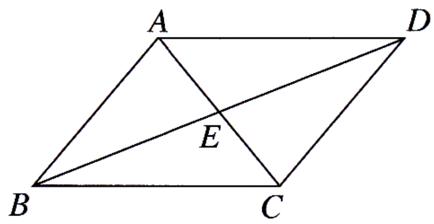
• quadrilateral with 4 equal sides and 4 right angles	
• rectangle with equal adjacent sides	
• rhombus with 1 right angle	

Example 1

The diagram shows a quadrilateral *ABCD* with its diagonals *AC* and *BD* bisecting each other at *E*.
Prove that

(i) triangle *ABE* is congruent to triangle *CDE*,

(ii) *ABCD* is a parallelogram. **(Pg255Q1)**



Proof

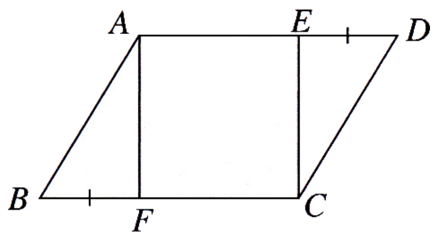
(i)	$AE = EC$	(Given)
	$\angle AEB = \angle CED$	(vert opp $\angle$s)
	$BE = ED$	(Given)
	$\Delta ABE \equiv \Delta CDE$	(SAS)
(ii)	$AB = CD$	(corr sides of congruent Δs)
	$\angle BAE = \angle DCE$	(corr $\angle$s of congruent Δs)
	$\therefore AB // DC$	(alt $\angle$s are equal)
	Hence *ABCD* is a parallelogram.	

Example 2

In the diagram, *AFCE* is a square and *BF* = *ED*.
Prove that

(i) triangle *ABF* is congruent to triangle *CDE*,

(ii) *ABCD* is a parallelogram. **(Pg255Q2)**



Proof

(i)	$AF = CE$	(Sides of a square)
	$\angle AFB = \angle CED = 90°$	(Adj $\angle$s on a st line)
	$BF = ED$	(Given)
	$\Delta ABF \equiv \Delta CDE$	(SAS)
(ii)	$AB = DC$	(corr sides of congruent Δs)
	$AD // BC$	(opp sides of a square are //)
	Hence *ABCD* is a parallelogram	

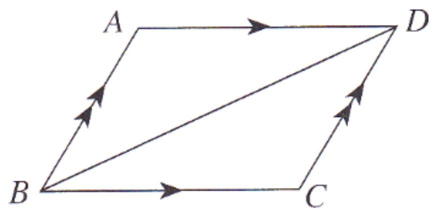
Example 3

The diagram shows a parallelogram $ABCD$ with its diagonal BD.

(i) Using the property of parallel lines in a parallelogram, prove that ΔABD is congruent to ΔCDB.

(ii) Hence prove that the parallelogram $ABCD$ has two pairs of equal sides. **(Pg255Q3)**

Proof



(i) $ADB = \angle CBD$ (alt $\angle$s)

$\angle ABD = \angle CDB$ (alt $\angle$s)

$BD = DB$ (Common sides)

$\Delta ABD \equiv \Delta CDB$ (AAS)

(ii) $AB = DC$ (corr sides of congruent Δs)

$AD = BC$ (corr sides of congruent Δs)

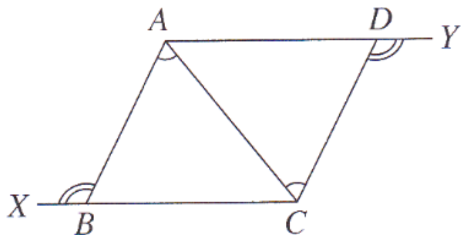
Hence $ABCD$ has two pairs of equal sides.

Example 4

In the diagram, $ABCD$ is a quadrilateral. X and Y are on CB and AD produced respectively, $\angle BAC = \angle ACD$ and $\angle XBA = \angle YDC$.

Show that $ABCD$ is a parallelogram. **(Pg255Q5)**

Proof



$\angle BAC = \angle ACD$ (Given)

$\therefore AB//DC$ (Alt $\angle$s are equal)

$\angle ABC = \angle CDA$ (adj $\angle$s on a st line)

$AC = CA$ (common sides)

$\Delta ABC \equiv \Delta CDA$ (AAS)

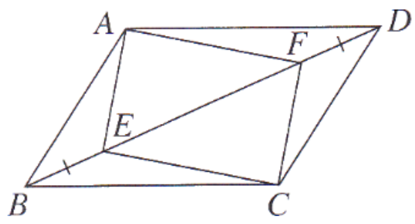
$\angle BCA = \angle DAC$ (Corr $\angle$s of congruent Δs)

$\therefore AD//BC$ (Alt $\angle$s are equal)

Hence $ABCD$ is a parallelogram.

Example 5

The diagram shows a parallelogram *ABCD* and its diagonal *BEFD*.
$BE = FD = 4$ cm and $CF = 7$ cm.



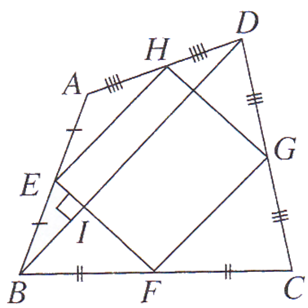
(i) Show that triangle *ABE* is congruent to triangle *CDF*.
(ii) Show that *AECF* is a parallelogram.
(iii) Find the length of *AE*. **(Pg255Q7)**

Proof

(i)	$AB = CD$	(opp sides of a //gram)
	$\angle ABE = \angle CDF$	(alt $\angle$s)
	$BE = FD$	(Given)
	$\Delta ABE \equiv \Delta CFE$	(SAS)
(ii)	$CF = AE = 7$ cm	(corr sides of congruent Δs)
	$\angle AEF = \angle CDE$	(corr $\angle$s of congruent Δs, adj $\angle$s on a st line)
	$EF = FE$	(common sides)
	$\Delta AEF \equiv \Delta CFE$	(SAS)
	$\angle AFE = \angle CFE$	(corr $\angle$s of congruent Δs)
	$\angle AEF = \angle CEF$	(corr $\angle$s of congruent Δs)
	Hence *AF*//*EC* and *AE*//*FC*.	(alt $\angle$s are equal)
	Hence *AECF* is a parallelogram.	
(iii)	$AE = CF = 7$ cm	(corr sides of congruent Δs)

Example 6

The diagram shows a quadrilateral *ABCD* with its diagonal *BD*. The points *E*, *F*, *G* and *H* are the mid-points of the sides *AB*, *BC*, *CD* and *DA* respectively.



If *BD* is perpendicular to *EF* at the point *I* show that
(i) *EH* is parallel to *FG* and $\angle EFG = 90°$,
(ii) *EFGH* is a rectangle. **(Pg255Q10)**

Proof

(i)	*EH*//*BD*	(Midpoint Thm)
	FG//*BD*	(Midpoint Thm)
	Hence *EF*//*FG*.	
	$\angle EIB = \angle EFG = 90°$.	(alt $\angle$s)
(ii)	*HG*//*AC*	(Midpoint Thm)
	EF//*AC*	(Midpoint Thm)
	Hence *HG*//*EF*.	
	$\angle EFG = 90°$	(corr $\angle$s of congruent Δs)
	$\angle AEF = \angle HFG = 90°$	(int $\angle$s)
	Hence *EFGH* is a rectangle.	

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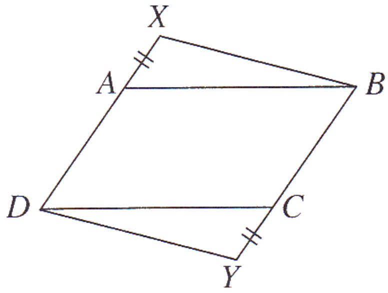
<u>Assignment 6: Angle Properties of Quadrilaterals</u>



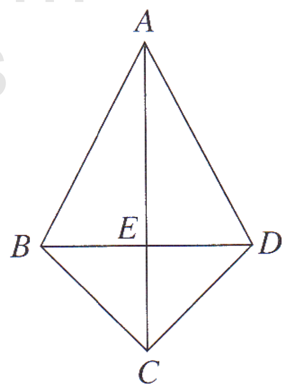
1. In the diagram, *ABCD* is a parallelogram. The sides *DA* and *BC* are produced to points *X* and *Y* respectively such that *AX* = *CY*. Prove that *XBYD* is a parallelogram. **(Pg255Q6)**

<u>Proof</u>

AD = *BC* (opp. sides of parallelogram)
AX = *CY* (given)
$\therefore$ *DX* = *BY*

AD // *BC* (opp. sides of parallelogram)
$\therefore$ *DX* // *BY*

Since *DX* = *BY* and *DX* // *BY*, *XBYD* is a parallelogram.



2. The diagram shows a kite *ABCD* with its diagonals *AC* and *BD* intersecting at *E*.
Prove that

(i) ΔABC is congruent to ΔADC,
(ii) ΔBCE is congruent to ΔDCE,
(iii) *BD* is perpendicular to *AC*. **(Pg255Q8)**

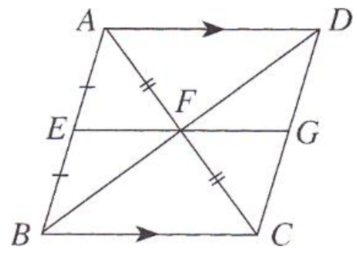
<u>Proof</u>

(i) In ΔABC and ΔADC,
AB = *AD* (adj. sides of kite)
BC = *DC* (adj. sides of kite)
AC = *AC* (common side)
Hence ΔABC is congruent to ΔADC. (SSS)

(ii) In ΔBCE and ΔDCE,
BC = *DC* (adj. sides of kite)
$\angle BCE = \angle DCE$ (corr $\angle$s of congruent Δs)
EC = *EC* (common side)

Hence ΔBCE is congruent to ΔDCE. (SAS)

3. In the diagram, AD is parallel to BC, $AE = EB$ and $AF = FC$. EFG is a straight line, where G lies on CD.



Prove that
(i) F is the mid-point of BD,
(ii) G is the mid-point of CD,
(iii) $ABCD$ is a parallelogram. **(Pg255Q11)**

Proof

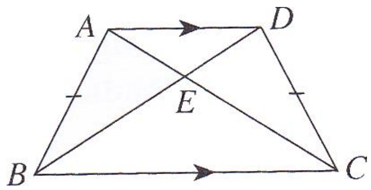
(i) $AF = CF$ (given)
$\angle ADF = \angle CBF$ (alt. $\angle$s, AD // BD)
$\angle AFD = \angle CFB$ (vert. opp. $\angle$s)
Hence $\triangle ADF$ is congruent to $\triangle CBF$. (AAS)
$\therefore DF = BF$ (corr sides of congruent $\triangle$s)
Hence F is the mid-point of BD.

(ii) EFG // BC (Midpt Thm)
$\angle DFG = \angle DBC$ (corr $\angle$ s)
$\angle FDG = \angle BDC$ (common$\angle$ s)
ΔFDG is similar to ΔBDC. (AA)
$\therefore \dfrac{DF}{DB} = \dfrac{DG}{GC} = \dfrac{1}{2}$
Hence, G is the mid-point of CD.

(iii) From (i), ΔADF is congruent to ΔCBF.
$\therefore AD = CB$
Since $AD = BC$ and AD // BC, $ABCD$ is a parallelogram.

---

4. In the diagram, $ABCD$ is a trapezium with $AB = CD$.



Prove that
(i) ΔBAD is congruent to ΔCDA.
(ii) $AE = DE$,
(iii) ΔBEC is isosceles. **(Pg256Q12)**

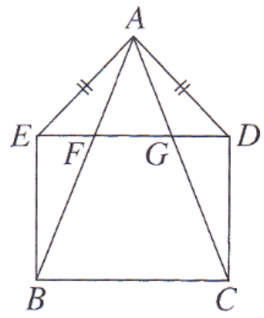
Proof

(i) Since AD // BC and $AB = CD$,
$\angle BAD = \angle CDA$
$AB = CD$ (given)
$AD = AD$ (common side)
Hence ΔBAD is congruent to ΔCDA. (SAS)

(ii) $\angle BDA = \angle CAD$. (corr $\angle$s of congruent Δs)
Hence $AE = DE$. (base $\angle$s of isos Δ)

(iii) $\angle BCE = \angle EBC$ (alt. $\angle$s, AD // BC)
$\therefore BE = EC$ (base $\angle$s of isos Δ)
Hence ΔBEC is isosceles.

5. In the diagram, $BCDE$ is a rectangle. ΔAED is isosceles with $AD = AE$. AFB and AGC are straight lines.



Show that
(i) ΔABE is congruent to ΔACD,
(ii) ΔAEF is congruent to ΔADG. **(Pg256Q13)**

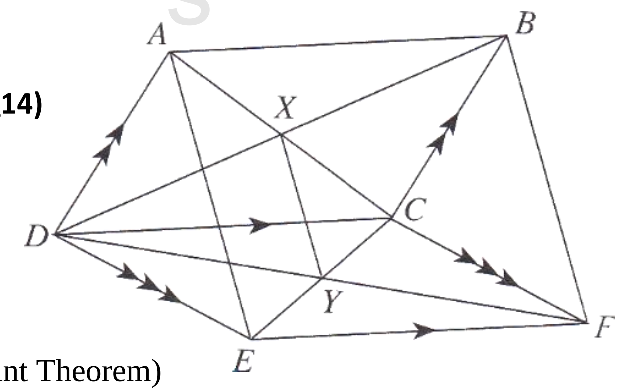
<u>Proof</u>

(i)	$\angle AED = \angle ADE$	(base $\angle$s of isos. ΔAED)
	$\angle BEF = \angle CDG = 90°$	($\angle$s of rectangle)
	$\angle AED + \angle BEF = \angle ADE + \angle CDG$	
	Hence, $\angle AEB = \angle ADC$.	
	$AE = AD$	(given)
	$BE = CD$	(opp sides of rectangle)
	Hence ΔABE is congruent to $\triangle ACD$.	(SAS)
(ii)	$\angle EAF = \angle DAG$	(corr $\angle$s of congruent Δs)
	$AE = AD$	(given)
	$\angle AEF = \angle ADG$	(base $\angle$s of isos Δ)
	Hence ΔAEF is congruent to ΔADG.	(ASA)

---

6. In the diagram, $ABCD$ and, $CDEF$ are parallelograms. The point X is the mid-point of AC and of BD, and the point Y is the mid-point of CE and of DF.

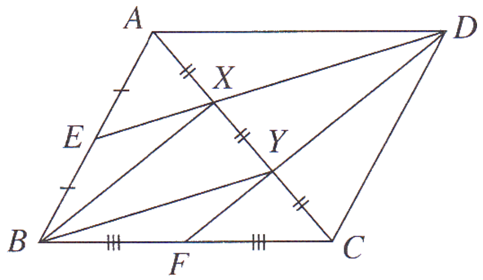
Prove that
(i) AE is parallel to XY,
(ii) AE is parallel to BF,
(iii) $BAEF$ is a parallelogram. **(Pg256Q14)**



<u>Proof</u>

(i)	AE // XY	(Mid-point Theorem)
(ii)	XY// BF	(Mid-point Theorem)
	Hence, AE // XY // BF.	
(iii)	AB // CD and CD // EF	
	Hence, AB // EF.	
	$\therefore$ $BAEF$ is a parallelogram.	

7. In the diagram below, $ABCD$ is a quadrilateral. $AE = BE$ and $BF = FC$. The diagonal AC intersects DE at X and DF at Y, so that $AX = XY = YC$.



Prove that
(i) $BXDY$ is a parallelogram,
(ii) ΔAXB is similar to ΔCYD.
(iii) $ABCD$ is a parallelogram. **(Pg270Q5)**

Proof

(i)	$EX // BY$	(Mid-point Theorem)
	$FY // BX$	(Mid-point Theorem)
	Hence, $BXDY$ is a parallelogram.	
(ii)	$\angle AXB = \angle XYF$	(corr. $\angle$s, $BX // FY$)
	$= \angle CYD$	(vert. opp. $\angle$s)
	$AX = CY$	(given)
	$BX = DY$	(opp. sides of parallelogram)
	$\therefore\ \Delta AXB$ is congruent to ΔCYD.	(SAS)
(iii)	$\angle BAX = \angle DCY$.	(corr $\angle$s of congruent Δs)
	$AB // CD$	(alt $\angle$s are equal)
	$AB = CD$	(corr sides of congruent Δs)
	$ABCD$ is a parallelogram.	

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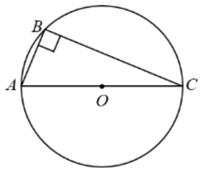
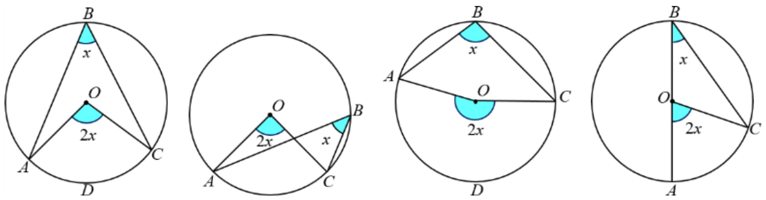
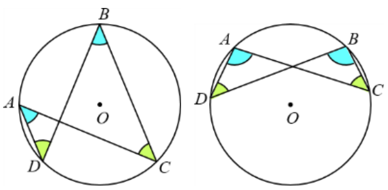
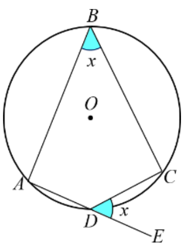
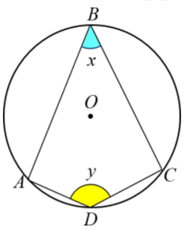
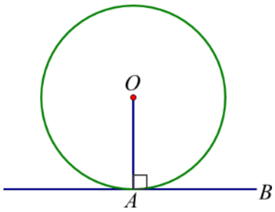
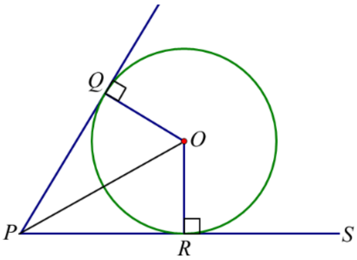
Worksheet 7: Angle Properties of Circles

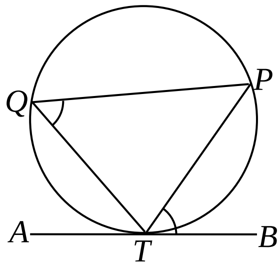
Some Angle Properties of Circles

Property	Abbreviation
1. An angle in a semicircle is a right angle. 	$\angle$ in a semicircle
2. An angle at the centre is twice any angle at the circumference. 	$\angle$ at centre = 2$\angle$ at $\odot^{ce}$
3. Angles in the same segment are equal. 	$\angle$s in the same segment
4. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. 	Ext $\angle$ of a cyclic quad
5. Angles in opposite segments are supplementary.  $\angle x + \angle y = 180°$	$\angle$s in opp segments (or opp $\angle$s of a cyclic quad)

<table>
<tr><th>Property</th><th>Abbreviation</th></tr>
<tr><td>6. A tangent of a circle is perpendicular to the radius drawn to the point of tendency.

</td><td>tan ⊥ rad</td></tr>
<tr><td>7. If PQ and PR are two tangents to a circle centred at O, then
$PQ = PR$
$\angle OPQ = \angle OPR$

</td><td>tan from an ext pt</td></tr>
</table>

<u>The Alternate Segment Theorem (Tangent-Chord Theorem)</u>

An angle between a tangent, *ATB* and a chord, *TP* through the point of contact, *T*, is equal to the angle in the alternate segment.

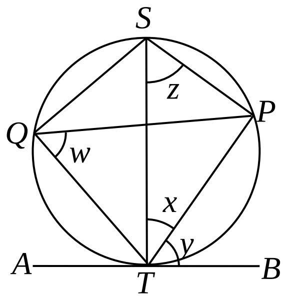


i.e. $\angle PTB = \angle PQT$.

<u>Abbreviation:</u> alt seg thm

<u>Proof</u>

Let *SOT* and *O* be the diameter and centre of the circle.
Insert the diameter *TS* and join *PS*.

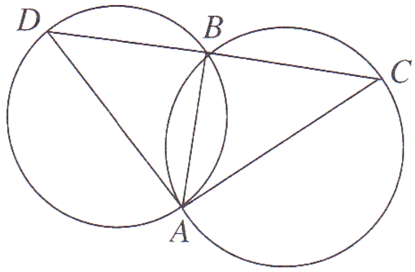


$\angle SPT = 90°$	($\angle$ in a semicircle)
$\angle x + \angle y = 90°$	(tan ⊥ rad)
$\angle z + \angle x = 90°$	($\angle$ sum of Δ)
$\therefore \angle z = \angle y$	
$\angle w = \angle z$	($\angle$s in the same segment)

Therefore, $\angle PTB = \angle PQT$.

Example 1

In the diagram, two circles intersect at the points A and B. AC and AD are diameters of the circles. Prove that C, B and D lie on a straight line. **(Pg264Q1)**



Proof

$\angle DBA = \angle CBA = 90°$ (∠ in a semicircle)

$\therefore DB // BC$

Since $DB//BC$ and B is a common point, $\therefore$ C, B and D lie on a straight line.

Example 2

In the diagram, CB and DA produced meet at the point P and $AB = AP$. Prove that $CD = CP$.

(Pg264Q2)

Proof

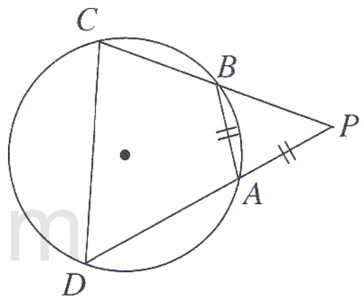
$\angle APB = \angle ABP$ (Base ∠s of an isos Δ)

$\angle CDP = \angle ABP$ (ext ∠ of a cyclic quad)

$= \angle APB$

$= \angle DPC$

$\therefore CD = CP$



Example 3

In the diagram, the circles $ABCD$ and $AEFB$ intersect at A and B. Prove that

(ii) $\angle AEF + \angle CDA = 180°$,

(iii) CD is parallel to FE. **(Pg264Q3)**

Proof

(i) $\angle AEF = \angle ABC$ (ext ∠ of a cyclic quad)

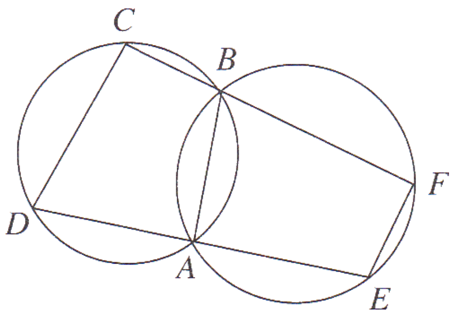
$\angle CDA = \angle ABF$ (ext ∠ of a cyclic quad)

$\angle AEF + \angle CDA = \angle ABC + \angle ABF$ (adj ∠s on a st line)

$= 180°$

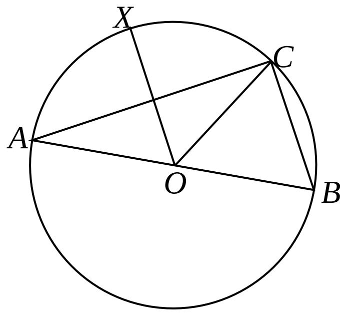
(ii) $\angle AEF + \angle CDA = 180°$

$\therefore CD$ is parallel to FE. (int ∠s are supp)



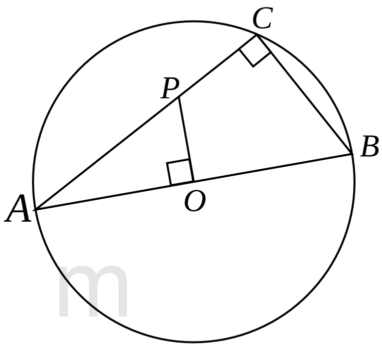
Example 4
In the diagram, O is the centre and AB is the diameter of the circle. Given that OX is parallel to BC. Prove that OX is perpendicular to AC.

Proof



$\angle ACB = 90°$ (∠ in a semicircle)
$AC \perp CB$
Hence, $OX \perp AC$. ($CB \,//\, OX$)

Example 5
AB is a diameter and AC is a chord of a circle centre O and radius r. P is a point on the chord AC and O is the foot of the perpendicular from P to AB.
(i) Name the two similar triangles.
(ii) Hence, prove that $AP \times AC = 2r^2$.

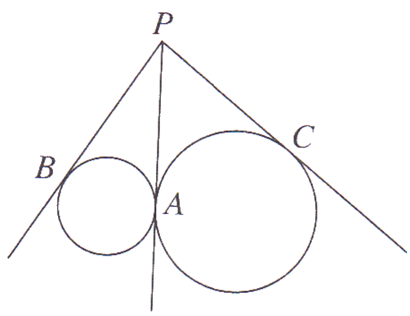


Proof
(i) $\angle ACB = 90°$ (∠ in a semicircle)
$= \angle AOP$
$\angle PAO = \angle BAC$ (Common ∠)
$\therefore$ ΔACB is similar to ΔAOP. (AA)

(ii) $\dfrac{AP}{AB} = \dfrac{AO}{AC} \Rightarrow \dfrac{AP}{2r} = \dfrac{r}{AC}$ (Corr sides of similar Δs)
$\therefore AP \times AC = 2r^2$

Example 6
In the diagram, PA, PB and PC are tangents to the circles at A, B and C respectively.
Prove that $PB = PC$. **(Pg264Q3)**

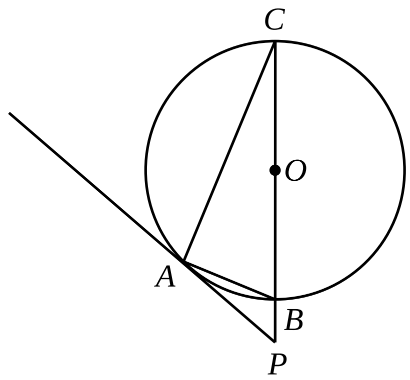
Proof



$PB = PA$ (tan from an ext pt)
$PA = PC$ (tan from an ext pt)
$\therefore$ $PB = PC$

Example 7

In the diagram, COB is a chord of a circle with centre O and it is produced to a point P where PA is a tangent at A. Prove that $PA \times AC = AB \times PC$.



Proof

$\angle BAP = \angle ACP$ (alt seg thm)

$\angle APB = \angle CPA$ (common $\angle$)

ΔAPB is similar to ΔCPA. (AA)

$\frac{PA}{PC} = \frac{AB}{AC}$ (Corr sides of similar Δs)

$\therefore PA \times AC = AB \times PC$

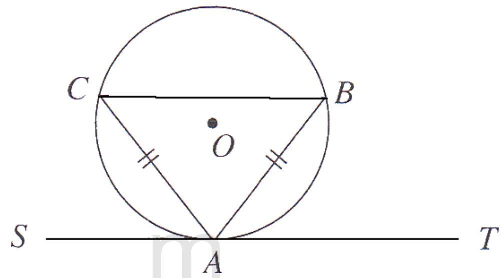
Example 8

In the diagram, ABC is a triangle inscribed in the circle with centre O, and SZ is a tangent to the circle at A.

Given that $AB = AC$, prove that

(i) $\angle ABC = \angle BAT$,

(ii) CB is parallel to ST. **(Pg264Q5)**



Proof

(i) $\angle BAT = \angle BCA$ (alt seg thm)

$= \angle ABC$ (base $\angle$s of an isos Δ)

(ii) $\angle ABC = \angle BAT$

$\therefore$ CB is parallel to ST. (alt $\angle$s are equal)

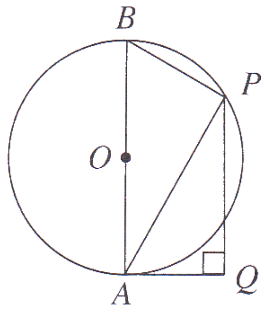
Example 9

The diagram shows a circle with centre O. AB is a diameter and AQ is a tangent to the circle at A so that $\angle PQA$ is a right angle.

Prove that

(i) ΔABP is similar to ΔPAQ,

(ii) $AP = \sqrt{AB \times QP}$. **(Pg265Q8)**



Proof

(i) $\angle PAB = \angle PBA$ (alt seg thm)

$\angle BPA = 90°$ ($\angle$ in a semicircle)

$= \angle PBA$

$\therefore \Delta ABP$ and ΔPAQ are similar. (AA)

(ii) $\frac{AP}{PQ} = \frac{AB}{AP}$ (Corr sides of similar Δs)

$AP^2 = AB \times QP$

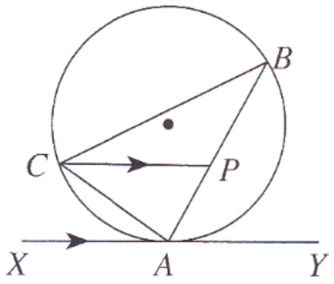
$\therefore AP = \sqrt{AB \times QP}$

Example 10

In the diagram, ABC is a triangle inscribed in a circle, XY is a tangent at A. and CP is parallel to XY,

(i) Prove that ΔABC and ΔACP are similar.

(ii) Deduce that $PA = \dfrac{AC^2}{AB}$. **(Pg266Q14)**



Proof

(i) $\angle XAC = \angle ACP$ (Corresponding $\angle$s, $PC//XY$)

$= \angle ABC$ (alt seg thm)

$\angle CAP = \angle BAC$ (Common $\angle$)

ΔACP is similar to ΔABC. (AA)

(ii) $\dfrac{PA}{AC} = \dfrac{AC}{AB}$ (Corr sides of similar Δs)

$\therefore PA = \dfrac{AC^2}{AB}$

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Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Plane Geometry

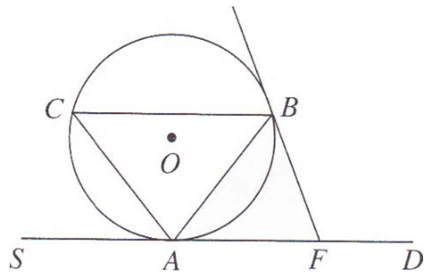
Name: ________________________________ (____) Date: ______________
Class: Sec 4_____

Assignment 8: Angle Properties of Circles

1. In the diagram, AF and BF are tangents to the circle at A and B respectively.
Prove that $\angle BFD = 2\angle ACB$. **(Pg265Q8)**

Proof

$\angle FAB = \angle ACB$	(alt seg thm)
$AF = BF$	(tan from an ext pt)
$\angle BFD = \angle FAB + \angle ABF$	(ext $\angle$ of a Δ)
$\quad = 2\angle FAB$	(base $\angle$s of an isos Δ)



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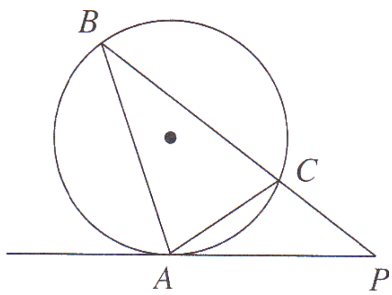
2. In the diagram, triangle ABC is inscribed in the circle. The tangent at A meets the line BC produced at P.
Prove that
(i) ΔAPC and ΔBPA are similar,
(ii) $BP \times CP = AP^2$. **(Pg265Q10)**

Proof

(i) $\angle CAP = \angle ABP$	(alt seg thm)
$\angle CPA = \angle APB$	(common $\angle$)
$\therefore$ ΔAPC and ΔBPA are similar.	(AA)
(ii) $\dfrac{AP}{BP} = \dfrac{CP}{AP}$	(Corr sides of similar Δs)
$\therefore BP \times CP = AP^2$	



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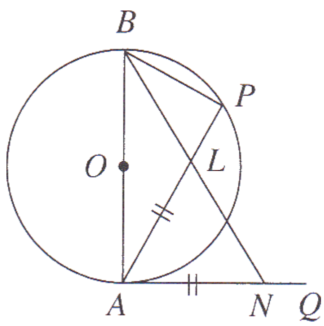
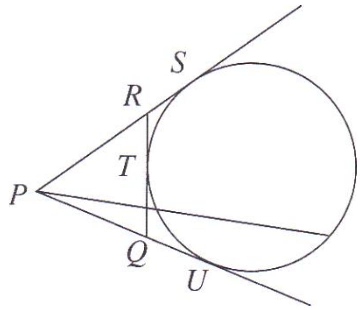
3. The diagram shows a circle with centre O and diameter AB. AQ is a tangent to the circle at A, and $AL = AN$. Prove that $\angle ABL = \angle PBL$. **(Pg265Q11)**

Proof

$\angle ANB = \angle ALN$	(base $\angle$s of an isos Δ)
$\quad = \angle PLB$	(vert opp $\angle$s)
$\angle APB = \angle BAN$	($\angle$ in a semicircle, tan $\perp$ rad)
ΔABN and ΔPBL are similar.	(AA)
$\therefore$ $\angle ABL = \angle PBL$.	(Corr $\angle$s of similar Δs)



4. In the diagram, PS and PU are tangents to the circle at S and U respectively. QR is a tangent to the circle at T.
Prove that $PQ + QR + RP = 2PU$. **(Pg265Q13)**



Proof

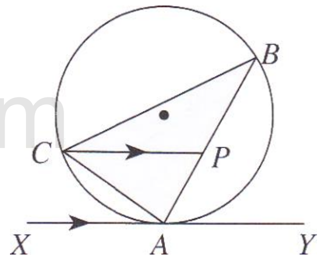
$QU = QT$	(tan from an ext pt)
$RS = RT$	(tan from an ext pt)
$RS = PU$	(tan from an ext pt)

$PQ + QR + RP$
$= (PU - QU) + (QT + RT) + (PS - RS)$
$= (PU - QU) + (QU + RS) + (PS - RS)$
$= PU + PS$
$= 2PU$

Hence $PQ + QR + RP = 2PU$.

---

5. In the diagram, triangle ABC is inscribed in the circle, and XY is a tangent at A. Given that PC bisects $\angle BCA$ and that CP is parallel to XY, prove that $PB = PC$. **(Pg265Q15)**

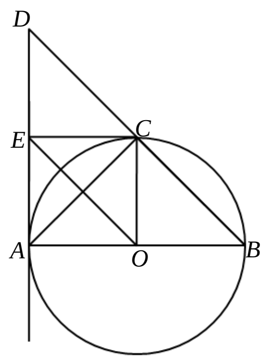


Proof

$\angle XAC = \angle ACP$	(alt $\angle$s)
$\quad\quad = \angle PCB$	(given)
$\angle XAC = \angle PBC$	(alt seg thm)
$\therefore\ \angle PCB = \angle PBC$	
Hence $PB = PC$.	

---

6. The diagram shows a circle, centre O, with diameter AB. The point C lies on the circle. The tangent to the circle at A meets BC extended at D. The tangent to the circle at C meets the line A at E.
(i) Prove that triangles AEO and CEO are congruent.
(ii) Prove that E is the mid-point of AD. **(Nov 2008)**



Proof

(i) OE is common.

$\angle EAO = \angle ECO = 90°$	(tangent $\perp$ radius)
$OC = OA$	(radius)
$\Delta AEO \equiv \Delta CEO$	(RHS)

(ii)

$\angle EOA = \angle EOC$	(corr $\angle$s of congruent Δ)
$\angle OCB = \angle OBC$	(base $\angle$s of an isos Δ)
$\therefore\ \angle EOA = \angle CBO$	
$\angle EAO = \angle ABD$	(common $\angle$)

Hence, ΔEAO and ΔDAB are similar. (AA)

$\dfrac{EA}{AD} = \dfrac{OA}{BA} = \dfrac{1}{2}$ (corr $\angle$s of similar Δs)

Therefore E is the mid-point of AD.

Alternative Method

(i) $OC = AO$ (Radius)

EO is common.

$\angle ECO = \angle EAO = 90°$ (tan $\perp$ radius)

ΔAEO and ΔCEO are congruent. (RHS)

(ii) $\angle EAO = \angle DAB$ (Common $\angle$)

$\angle COA = 2\angle CBO$ ($\angle$ at centre)

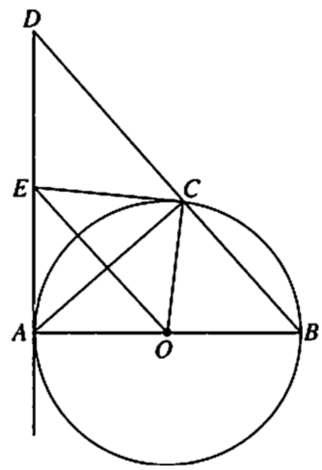
$2\angle EOA = 2\angle CBO$ ($\angle EOA = \angle EOC$, Corr $\angle$s of congruent Δs)

$\therefore \angle EOA = \angle CBO$

ΔEAO and ΔDAB are similar. (AA)

$$\frac{AE}{AD} = \frac{AO}{AB} = \frac{1}{2}$$

Hence, E is the midpoint of AD.



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7. In the diagram, PQ is the common chord of the two circles PQR and $PQBC$, and QC is a tangent to the circle PQR at Q. Prove that BC is parallel to QR. **(Pg270Q7)**

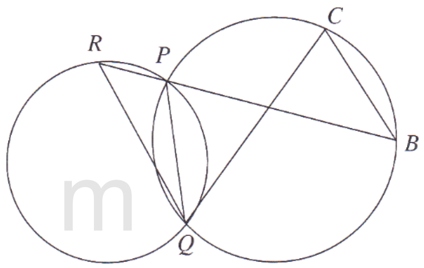
Proof

$\angle PQC = \angle PBC$ ($\angle$s in the same seg)

$= \angle QRP$ (alt seg thm)

Hence $BC // QR$. (alt $\angle$s are equal)

Hence $PB = PC$.



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8. In the diagram SPT is a tangent to a circle at the point P. The points Q and R lie on the circle. The line PM is perpendicular to the chord QR and the line RN is perpendicular to the tangent SPT.
   (i) By considering QP as a chord of the circle, find, with explanation, an angle equal to angle QPT.
   (ii) Explain why a circle with PR as diameter passes through M and N.
   (iii) Prove that the lines MN and QP are parallel. **(Specimen Paper)**

Proof

(i) $\angle QPT = QRP$ (alt seg thm)

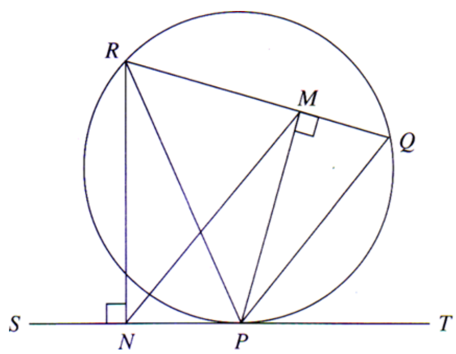
(ii) $\angle RNP = \angle RMP = 90°$ ($\angle$ in a semi-circle)

$\therefore$ a circle with PR as diameter passes through M and N.

(iii) $\angle MRP = \angle MNP$ (alt seg thm)

$= \angle QPT$ (from (ii))

Since the corresponding angles are equal, $MN // QP$.

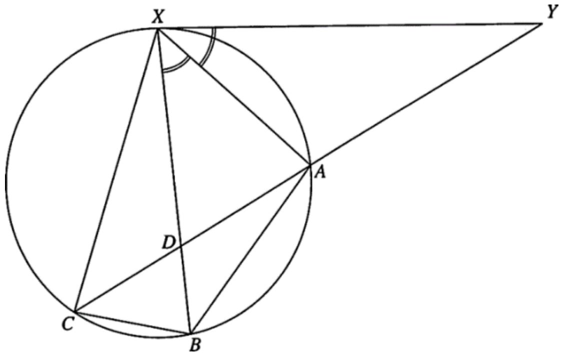


9. The diagram shows a point X on a circle and XY is a tangent to the circle. Points A, B and C lie on the circle such that XA bisects angle YXB and YAC is a straight line. The lines YC and XB intersect at D.

(i) Prove that $AX = AB$.

(ii) Prove that CD bisects angle XCB.

(iii) Prove that triangles CDX and CBA are similar.



Proof

(i) $\angle YXA = \angle XBA$ (alt seg thm)

$= \angle BXA$

Hence, AXB is an isosceles Δ and $AX = AB$.

(ii) $\angle YXB = \angle XCB$ (alt seg thm)

$= 2\angle XBA$ (from (i))

$\angle XBA = \angle XCD$ (alt seg thm)

$2\angle XBA = \angle XBA + \angle DCB$

$\therefore \angle DCB = \angle XBA = \angle DCX$

Hence, CD bisects XCB.

(iii) $\angle CXB = \angle CBA$ (alt seg thm)

$\angle ACB = \angle XCD$ (from (ii))

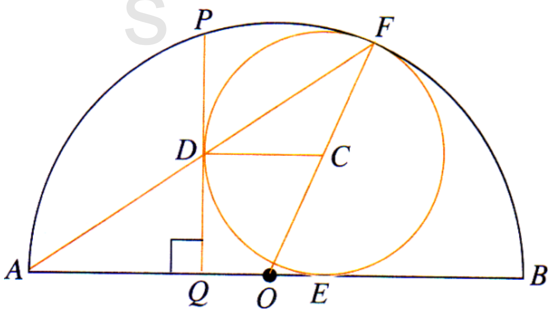
Hence, triangles CDX and CBA are similar. (AA)

---

10. In the diagram, P is any point on the semicircle centre O, and PQ is perpendicular to AB. The inscribed circle centre C touches PQ, AB and the semicircle at D, E and F respectively.
Prove that

(i) A, D and F lie on the same straight line,

(ii) $AD \times AF = AQ \times AB$. **(Pg269Q12)**



Proof

(i) $\angle FAO = \angle AFO$ (Base $\angle$s of an isos Δ)

$= \angle CDF$ (Base $\angle$s of an isos Δ)

$\therefore CD // OA$ (corr$\angle$s)

Hence, A, D and F lie on the same straight line.

(ii) $\angle DAQ = \angle BAF$ (Common $\angle$)

$\angle AQD = 90°$

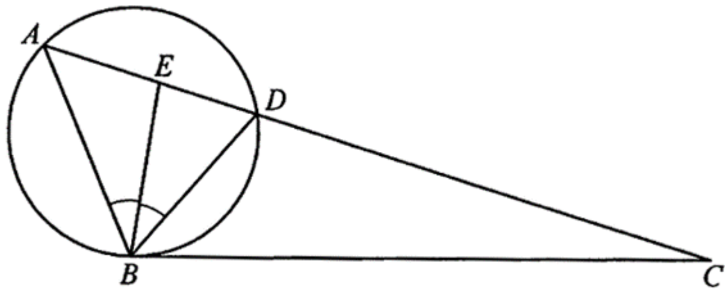
$= \angle AFB$ ($\angle$ in a semicircle)

$\therefore \Delta ADQ$ is similar to ΔABF. (AA)

$$\frac{AD}{AB} = \frac{AQ}{AF}$$

$\therefore AD \times AF = AQ \times AB$

11. In the diagram *AC* is a straight line intersecting a circle at *A* and *D*. The point *B* lies on the circle and *BC* is a tangent to the circle. The point *E* lies on *AC* such that the line *BE* bisects angle *ABD*. Prove that triangle *BCE* is isosceles. **(2013)**



<u>Proof</u>

$\angle DBC = \angle BAC$ (Alt Seg Thm)

$\angle BED = \angle ABE + \angle BAC$ (Ext $\angle$ of Δ)

$\quad = \angle EBD + \angle DBC$ ($\angle EBD = \angle ABE$, *BE* bisects $\angle ABD$)

$\quad = \angle BEC$

$\therefore$ *BCE* is an isos Δ.

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 3

Name: ________________________________(____)   Date: ________________

Class: Sec 4_____

Revision 3: Quadratic Equations and Inequalities

1. Solving Quadratic Inequalities

- If $a(x-\alpha)(x-\beta) > 0$ where $\alpha < \beta$ then $x < \alpha$ or $x > \beta$.
- If $a(x-\alpha)(x-\beta) \geq 0$ where $\alpha \leq \beta$ then $x \leq \alpha$ or $x \geq \beta$.
- If $a(x-\alpha)(x-\beta) < 0$ where $\alpha < \beta$ then $\alpha < x < \beta$.
- If $a(x-\alpha)(x-\beta) \leq 0$ where $\alpha \leq \beta$ then $\alpha \leq x \leq \beta$.

2. Nature of roots of a quadratic equation, $ax^2 + bx + c = 0$

Description	Number of real roots	Conditions
• The equation $ax^2+bx+c=0$ has two equal roots.	1	$b^2-4ac=0$
• The equation $ax^2+bx+c=0$ has two distinct real roots.	2	$b^2-4ac>0$
• The equation $ax^2+bx+c=0$ has no real roots.	0	$b^2-4ac<0$
• The equation $ax^2+bx+c=0$ has real roots.	1 or 2	$b^2-4ac\geq 0$
• The expression $ax^2+bx+c>0$.	0	$b^2-4ac<0$ and $a>0$
• The expression $ax^2+bx+c<0$.	0	$b^2-4ac<0$ and $a<0$
• The expression $ax^2+bx+c\geq 0$ (or is non-negative)	0 or 1	$b^2-4ac\leq 0$ and $a>0$
• The expression $ax^2+bx+c\leq 0$ (or is non-positive)	0 or 1	$b^2-4ac\leq 0$ and $a<0$

3. Intersection of a line and a curve

A straight line intersects a curve. The solutions from the equation $ax^2+bx+c=0$ give the x-coordinates of the points of intersection of these two graphs.

Description	Number of real roots	Conditions
• The line is a tangent to the curve.	1	$b^2-4ac=0$
• The line touches the curve.	1	$b^2-4ac=0$
• The line intersects the curve.	1 or 2	$b^2-4ac\geq 0$
• The line meets the curve.	1 or 2	$b^2-4ac\geq 0$
• The line does not intersect the curve.	0	$b^2-4ac<0$

Example

1. Find the smallest value of the integer a for which $ax^2 + 5x + 2$ is positive for all values of x.

$ax^2 + 5x + 2 > 0$
Discriminant < 0 and $a > 0$
$5^2 - 4a(2) < 0$
$a > \frac{25}{8}$
Smallest integer value of $a = 4$

---

2. Find the smallest value of the integer b for which $-5x^2 + bx - 2$ is negative for all values of x.

$-5x^2 + bx - 2 < 0$
Discriminant < 0
$b^2 - 4(-5)(-2) < 0$
$b^2 - 40 < 0$
$(b + \sqrt{40})(b - \sqrt{40}) < 0$
$-\sqrt{40} < b < \sqrt{40}$
Smallest integer value of $b = -6$

- Solve inequality by factorisation
- $b < \pm\sqrt{40}$ is mathematically wrong and **should not be written**.

---

3. Find the value of p and q for which the quadratic inequality $x^2 + px \geq q$ is satisfied by $x \geq 2$ or $x \leq -4$.

$x \geq 2$ or $x \leq -4$
$(x-2)(x+4) \geq 0$
$x^2 + 2x - 8 \geq 0$
$x^2 + 2x \geq 8$

$p = 2$
$q = 8$

---

4. The speed v ms^{-1} of a particle travelling from A to B, at time t seconds after leaving A, is given by $v = 10t - t^2$. The particle starts from rest at A and comes to rest at B. Show that the particle has a speed of 5 ms^{-1} or greater for exactly $4\sqrt{5}$ s.

$10t - t^2 \geq 5$
$t^2 - 10t + 5 \leq 0$
Let $t^2 - 10t + 5 = 0$
$$t = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(5)}}{2(1)}$$
$= 5 \pm 2\sqrt{5}$
$t^2 - 10t + 5 \leq 0 \Rightarrow 5 - 2\sqrt{5} \leq t \leq 5 - 2\sqrt{5}$
Duration $= 5 + 2\sqrt{5} - (5 - 2\sqrt{5})$
$= 4\sqrt{5}$ s

- Solve $v \geq 5$
- The quadratic expression could not be factorised.
- Assume that $t^2 - 10t + 5 = 0$.
- Apply $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the t-intercepts of the curve of $y = t^2 - 10t + 5$.

4. Roots of Quadratic Equations

If the roots of a quadratic equation $ax^2+bx+c=0$ are α and β, then

$$ax^2+bx+c=0 \Leftrightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$\text{i.e. } \alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

Some useful identities

- $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
- $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)=(\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]$
- $\alpha^3-\beta^3=(\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)=(\alpha+\beta)[(\alpha+\beta)^2-\alpha\beta]$

Example

5. The equation $x^2+4x+k=0$ has two real roots, α and β, where $\alpha > \beta$. If $\alpha-\beta=2\sqrt{7}$, find the value of k.

$\alpha+\beta=-4$

$\alpha\beta=4$

$(\alpha-\beta)^2=(2\sqrt{7})^2$

$\alpha^2+\beta^2-2\alpha\beta=28$

$(\alpha+\beta)^2-2\alpha\beta-2\alpha\beta=28$

$16-4k=28$

$k=-3$

- Do not attempt to find the value of α and β.
- Make use of the values of $\alpha+\beta$ and $\alpha\beta$.

---

6. The quadratic equation $2x^2+5x+4=0$ has roots α and β.
Find the quadratic equation whose roots are α^3 and β^3.

$\alpha+\beta=-\frac{5}{2}$

$\alpha\beta=2$

$$\begin{aligned}\alpha^2+\beta^2&=(\alpha+\beta)^2-2\alpha\beta\\&=\left(-\frac{5}{2}\right)^2-2(2)\\&=\frac{9}{4}\end{aligned}$$

$$\begin{aligned}\alpha^3+\beta^3&=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)\\&=-\frac{5}{2}\left(\frac{9}{4}-2\right)\\&=-\frac{5}{8}\end{aligned}$$

$\alpha^3\beta^3=8$

The required equation is $x^2+\frac{5}{8}x+8=0$ or $8x^2+5x+64=0$

- Always find the value of $\alpha+\beta$, $\alpha\beta$ and $\alpha^2+\beta^2$ before attempting to find the value of $\alpha^3+\beta^3$ and $\alpha^3\beta^3$.

Exercise

1. A ball is thrown from the top of a building. Its height, h metres, from the ground after time t seconds is given by $h = 50 + 10t - 5t^2$.
   (i) Find the range of t for which the height reached is more than 50 m.
   (ii) Use the discriminant to determine whether the ball could reach a height of 60 m.

(i) $h > 50$

$50 + 10t - 5t^2 > 50$

$5t^2 - 10t < 0$

$5t(t-2) < 0$

$0 < t < 2$

(ii) When $h = 60$, $\quad 50 + 10t - 5t^2 = 60$

$5t^2 - 10t + 10 = 0$

Discriminant $= (-10)^2 - 4(5)(10)$

$= -100 < 0$

$50 + 10t - 5t^2 = 60$ has no real solution.

$\therefore$ the ball could not reach a height of 60 m.

---

2. Find the range of values of k such that $kx^2 + 3x + k > 2x^2 + kx + 3$ for all real values of x.

$kx^2 + 3x + k > 2x^2 + kx + 3$

$(k-2)x^2 + (3-k)x + (k-3) > 0$

Discriminant < 0 and $k - 2 > 0$

$(3-k)^2 - 4(k-2)(k-3) < 0$ $\qquad k > 2$

$(k-3)^2 - 4(k-2)(k-3) < 0$

$(k-3)(k-3-4k+8) < 0$

$(k-3)^2 - 4(k-2)(k-3) < 0$

$(k-3)(5-3k) < 0$

$k < \frac{5}{3}$ or $k > 3$

$\therefore k > 3$

s

---

3. (a) Find the range of values of p for which the expression $(p-6)x^2-8x+p$ is always positive for all real values of x.

(b) Show that the line $y=\dfrac{x}{k}+\dfrac{k}{2}$ is a tangent to the curve $y^2=2x$ for all real values of k.

(a) $(p-6)x^2-8x+p>0$

Discriminant <0 and $p-6>0$

$8^2-4p(p-6)<0$ $\qquad$ $p>2$

$64-4p^2+24p<0$

$p^2-6p-16>0$

$(p-8)(p+2)>0$

$p<-2$ or $p>8$

$\therefore p>8$

(b) $y=\dfrac{x}{k}+\dfrac{k}{2}$ ......(1)

$y^2=2x$ ......(2)

Sub (1) into (2) $\left(\dfrac{x}{k}+\dfrac{k}{2}\right)^2=2x$

$$\frac{x^2}{k^2}+x+\frac{k^2}{4}=2x$$

$$\frac{x^2}{k^2}-x+\frac{k^2}{4}=0$$

Discriminant

$=(-1)^2-4\left(\dfrac{1}{k^2}\right)\left(\dfrac{k^2}{4}\right)$

$=0$

$\therefore$ $y=\dfrac{x}{k}+\dfrac{k}{2}$ is a tangent to the curve $y^2=2x$ for all real values of k.

---

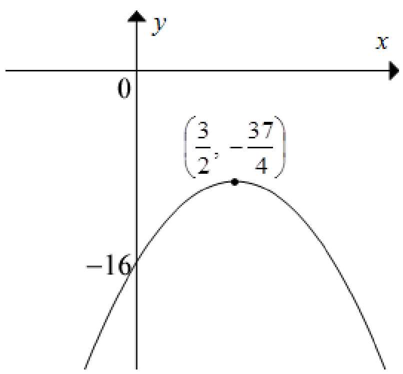
4. Express $y=-3x^2+9x-16$ in the form $y=a(x+b)^2+c$ where a, b and c are constants. Hence state the maximum value of y and the value of x at which this maximum value occurs. Sketch the graph of $y=-3x^2+9x-16$.

$y=-3x^2+9x-16$

$=-3\left[x^2-3x+\left(\dfrac{3}{2}\right)^2\right]+3\left(\dfrac{3}{2}\right)^2-16$

$=-3\left(x-\dfrac{3}{2}\right)^2-\dfrac{37}{4}$

maximum $y=-\dfrac{37}{4}$



5. (a) Find the range of values of k for which the line $y = kx + 3$ will intersect the curve $y = 7x^2 + 5$ at two real distinct points.

(b) Find the range of values of x for which $x^2 < 4x + 12$.

(c) Find the two possible values of k for which the line $y = 3kx - 2k + 4$ is a tangent to the curve $y = (2k+1)x^2$.

(a) $y = kx + 3$ ......(1)

$y = 7x^2 + 5$ ......(2)

Sub (1) into (2) $kx + 3 = 7x^2 + 5$

$7x^2 - kx + 2 = 0$

Discriminant > 0

$k^2 - 4(7)(2) < 0$

$k^2 - 56 > 0$

$(k + \sqrt{56})(k - \sqrt{56}) > 0$

$(k + 2\sqrt{7})(k - 2\sqrt{7}) > 0$

$k < -2\sqrt{7}$ or $k > 2\sqrt{7}$

(b) $x^2 < 4x + 12$

$x^2 - 4x - 12 < 0$

$(x+2)(x-6) < 0$

$-2 < x < 6$

(c) $y = 3kx - 2k + 4$ ......(1)

$y = (2k+1)x^2$ ......(2)

Sub (1) into (2) $3kx - 2k + 4 = (2k+1)x^2$

$(2k+1)x^2 - 3kx + (2k-4) = 0$

Discriminant = 0

$9k^2 - 4(2k+1)(2k-4) = 0$

$9k^2 - 4(4k^2 - 6k - 4) = 0$

$9k^2 - 16k^2 + 24k + 16 = 0$

$7k^2 - 24k - 16 = 0$

$(7k+4)(k-4) = 0$

$k = -\frac{4}{7},\ 4$

6. (a) A cone has a circular base area of $(2x+5)\,\text{cm}^2$ and a perpendicular height of x cm. Calculate the values of x for which the volume of the cone is greater than 4 cm^3.

(b) Find the range of values of m such that the curve $y = x^2 + 2mx + m + 12$ always lies above the x-axis.

(a) Volume > 4 cm^3

$\frac{1}{3}\times(2x+5)\times x > 4$ and $2x+5>0$

$2x^2+5x>12$ $\quad x > -\frac{5}{2}$

$2x^2+5x-12>0$

$(2x-3)(x+4)>0$

$x<-4,\ x>\frac{3}{2}$

$\therefore x>\frac{3}{2}$

(b) $x^2+2mx+m+12>0$

$4m^2-4(1)(m+12)<0$

$m^2-m-12<0$

$*m+3)(m-4)<0$

$-3<m<4$

---

7. (a) Given that $x<-1$ or $x>5$ is the solution set to the inequality $(x+2)(x-6)>k$, find the value of k.

(b) A straight line passes through (1, 3) with gradient m. Find the limits within which m must lie if the straight line is not to touch the curve $y=(x+1)^2$.

(c) Prove that the equation $(x+a)(x+b)=c^2$ always has real roots.

(a) $x<-1$ or $x>5$

$(x+1)(x-5)>0$

$x^2-4x-5>0$

$x^2-4x-12>-7$

$(x+2)(x-6)>-7$

$\therefore k=-7$

(b) $y-3=m(x-1)$

$y=mx+(3-m)$ ……(1)

$y=(x+1)^2$ ……(2)

Sub (1) into (2) $\quad mx+(3-m)=(x+1)^2$

$mx+(3-m)=x^2+2x+1$

$x^2+(2-m)x+(m-2)=0$

Discriminant < 0

$(2-m)^2-4(1)(m-2)<0$

$m^2-4m+4-4m+8<0$

$m^2-8m+12<0$

$(m-2)(m-6)<0$

$2<m<6$

(c) $(x+a)(x+b)=c^2$

$x^2+x(a+b)+ab-c^2=0$

Discriminant

$=(a+b)^2-4(ab-c^2)$

$=a^2+2ab+b^2-4ab+4c^2$

$=a^2-2ab+b^2+4c^2$

$=(a-b)^2+4c^2$

Since $(a-b)^2 \geq 0, c^2 \geq 0$, $\therefore (a-b)^2+4c^2 \geq 0$

Hence, $(x+a)(x+b)=c^2$ always has real roots.

---

8. Using a separate diagram for each part, represent on the number line the solution set of

(i) $3(2-x)<x+18$,

(ii) $6(x^2-2)>3x-9$.

State the set of values of x which satisfy both of the inequalities.

(i) $3(2-x)<x+18$

$6-3x<x+18$

$4x>-12$

$x>-3$

(ii) $6(x^2-2)>3x-9$

$6x^2-3x+3>0$

$2x^2-x+1>0$

$(2x+1)(x-1)>0$

$x<-\frac{1}{2}$ or $x>1$

---

9. (i) Find the range of values of c for which the straight line $y=4x-2c$ meets the curve $y=2x^2-6x+9$.

(ii) State the value of c for which the straight line is tangent to the curve.

(i) $y=2x^2-6x+9$ ……(1)

$y=4x-2c$ ……(1)

Sub (1) into (2) $4x-2c=2x^2-6x+9$

$2x^2-10x+9-2c=0$

Discriminant ≥ 0

$100-4(2)(9-2c)>0$

$25-18+4c>0$

$4c>-7$

$c>-\frac{7}{4}$

(ii) $c=-\frac{7}{4}$

10. (a) Find the range of values of h for which the equation $x^2 - 8x + h^2 - 9 = 0$ has real roots.
If h is a positive integer, list the values of h for which the roots of the equation are real and of the same sign, stating the reasons for your answer.

$x^2 - 8x + h^2 - 9 = 0$

Equation has real roots $\Rightarrow$ discriminant ≥ 0

$8^2 - 4(1)(h^2 - 9) \geq 0$

$64 - 4h^2 + 36 \geq 0$

$4h^2 - 100 \leq 0$

$h^2 - 25 \leq 0$

$(h+5)(h-5) \leq 0$

$-5 \leq h \leq 5$

If the roots are real and of the same sign, **the product of roots > 0**.

$h^2 - 9 > 0$

$(h-3)(h+3) > 0$

$h < -3$ or $h > 3$

As h is a positive integer, $h > 0$, $h = 4, 5$

(b) Find the value of b and of c for which $x < -3$ or $x > 8$ is the solution set of $x^2 > c - bx$.

$x < -3$ or $x > 8$

$(x+3)(x-8) > 0$

$x^2 - 5x - 24 > 0$

$x^2 > 24 + 5x$

$\therefore b = -5,\ c = 24$

11. The roots of the quadratic equation $x^2 - 5x + 3 = 0$ are α and β.

(i) Express $\alpha^2 - \alpha\beta + \beta^2$ in terms of $(\alpha + \beta)$ and $\alpha\beta$.

(ii) Find the quadratic equation whose roots are α^3 and β^3.

(i) $\alpha^2 - \alpha\beta + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta - \alpha\beta$

$= (\alpha+\beta)^2 - 3\alpha\beta$

(ii) $\alpha + \beta = 5$

$\alpha\beta = 3$

$\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$

$= (\alpha+\beta)[(\alpha+\beta)^2 - 3\alpha\beta]$

$= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= 125 - 3(3)(5)$

$= 80$

$\alpha^3\beta^3 = 3^3$

$= 27$

The required quadratic equation is $x^2 - 80x + 27 = 0$

12. The equation $6x^2+7x-3=0$ has roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

(i) Find the value of $(\alpha+\beta)$ and $\alpha\beta$.

(ii) Hence, or otherwise, find the exact value of $\alpha^3+\beta^3$.

(i)
$$\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{7}{6}$$
$$\frac{\alpha+\beta}{\alpha\beta}=-\frac{7}{6}$$
$$\alpha+\beta=-\frac{7}{6}\alpha\beta \quad \ldots\ldots(1)$$

$$\frac{1}{\alpha}\cdot\frac{1}{\beta}=-\frac{1}{2}$$
$$\frac{1}{\alpha\beta}=-\frac{1}{2}$$
$$\alpha\beta=-2 \quad \ldots\ldots(2)$$

Sub (2) into (1) $\quad \alpha+\beta=-\dfrac{7}{6}(-2)$

$$\alpha+\beta=\frac{7}{3}$$

(ii)
$$\begin{aligned}\alpha^3+\beta^3 &= (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)\\ &= (\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]\\ &= (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)\\ &= \left(\frac{7}{3}\right)^3-3(-2)\left(\frac{7}{3}\right)\\ &= \frac{721}{27}\end{aligned}$$

---

13. The roots of the quadratic equation $5x^2 - 4x + 2 = 0$ are α and β.

(i) Find the values of $\alpha^2 + \beta^2$ and $\alpha^4 + \beta^4$.

(ii) Find the quadratic equation with integer coefficients and whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$.

(i) $\alpha + \beta = \dfrac{4}{5}$

$\alpha\beta = \dfrac{2}{5}$

$$\begin{aligned}\alpha^2 + \beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta \\ &= \left(\frac{4}{5}\right)^2 - 2\left(\frac{2}{5}\right) \\ &= -\frac{4}{25}\end{aligned}$$

$$\begin{aligned}\alpha^4 + \beta^4 &= (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2 \\ &= \left(-\frac{4}{25}\right)^2 - 2\left(\frac{2}{5}\right)^2 \\ &= -\frac{384}{625}\end{aligned}$$

(ii) $$\begin{aligned}\alpha^2 + 2 + \beta^2 + 2 &= -\frac{4}{25} + 4 \\ &= \frac{96}{25}\end{aligned}$$

$$\begin{aligned}(\alpha^2+2)(\beta^2+2) &= \alpha^2\beta^2 + 2(\alpha^2+\beta^2) + 4 \\ &= \left(\frac{2}{5}\right)^2 + 2\left(-\frac{4}{25}\right) + 4 \\ &= \frac{96}{25}\end{aligned}$$

The required quadratic equation is $x^2 - \dfrac{96}{25}x + \dfrac{96}{25} = 0$ or $25x^2 - 96x + 96 = 0$.

14. The roots of the quadratic equation $4x^2 - x + 16 = 0$ are α^2 and β^2.
Find the quadratic equation whose roots are α and β, where $\alpha + \beta > 0$.

$$\alpha^2 + \beta^2 = \frac{1}{4}$$

$$(\alpha+\beta)^2 - 2\alpha\beta = \frac{1}{4} \quad \ldots\ldots.(1)$$

$$\alpha^2\beta^2 = 4$$

$$\alpha\beta = \pm 2 \quad \ldots\ldots.(2)$$

Sub $\alpha\beta = 2$ into (1) $\quad (\alpha+\beta)^2 - 4 = \frac{1}{4}$

$$(\alpha+\beta)^2 = \frac{17}{4}$$

Since $\alpha + \beta > 0$, $\quad \alpha + \beta = \frac{\sqrt{17}}{2}$

Sub $\alpha\beta = -2$ into (1) $\quad (\alpha+\beta)^2 + 4 = \frac{1}{4}$

$$(\alpha+\beta)^2 = -\frac{17}{4} \text{ (NA)}$$

The required quadratic equation is $x^2 - \frac{\sqrt{17}}{2}x + 2 = 0$ or $2x^2 - \sqrt{17}x + 4 = 0$.

---

15. The roots of the quadratic equation $3x^2 - 4x + 5 = 0$ are α and β.

(i) If $\alpha + \beta = p$ and $\alpha\beta = q$, find the value of $\frac{p}{q}$.

(ii) Find the quadratic equation whose roots are $\frac{\alpha}{\alpha+2}$ and $\frac{\beta}{\beta+2}$.

(i) $p = \frac{4}{3}$

$q = \frac{5}{3}$

$\frac{p}{q} = \frac{4}{5}$

(ii)
$$\frac{\alpha}{\alpha+2} + \frac{\beta}{\beta+2} = \frac{\alpha(\beta+2)+\beta(\alpha+2)}{(\alpha+2)(\beta+2)}$$
$$= \frac{2\alpha\beta + 2(\alpha+\beta)}{\alpha\beta + 2(\alpha+\beta)+4}$$
$$= \frac{2\left(\frac{5}{3}\right)+2\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)+2\left(\frac{4}{3}\right)+4}$$
$$= \frac{18}{25}$$

$$\left(\frac{\alpha}{\alpha+2}\right)\left(\frac{\beta}{\beta+2}\right) = \frac{\alpha\beta}{(\alpha+2)(\beta+2)}$$
$$= \frac{\frac{5}{3}}{\left(\frac{5}{3}\right)+2\left(\frac{4}{3}\right)+4}$$
$$= \frac{1}{5}$$

The required quadratic equation is $x^2 - \frac{18}{25}x + \frac{1}{5} = 0$ or $25x^2 - 18x + 5 = 0$.

16. The roots of the equation $x^2 - x\sqrt{12} + 2 = 0$ are p and q.

(i) Evaluate $\frac{1}{p} + \frac{1}{q}$.

(ii) Hence form the quadratic equation whose roots are $\frac{1}{p}, \frac{1}{q}$,

(i) $p + q = \sqrt{12}$
$pq = 2$

$$\frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq}$$
$$= \frac{\sqrt{12}}{2}$$
$$= \sqrt{3}$$

(ii) $$\left(\frac{1}{p}\right)\left(\frac{1}{q}\right) = \frac{1}{pq}$$
$$= \frac{1}{2}$$

The required quadratic equation is $x^2 - \sqrt{3}x + \frac{1}{2} = 0$ or $2x^2 - 2\sqrt{3}x + 1 = 0$.

---

17. Consider the equation $kx^2 - k^2x - 3x + 4 = k$.

(a) If β is a root of the equation and the roots are reciprocal of each other,

(i) find the value of k,

(ii) show that $\frac{7}{\beta^2 + 1} = \frac{2}{\beta}$.

(a) (i) $$kx^2 - k^2x - 3x + 4 = k$$
$$kx^2 - (k^2 + 3)x + 4 - k = 0$$
$$\beta + \frac{1}{\beta} = \frac{k^2 + 3}{k}$$
$$\beta\left(\frac{1}{\beta}\right) = \frac{4-k}{k}$$
$$k = 4 - k$$
$$2k = 4$$
$$k = 2$$

(ii) $$\beta + \frac{1}{\beta} = \frac{k^2 + 3}{k}$$
$$\frac{\beta^2 + 1}{\beta} = \frac{4+3}{2}$$
$$\therefore \frac{7}{\beta^2 + 1} = \frac{2}{\beta}$$

(b) If the roots are λ and μ, find the quadratic equation, in terms of k, whose roots are $\frac{1}{\lambda}$ and $\frac{1}{\mu}$.

$$kx^2 - (k^2+3)x + 4 - k = 0$$
$$\lambda + \mu = k^2 + 3$$
$$\lambda\mu = 4 - k$$

$$\begin{aligned}\frac{1}{\lambda} + \frac{1}{\mu} &= \frac{\lambda+\mu}{\lambda\mu} \\ &= \frac{k^2+3}{k} \times \frac{k}{4-k} \\ &= \frac{k^2+3}{4-k}\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{\lambda}\right)\left(\frac{1}{\mu}\right) &= \frac{1}{\lambda\mu} \\ &= \frac{k}{4-k}\end{aligned}$$

The required quadratic equation is $x^2 - \frac{k^2+3}{4-k}x + \frac{k}{4-k} = 0$ or $(4-k)x^2 - (k^2+3)x + k = 0$.

---

18. The equation $x^2 + 4x + k = 0$ has two real roots, α and β, where $\alpha > \beta$.
 (i) Find the range of possible values of k.
 (ii) If $\alpha - \beta = 2\sqrt{7}$, find the value of k.
 (iii) Hence, find the exact value of $\alpha^3 - \beta^3$.

(i) Discriminant > **0**

$$4^2 - 4(1)(k) > 0$$
$$4 - k > 0$$
$$k < 4$$

(ii) $\alpha + \beta = -4$

$$\alpha\beta = k$$

$$\alpha - \beta = 2\sqrt{7}$$
$$(\alpha-\beta)^2 = 28$$
$$\alpha^2 - 2\alpha\beta + \beta^2 = 28$$
$$(\alpha-\beta)^2 - 4\alpha\beta = 28$$
$$16 - 4k = 28$$
$$4k = -12$$
$$k = -3$$

(iii) $\alpha^3 - \beta^3 = (\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)$

$$= 2\sqrt{7}(4^2+3)$$
$$= 38\sqrt{7}$$

Name: ______________________________ (____)   Date: ______________

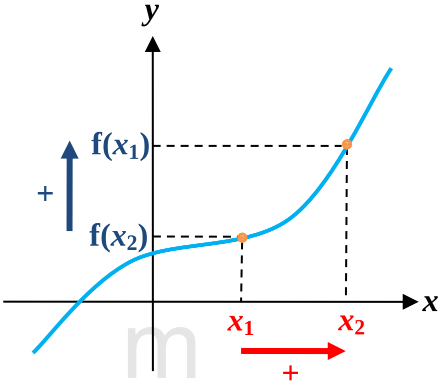
Class: Sec 4_____

Worksheet 13: Increasing and Decreasing Functions

Increasing Function

A function $y = \mathrm{f}(x)$ is increasing on an interval $a < x < b$, if

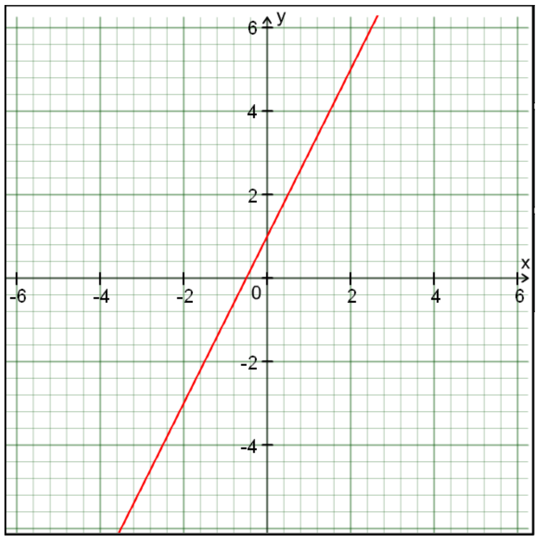
- $\mathrm{f}(x_1) < \mathrm{f}(x_2)$ where $a < x_1 < x_2 < b$.
  In other words, the value of $\mathrm{f}(x)$ becomes larger as x increases over the interval $a < x < b$.
- $\dfrac{dy}{dx} > 0$.



Example 1

Consider the graph and the gradient function of

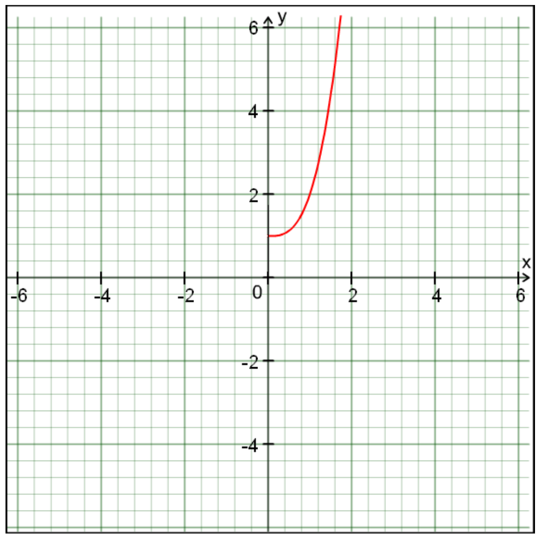
(a) $y = 2x + 1$



$\dfrac{dy}{dx} = 2$

$\dfrac{dy}{dx} > 0$ for all values of x

(b) $y = x^3 + 1,\ x > 0$



$\dfrac{dy}{dx} = 3x^2$

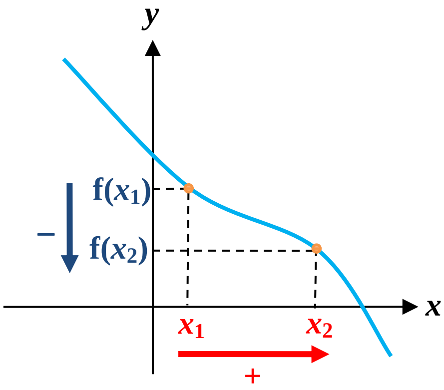
Since $x^2 > 0$ for all $x > 0$, $\dfrac{dy}{dx} > 0$

The values of y for both graphs become larger as x increases and $\dfrac{dy}{dx} > 0$.

Hence, $y = 2x + 1$ and $y = x^3 + 1,\ x > 0$ are both increasing functions.

Decreasing Function

A function $y = \mathrm{f}(x)$ is decreasing on an interval $a < x < b$ if

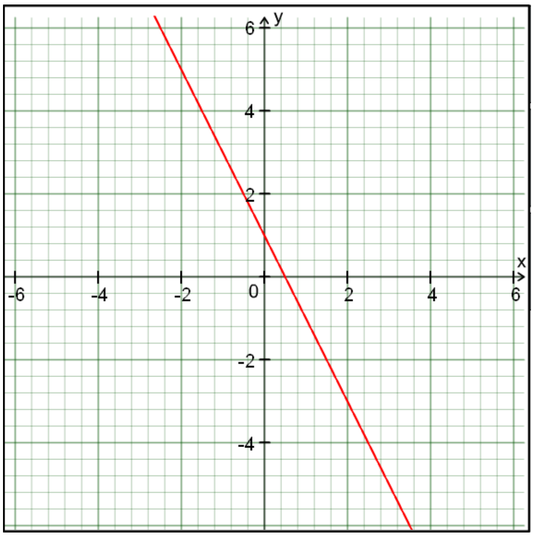


- $\mathrm{f}(x_1) > \mathrm{f}(x_2)$ where $a < x_1 < x_2 < b$.
  In other words, the value of $\mathrm{f}(x)$ becomes smaller as x increases over the interval $a < x < b$.
- $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.

Example 2

Consider the graph and the gradient function of

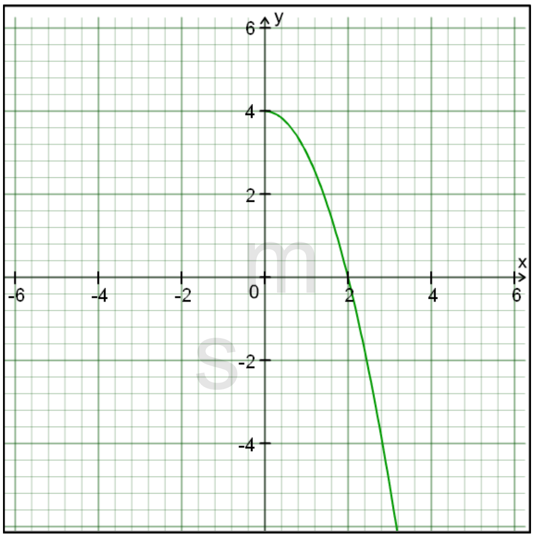
(a) $y = 1 - 2x$



$\frac{\mathrm{d}y}{\mathrm{d}x} = -2$

$\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ for all values of x

(b) $y = 4 - x^2,\ x > 0$



$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x$

Since $-2x < 0$ for all $x > 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$

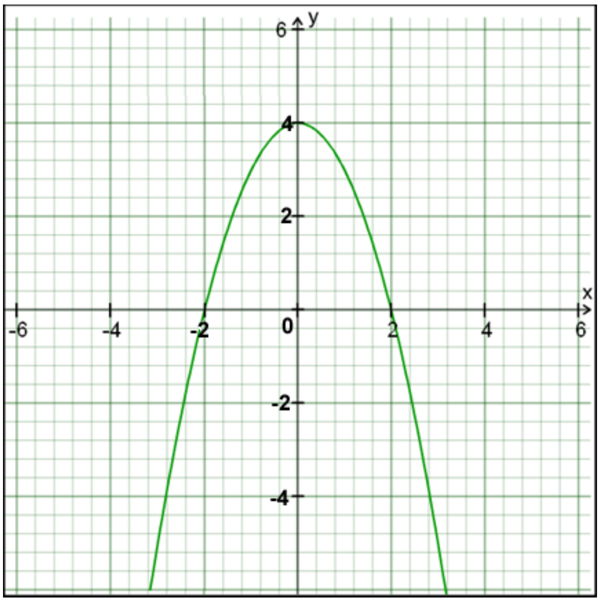
The values of y of both graphs become smaller as x increases and $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.

Hence, $y = 1 - 2x$ and $y = -x^2,\ x > 0$ are both decreasing functions.

Example 3

A function can be an increasing or a decreasing function, depending on the interval under analysis or consideration.

Consider the graph and the gradient function of $y = 4 - x^2$.



For all $x > 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2x < 0$

For all $x < 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2x > 0$

Hence

- $y = 4 - x^2$ is **<u>an increasing</u>** function for $x < 0$ and
- $y = 4 - x^2$ is **<u>a decreasing</u>** function for $x > 0$.

Example 4

A curve has the equation $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = x^3 + 3x$.

(i) Obtain an expression for $\mathrm{f}'(x)$.

(ii) Determine, with explanation, whether f is an increasing or decreasing function.

(i) $\mathrm{f}'(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}(3x^2 + 3)$

$= 3x^2 + 3$

(ii) Since $x^2 \geq 0$, $3x^2 + 3 \geq 3$

$\mathrm{f}'(x) > 0$

Hence, f is an increasing function for all real values of *x*.

Example 5

A curve has the equation $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = \dfrac{x-1}{x+1}$ for $x > -1$.

(i) Show that $\mathrm{f}'(x)$ can be expressed in the form $\dfrac{k}{(x+1)^2}$ where k is a constant to be found.

(ii) Determine, with explanation, whether f is an increasing or decreasing function.

(iii) Showing full working, determine whether the gradient of the curve is increasing or decreasing.

(i) $$\mathrm{f}'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-1}{x+1}\right)$$
$$= \frac{x+1-(x-1)}{(x+1)^2}$$
$$= \frac{2}{(x+1)^2}$$

(ii) Since $x > -1$, $\quad (x+1)^2 > 0$

$$\therefore \mathrm{f}(x) = \frac{2}{(x+1)^2} > 0$$

Hence, f is an increasing function for all real values of *x*.

(iii) $$\mathrm{f}''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{2}{(x+1)^2}\right]$$
$$= 2(-2)(x+1)^{-3}$$
$$= -\frac{4}{(x+1)^3}$$

Since $x > -1$, $\quad (x+1)^2 > 0$

$$\therefore \mathrm{f}''(x) - \frac{4}{(x+1)^3} < 0$$

Hence, the gradient of f is decreasing for $x > -1$.

Example 6

A function is defined by $y=\dfrac{2x^2}{x-1}$ where $x>2$. Determine, with explanation, whether $y=\dfrac{2x^2}{x-1}$ is an increasing or a decreasing function.

$$\begin{aligned}\frac{dy}{dx} &= \frac{4x(x-1)-2x^2}{(x-1)^2}\\ &= \frac{2x^2-4x}{(x-1)^2}\\ &= \frac{2x(x-2)}{(x-1)^2}\end{aligned}$$

Since $x>2$, $\quad 2x>0$, $x-2>0$ and $(x-1)^2>0$

$$\therefore \frac{dy}{dx}=\frac{2x(x-2)}{(x-1)^2}>0$$

Hence $y=\dfrac{2x^2}{x-1}$ is an increasing function.

Example 7

A function is defined by $y=\dfrac{x}{x^2+1}$. Find the values of x for which y is an increasing function.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+1)\frac{d}{dx}(x)-x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}\\ &= \frac{(x^2+1)(1)-x(2x)}{(x^2+1)^2}\\ &= \frac{x^2+1-2x^2}{(x^2+1)^2}\\ &= \frac{1-x^2}{(x^2+1)^2}\end{aligned}$$

When y is an increasing function, $\dfrac{dy}{dx}>0$,

$$\begin{aligned}\frac{1-x^2}{(x^2+1)^2}&>0\\ 1-x^2&>0\\ x^2-1&<0\\ (x+1)(x-1)&<0\\ -1<x&<1\end{aligned}$$

Example 8

It is given that $y = 18 + px + qx^2 - x^3$ where p and q are integers. The only values of x for which y is a decreasing function of x are those values for which $x < \frac{7}{3}$ or $x > 3$.

Find the value of p and of q.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p + 2qx - 3x^2$$

When y is a decreasing function, $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$,

$$\begin{aligned} p + 2qx - 3x^2 < 0 &\Leftrightarrow x < \frac{7}{3} \text{ or } x > 3 \\ 3x^2 - 2qx - p > 0 &\Leftrightarrow \left(x - \frac{7}{3}\right)(x-3) > 0 \\ &\Leftrightarrow (3x-7)(x-3) > 0 \\ &\Leftrightarrow 3x^2 - 16x + 21 > 0 \end{aligned}$$

Comparing coefficient of x, $-2q = -16$

$q = 8$

Comparing constant term, $-p = 21$

$p = -21$

Singapore Chinese Girls' School
Secondary 4
Mathematics II
Rate of Change

Name: ________________________________(____)   Date: ________________

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<u>Assignment 14: Increasing and Decreasing Functions</u>

1. Show that the function $y=\dfrac{1}{(2x-3)^2}$, $x>\dfrac{3}{2}$, is a decreasing function.

$$y=(2x-3)^{-2}$$
$$\frac{dy}{dx}=-2(2x-3)^{-3}(2)$$
$$=-\frac{4}{(2x-3)^3}$$

Since $x>\dfrac{3}{2}$, $\quad 2x-3>0$

$$(2x-3)^3>0$$
$$\frac{1}{(2x-3)^3}>0$$
$$\therefore \frac{dy}{dx}=-\frac{4}{(2x-1)^3}<0$$

Hence $y=\dfrac{1}{(2x-3)^2}$, $x>\dfrac{3}{2}$ is an increasing function.

---

2. Determine, with explanation, whether $y=\sqrt{x-1}$, $x>1$ is an increasing or a decreasing function.

$$y=(x-1)^{\frac{1}{2}}$$
$$\frac{dy}{dx}=\frac{1}{2}(x-1)^{-\frac{1}{2}}$$
$$=\frac{1}{2\sqrt{x-1}}$$

Since $x>1$, $\quad x-1>0$

$$\sqrt{x-1}>0$$
$$\therefore \frac{dy}{dx}=\frac{1}{2\sqrt{x-1}}>0$$

Hence $y=\sqrt{x-1}$, $x>1$ is an increasing function.

3. A function is defined by $y = \dfrac{1}{1+x^2}$, determine whether y is an increasing or a decreasing function

(a) for $x < 0$,

(b) for $x > 0$.

(a) $y = (1+x^2)^{-1}$

$$\frac{dy}{dx} = -2x(1+x^2)^{-2}$$

$$= -\frac{2x}{(1+x^2)^2}$$

Since $x < 0$, $\quad -2x > 0$

$(1+x^2)^2 > 1$

$\therefore \dfrac{dy}{dx} = -\dfrac{2x}{(1+x^2)^2} > 0$

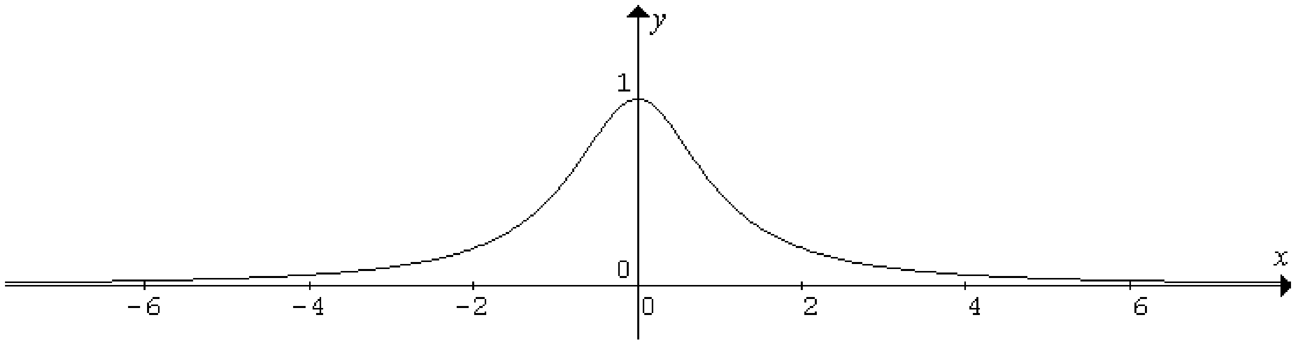
Hence $y = \dfrac{1}{1+x^2}$, $x < 0$ is an increasing function.

(b) Since $x > 0$, $-2x < 0$

$1 + x^2 > 1$

$\therefore \dfrac{dy}{dx} = -\dfrac{2x}{(1+x^2)^2} < 0$

Hence $y = \dfrac{1}{1+x^2}$, $x > 0$ is a decreasing function.



4. Given that $y=\dfrac{3x+4}{\sqrt{2x-1}}$, where $x>\dfrac{1}{2}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and hence find the range of values of x for which y is

(i) increasing,

(ii) decreasing.

$$y=\frac{3x+4}{\sqrt{2x-1}}$$
$$=\frac{3x+4}{(2x-1)^{\frac{1}{2}}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(2x-1)^{\frac{1}{2}}\dfrac{\mathrm{d}}{\mathrm{d}x}(3x+4)-(3x+4)\dfrac{\mathrm{d}}{\mathrm{d}x}(2x-1)^{\frac{1}{2}}}{(\sqrt{2x-1})^2}$$
$$=\frac{3(2x-1)^{\frac{1}{2}}-\dfrac{1}{2}(3x+4)(2x-1)^{-\frac{1}{2}}(2)}{2x-1}$$
$$=\frac{(2x-1)^{-\frac{1}{2}}(6x-3-3x-4)}{2x-1}$$
$$=(2x-1)^{-\frac{3}{2}}(3x-7)$$
$$=\frac{3x-7}{(2x-1)^{\frac{3}{2}}}$$

(i) When $\dfrac{\mathrm{d}y}{\mathrm{d}x}>0$, $\quad \dfrac{3x-7}{(2x-1)^{\frac{3}{2}}}>0$

$3x-7>0 \quad$ and $\quad 2x-1>0$

$x>\dfrac{7}{3} \qquad\qquad x>\dfrac{1}{2}$

$\therefore x>\dfrac{7}{3}$

y is an increasing function when $x>\dfrac{7}{3}$

(ii) When $\dfrac{\mathrm{d}y}{\mathrm{d}x}<0$, $\quad \dfrac{3x-7}{(2x-1)^{\frac{3}{2}}}<0 \quad$ and $\quad 2x-1>0$

$3x-7<0 \qquad\qquad x>\dfrac{1}{2}$

$x<\dfrac{7}{3}$

$\therefore \dfrac{1}{2}<x<\dfrac{7}{3}$

Since $x>\dfrac{1}{2}$, y is a decreasing function when $\dfrac{1}{2}<x<\dfrac{7}{3}$.

5. It is given that $y = -x^3 + ax^2 + bx + 4$ where a and b are integers. The only values of x for which y is an increasing function of x are those values for which $\frac{4}{3} < x < 4$.

Find the value of a and of b.

$$\frac{dy}{dx} = -3x^2 + 2ax + b$$

When y is a increasing function, $\frac{dy}{dx} > 0$, $\quad -3x^2 + 2ax + b > 0 \Leftrightarrow \frac{4}{3} < x < 4$

$$3x^2 - 2ax - b < 0 \Leftrightarrow \left(x - \frac{4}{3}\right)(x-4) < 0$$

$$\Leftrightarrow (3x-4)(x-4) < 0$$

$$\Leftrightarrow 3x^2 - 16x + 16 < 0$$

Comparing coefficient of x, $\quad -2a = -16$

$a = 8$

Comparing constant term, $\quad -b = 16$

$b = -16$

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Rate of Change

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Worksheet 15: Rate of Change

Rates of Change

(a) If a variable x varies with time t, then $\frac{\mathrm{d}x}{\mathrm{d}t}$ is the rate of change of x with respect to time.

(b) If $\frac{\mathrm{d}x}{\mathrm{d}t}$ is a constant, x changes at a constant rate.

(c) If $\frac{\mathrm{d}x}{\mathrm{d}t}$ is a function of t, x changes at a non-uniform rate. The value of $\frac{\mathrm{d}x}{\mathrm{d}t}$ at $t = a$, is the instantaneous rate of change of x at the instance $t = a$.

(d) If $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$, then a decrease in magnitude of x is observed as the value of t increases.

(e) If $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$, then an increase in magnitude of x is observed as the value of t increases.

Example 1

The height of a rocket above the ground, h, kilometres, with respect to time t seconds from its time of launch to its return, is given by

$$\mathrm{h}(t) = \frac{t(10-t)}{4}.$$

Find the rate of change of h with respect to t when
(i) $t = 2$,
(ii) $t = 6$. **(Pg386Q1)**

Rate of change of h $= \frac{\mathrm{dh}}{\mathrm{d}t}$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{10t - t^2}{4}\right)$$

$$= \frac{10-2t}{4} \text{ km s}^{-1}$$

$$= \frac{5-t}{2} \text{ km s}^{-1}$$

(i) When $t = 2$, $\frac{\mathrm{dh}}{\mathrm{d}t} = \frac{5-2}{2}$

$= 1.5 \text{ km s}^{-1}$

$\therefore$ the height is **increasing** when t = 2.

(ii) When $t = 6$, $\frac{\mathrm{dh}}{\mathrm{d}t} = \frac{5-6}{2}$

$= -0.5 \text{ km s}^{-1}$

$\therefore$ the height is **decreasing** when t = 6.

Example 2
The length, l mm, of an elastic string at time t seconds is given by

$$l = \frac{t^3}{3} - 4t + 10 .$$

Find the value of t when
(a) the length is increasing at a rate of 5 mms^{-1},
(b) the length is decreasing at a rate of 4 mms^{-1}. **(Pg386Q2)**

Rate of change of length, $\frac{\mathrm{d}l}{\mathrm{d}t} = (t^2 - 4)\ \text{mms}^{-1}$

(a) When $\frac{\mathrm{d}l}{\mathrm{d}t} = 5\ \text{mms}^{-1}$,

$$t^2 - 4 = 5$$
$$t^2 = 9$$
$$t = 3\ (t \geq 0)$$

(b) When $\frac{\mathrm{d}l}{\mathrm{d}t} = -4\ \text{mms}^{-1}$,

$$t^2 - 4 = -4$$
$$t^2 = 0$$
$$t = 0\ (t > 0)$$

Example 3
The radius, r cm, of a body changes with time t seconds, and they are related by the equation

$$r = t^2 + 2 .$$

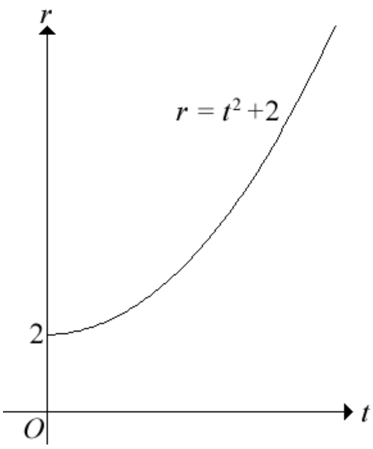
(i) Find $\frac{\mathrm{d}r}{\mathrm{d}t}$ and calculate the rate of change of r with respect to t at $t = 2$.
(ii) Sketch the graph of $r = t^2 + 2$ and explain the changes in r with respect to t. **(Pg386Q2)**

(i) $\frac{\mathrm{d}r}{\mathrm{d}t} = 2t$

When $t = 2$, $\frac{\mathrm{d}r}{\mathrm{d}t} = 4\ \text{cm s}^{-1}$

(iii) r increases non-uniformly with respect to t.

(ii)



Example 4

The radius, r cm, of a circle at time t seconds is given by

$$r=\sqrt{\frac{t^3}{3}-\frac{5t^2}{2}+6t}\ .$$

(i) Express its area, A cm^2, in terms of t.
(ii) Find the values of t for which the area is increasing.

$$\begin{aligned}A &= \pi r^2\\ &= \pi\left(\sqrt{\frac{t^3}{3}-\frac{5t^2}{2}+6t}\right)^2\\ &= \pi\left(\frac{t^3}{3}-\frac{5t^2}{2}+6t\right)\end{aligned}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \pi(t^2-5t+6)$$

When $\frac{\mathrm{d}A}{\mathrm{d}t}>0$, $\quad \pi(t^2-5t+6)>0$

$$\pi(t-2)(t-3)>0$$

$$t<2 \text{ or } t>3$$

Since $t\geq 0$, area is increasing when $0\leq t<2$ or $t>3$.

Example 5

The volume, V cm^3, of a cone of height h is $\frac{\pi h^3}{12}$. If h increases at a constant rate of 0.2 cm s^{-1} and the initial height is 2 cm, express
(i) h in terms of t,
(ii) V in terms of t and find the rate of change of V at time t.

(i) $h = 2+0.2t$

(ii) $V = \frac{\pi}{12}(2+0.2t)^3$

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\pi}{12}[3(2+0.2t)^2(0.2)]\\ &= 0.05\pi(2+0.2t)^2\ \mathrm{cm^3 s^{-1}}\end{aligned}$$

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Assignment 16: Rate of Change

1. The amount of water, $V\text{ cm}^3$, in a leaking tank at time t seconds is given by

$$V = (15 - t)^3 \text{ for } 0 \le t \le 15.$$

Find the rate at which the water leaves the tank at the instant when $t = 4$.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -3(15 - t)^2 \text{ cm}^3\text{s}^{-1}$$

When $t = 4$, $\frac{\mathrm{d}V}{\mathrm{d}t} = -3(15-4)^2 \text{ cm}^3\text{s}^{-1}$

$$= -363 \text{ cm}^3\text{s}^{-1}$$

m

The water leaves the tank at the rate of $363 \text{ cm}^3\text{s}^{-1}$

s

---

2. A rectangle has sides measuring x cm and $2x - 4$ cm. The length x cm at time t seconds is given by

$$x = 2 + 3t, \ (t \ge 0).$$

(i) Show that the area, $A\text{ cm}^2$, of the rectangle, in terms of t is $A = 12t + 18t^2$.
(ii) Find the rate of change of area at the instant when $t = 2$.

(i) $A = x(2x - 4)$

$$= (2 + 3t)[2(2 + 3t) - 4]$$
$$= 6t(2 + 3t)$$
$$= 12t + 18t^2$$

(ii) $\frac{\mathrm{d}A}{\mathrm{d}t} = 12 + 36t$

When $t = 2$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 12 + 36(2)$

$$= 84 \text{ cm}^2\text{s}^{-1}$$

3. The radius, r cm, of a spherical balloon at time t seconds is given by

$$r = 3 + \frac{2}{1+t}.$$

(i) What is its initial radius?

(ii) Find the rate of change of r with respect to t when $t = 3$. Suggest what may be happening to the balloon at this instant.

(i) When $t = 0$, $r = 3 + 2 = 5$

Initial radius = 5 cm

(ii) $r = 3 + 2(1+t)^{-1}$

$$\frac{dr}{dt} = 2(-1)(1+t)^{-2}(1)$$

$$= -\frac{2}{(1+t)^2}$$

When $t = 3$, $\frac{dr}{dt} = -\frac{2}{(1+3)^2}$

$$= -\frac{1}{8} \text{ cm s}^{-1}$$

The balloon is deflating at a rate of $\frac{1}{8}$ cm s^{-1}.

---

4. Mr Tan derived a function to estimate the number of people in his supermarket with respect to time. The time t is measured in minutes after 10 am, and the function is given by

$$f(t) = \frac{1}{10000000}(6t^2 + 5)^2.$$

(i) How many people will there be in the supermarket at 11 am?

(ii) At what rate are people entering the supermarket at 11 am?

(i) When $t = 60$, $f(60) = \frac{1}{10000000}[6(60)^2 + 5]^2$

$$= 46.7$$

There are 46 people

(ii) $\frac{d}{dt}[f(t)] = \frac{1}{10000000} \times 2(6t^2 + 5)(12t)$

$$= \frac{3t(6t^2 + 5)}{1250000}$$

When $t = 60$, $\frac{d}{dt}[f(t)] = \frac{3(60)[6(60)^2 + 5]}{1250000}$

$$= 3.11 \text{ people/min}$$

5. The stopping distance, s km, of a van moving at v km/h can be modelled by the formula

$$s = \frac{v}{8} + \frac{v^2}{80}.$$

(i) Find $\frac{\mathrm{d}s}{\mathrm{d}v}$ and calculate $\frac{\mathrm{d}s}{\mathrm{d}v}$ when $v = 60$.

(ii) Explain the meaning of your answer to (i).

(i) $\frac{\mathrm{d}s}{\mathrm{d}v} = \frac{1}{8} + \frac{v}{40}$

When $v = 60$, $\frac{\mathrm{d}s}{\mathrm{d}v} = \frac{1}{8} + \frac{60}{40}$

$= \frac{13}{8}$ km per km/h

(ii) If someone is travelling at 60 km/h, for every 1 km/h increase in speed, his stopping distance increases by $1\frac{5}{8}$ km.

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Worksheet 17: Connected Rates of Change (I)

Connected Rates of Change

If $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ is the rate of change of x with respect to time t and $y = \mathrm{f}(x)$, then the rate of change of y with respect to t is given by $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}y}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}t}$.

Example 1

Given that $y = \dfrac{2x-1}{x}$, find the rate of change of y when $x = 2$, if x is changing at the rate of 6 units per seconds at this instant.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x-(2x-1)}{x^2}\\ &= \frac{1}{x^2}\end{aligned}$$

When $x = 2$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 6$ unit/s,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}\\ &= \frac{1}{2^2} \times 6\\ &= \frac{3}{2}\ \text{unit/s}\end{aligned}$$

Example 2

Given that $y = (x-5)\sqrt{2x+5}$,

(i) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ can be expressed in the form $\dfrac{kx}{\sqrt{2x+5}}$,

(ii) find the rate of change of y when $x = 10$, if x is changing at the rate of 5 units per seconds at this instant.

(i)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{2x+5} + \frac{1}{2}(2)(x-5)(2x+5)^{-\frac{1}{2}}\\ &= \sqrt{2x+5} + \frac{x-5}{\sqrt{2x+5}}\\ &= \frac{2x+5+x-5}{\sqrt{2x+5}}\\ &= \frac{3x}{\sqrt{2x+5}}\end{aligned}$$

(ii) When $x = 10$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{30}{\sqrt{25}}\\ &= \frac{30}{5}\\ &= 6\end{aligned}$$
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}\\ &= 6 \times 5\\ &= 30\ \text{units s}^{-1}\end{aligned}$$

Example 3

Given that $y=\left(\frac{1}{3}x+2\right)^3$, find the rate of change of x when $x=9$, if y is changing at the rate of 5 units per seconds at this instant.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{1}{3}x+2\right)^2$$

When $x=9$ and $\frac{dy}{dt}=5$ unit/s,

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$5=\left(\frac{9}{3}+2\right)^2\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{5}\text{ unit/s}$$

Example 4

Given that $y=\frac{x}{3x+2}$,

(i) show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be expressed in the form $\frac{k}{(3x+2)^2}$,

(ii) find the rate of change of x when $x=0$, if y is changing at the rate of 5 units per seconds at this instant.

(i)
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x+2-3x}{(3x+2)^2}$$
$$=\frac{2}{(3x+2)^2}$$

(ii) When $x=10$,
$$\frac{\mathrm{d}y}{\mathrm{d}t}=5\text{ units s}^{-1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$$
$$5=\frac{1}{2}\times\frac{\mathrm{d}x}{\mathrm{d}t}$$
$$\frac{\mathrm{d}x}{\mathrm{d}t}=10\text{ units s}^{-1}$$

Example 5

A curve has equation $y=(2x-1)\sqrt{4x+1}$.

(i) Express $\dfrac{dy}{dx}$ in the form $\dfrac{kx}{\sqrt{4x+1}}$, where k is a constant.

Hence,

(ii) find the rate of change of x when $x=2$, given that y is changing at a constant rate of 2 units per second. (N09)

(i) $$\begin{aligned}\frac{dy}{dx} &= 2\sqrt{4x+1}+(2x-1)\left(\frac{1}{2\sqrt{4x+1}}\times 4\right)\\ &= \frac{8x+2+4x-2}{\sqrt{4x+1}}\\ &= \frac{12x}{\sqrt{4x+1}}\end{aligned}$$

(ii) When $x=2$, $\dfrac{dy}{dt}=2\text{ units s}^{-1}$,

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx}\times\frac{dx}{dt}\\ 2 &= \frac{12(2)}{\sqrt{4(2)+1}}\times\frac{dx}{dt}\\ \frac{dx}{dt} &= 2\div 8\\ &= \frac{1}{4}\text{ units s}^{-1}\end{aligned}$$

Example 6

A curve has the equation $y=\dfrac{2x-10}{x+1}$, where $x\neq -1$.

(i) Show that $\dfrac{dy}{dx}$ is always positive.

(ii) Given that both x and y vary with time t, find the value(s) of x for which $\dfrac{dy}{dt}=3\dfrac{dx}{dt}$.

(Pg391Q4)

(i) $$\begin{aligned}\frac{dy}{dx} &= \frac{(x+1)\frac{d}{dx}(2x-10)-(2x-10)\frac{d}{dx}(x+1)}{(x+1)^2}\\ &= \frac{(x+1)(2)-(2x-10)(1)}{(x+1)^2}\\ &= \frac{12}{(x+1)^2}\end{aligned}$$

Since $(x+1)^2>0$ for all values of x, $\therefore \dfrac{12}{(x+1)^2}>0$

Hence, $\dfrac{dy}{dx}>0$.

(ii) When $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 3\dfrac{\mathrm{d}x}{\mathrm{d}t}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$

$$\frac{12}{(x+1)^2} = 3$$

$$(x+1)^2 = 4$$

$$x+1 = \pm 2$$

$$x = 1,\ -3$$

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Mathematics II
Rate of Change

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<u>Assignment 18: Connected Rates of Change (I)</u>

1. Two variables, x and y, are related by the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$. Given that x is decreasing at a rate of 3.6 units per second when $x=5$, find the rate of change of y at this instant.

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$$

$$\begin{aligned}\frac{1}{y} &= \frac{1}{2}-\frac{1}{x}\\ &= \frac{x-2}{2x}\end{aligned}$$

$$y=\frac{2x}{x-2}$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x-4-2x}{(x-2)^2}\\ &= -\frac{4}{(x-2)^2}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= -\frac{4}{(x-2)^2}\times(-3.6)\\ &= \frac{14.4}{(x-2)^2}\end{aligned}$$

When $x=5$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{14.4}{9}$ units per second

= 1.6 units per second

2. Liquid is poured into a container at a rate of $12\ \text{cm}^3\text{s}^{-1}$. The volume of liquid in the container is V cm^3, where $V = \frac{1}{2}(h^2 + 4h)$, and h cm is the height of liquid in the container. Find, when $V = 16$,

(a) the value of h,

(b) the rate at which h is increasing.

(a) When $V = 16$,

$$\frac{1}{2}(h^2 + 4h) = 16$$
$$h^2 + 4h - 32 = 0$$
$$(h-4)(h+8) = 0$$
$$\therefore h = 4, -8\ (\text{NA})$$

(b) $$\frac{dV}{dh} = \frac{1}{2}(2h+4)$$
$$= h + 2$$

When $h = 4$, $\frac{dV}{dt} = 12\ \text{cm}^3\text{s}^{-1}$,

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$
$$12 = 6 \times \frac{dh}{dt}$$
$$12 = 6\frac{dh}{dt}$$
$$\therefore \frac{dh}{dt} = 2\ \text{cms}^{-1}$$

3. Two variable lengths x cm and y cm are related by the equation $y = \sqrt{\frac{22}{x} - x}$.

(a) Obtain an expression for $\frac{dy}{dx}$ in terms of x.

(b) Given that x and y are functions of t (seconds) and $\frac{dx}{dt} = 2$, find $\frac{dy}{dt}$ when $x = 2$.

(a) $$y = \left(\frac{22}{x} - x\right)^{\frac{1}{2}}$$

$$= (22x^{-1} - x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(22x^{-1} - x)^{-\frac{1}{2}} \times \left(-\frac{22}{x^2} - 1\right)$$

$$= \frac{-(22 + x^2)}{2x^2\sqrt{\frac{22}{x} - x}}$$

(b) $$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= -\frac{22 + x^2}{x^2\sqrt{\frac{22}{x} - x}}$$

When $x = 2$, $$\frac{dy}{dt} = -\frac{(22+4)}{4\sqrt{11-3}}$$

$$= -\frac{13}{6} \text{ cm s}^{-1}$$

4. A curve has the equation $y=(x-c)\sqrt{x+6}$.

(a) Find the value of c and of k for which $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{kx}{\sqrt{x+6}}$.

(b) Find the value of x for which $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{3}{2}\dfrac{\mathrm{d}x}{\mathrm{d}t}$.

(a)
$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{x+6}+\frac{1}{2}(x-c)(x+6)^{-\frac{1}{2}} \\
&= \sqrt{x+6}+\frac{x-c}{2\sqrt{x+6}} \\
&= \frac{2x+12+x-c}{2\sqrt{x+6}} \\
&= \frac{3x+12-c}{2\sqrt{x+6}} \\
&= \frac{3x}{2\sqrt{x+6}}+\frac{12-c}{2\sqrt{x+6}}
\end{aligned}
$$

When $\dfrac{3x}{2\sqrt{x+6}}+\dfrac{12-c}{2\sqrt{x+6}}=\dfrac{kx}{\sqrt{x+6}}$,
$$
\begin{aligned}
12-c&=0\\
c&=12\\
k&=\frac{3}{2}
\end{aligned}
$$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\mathrm{d}y}{\mathrm{d}x}\times\dfrac{\mathrm{d}x}{\mathrm{d}t}$

When $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{3}{2}\dfrac{\mathrm{d}x}{\mathrm{d}t}$,
$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{3}{2}\\
\frac{3x}{2\sqrt{x+6}}&=\frac{3}{2}\\
\frac{x}{\sqrt{x+6}}&=1\\
x^2&=x+6\\
x^2-x-6&=0\\
(x-3)(x+2)&=0\\
\therefore x&=3,\ -2\,(\text{NA})
\end{aligned}
$$

5. A particle moves along the curve $y = 2x^2 + 3x - 5$. At the point P, the x-coordinate of the particle is increasing at a rate of 0.04 units per second and the y-coordinate is increasing at 0.2 units per second. Find the coordinates of P. (N2014)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x + 3$$

When $\frac{\mathrm{d}y}{\mathrm{d}t} = 0.2$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.04$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$0.2 = 0.04(4x + 3)$$

$$4x + 3 = 5$$

$$4x = 2$$

$$x = \frac{1}{2}$$

$$y = -3$$

$$\therefore P\left(\frac{1}{2},\ -3\right)$$

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Worksheet 19: Connected Rates of Change (II)

Method for Solving Word Problems on Related Rates
• Form an equation that relates the dependent variables. Find the derivative of this equation. • Use the chain rule to get the related rates equation. • Substitute the given information into the related rates equation and solve for the unknown rate.

Example 1

The radius of a sphere increases at a rate of $2\,\text{cm s}^{-1}$. Find the rate of increase of its volume when the radius is 3 cm.

Let the radius and volume of the sphere be r cm and V cm^3 respectively.

$\therefore\ \dfrac{dr}{dt} = 2\,\text{cm s}^{-1}$

Step	
① Form an equation that relates the dependent variables.	$V = \dfrac{4}{3}\pi r^3$
② Use the chain rule to get the related rates equation.	$\dfrac{dV}{dr} = 4\pi r^2$
③ Substitute the given information into the related rates equation and solve for the unknown rate.	$\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$ $= 8\pi r^2$ When r = 3 cm, $\dfrac{dV}{dt} = 72\pi\,\text{cm}^3\text{s}^{-1}$

Example 2

The edge of a cube is increasing at the rate of $2\,\text{cms}^{-1}$. Find

(a) the rate of increase of the surface area, when each side is 4 cm.

(b) the rate of increase of the volume when each side is 8 cm.

Let the length, surface area and volume of the cube be x cm, A cm^2 and V cm^3 respectively.

$$\therefore \frac{\mathrm{d}x}{\mathrm{d}t} = 2\,\text{cms}^{-1}$$

(a) Step	
① Form an equation that relates the dependent variables.	$A = 6x^2$
② Use the chain rule to get the related rates equation.	$\frac{\mathrm{d}A}{\mathrm{d}x} = 12x$
③ Substitute the given information into the related rates equation and solve for the unknown rate.	$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = 24x$ When $x = 4$ cm, $\frac{\mathrm{d}A}{\mathrm{d}t} = 96\,\text{cm}^2\text{s}^{-1}$

(b) Step	
① Form an equation that relates the dependent variables.	$V = x^3$
② Use the chain rule to get the related rates equation.	$\frac{\mathrm{d}V}{\mathrm{d}x} = 3x^2$
③ Substitute the given information into the related rates equation and solve for the unknown rate.	$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = 6x^2$ When $x = 8$ cm, $\frac{\mathrm{d}V}{\mathrm{d}t} = 384\,\text{cm}^3\text{s}^{-1}$

Example 3

The area of a circle increases at a rate of $2\pi\,\text{cm}^2\text{s}^{-1}$. Find the rate of increase of the radius when the radius is 6 cm.

(a) the radius is 4 cm,

(b) the area is 9π cm^2.

Let the radius and surface area of the circle be r cm and A cm^2 respectively.

$$\therefore\ \frac{\mathrm{d}r}{\mathrm{d}t} = 2\,\text{cms}^{-1}$$

(a) $A = \pi r^2 \Rightarrow \dfrac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$$
$$= 4\pi r$$

When r = 4 cm, $\dfrac{\mathrm{d}A}{\mathrm{d}t} = 16\pi\,\text{cm}^2\text{s}^{-1}$

(b) When A = 9π cm^2, $\pi r^2 = 9\pi$

$r = 3 \quad (r > 0)$

When r = 3 cm, $\dfrac{\mathrm{d}A}{\mathrm{d}t} = 12\pi\,\text{cm}^2\text{s}^{-1}$

Example 4

The surface area of a sphere is decreasing at the rate of $20\,\text{cm}^2\text{s}^{-1}$ when the radius is 15 cm.

Calculate (a) the rate of change of the radius at this instant,

(b) the rate of change of the volume at this instant.

Let the radius, surface area and volume of the sphere be r cm, A cm^2 and V cm^3 respectively.

$$\therefore\ \frac{\mathrm{d}A}{\mathrm{d}t} = -20\,\text{cm}^2\text{s}^{-1}$$

$$A = 4\pi r^2 \quad \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r\ \text{cm}$$

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2\ \text{cm}^2$$

(a) Since $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{\mathrm{d}A}{\mathrm{d}r} \times \dfrac{\mathrm{d}r}{\mathrm{d}t}$, $\quad \therefore \dfrac{\mathrm{d}r}{\mathrm{d}t} = \dfrac{\mathrm{d}A}{\mathrm{d}t} \div \dfrac{\mathrm{d}A}{\mathrm{d}r}$

$$= -\frac{20}{8\pi r}$$
$$= -\frac{5}{2\pi r}\,\text{cms}^{-1}$$

When r =15, $\dfrac{\mathrm{d}r}{\mathrm{d}t} = -\dfrac{5}{2\pi(15)}$

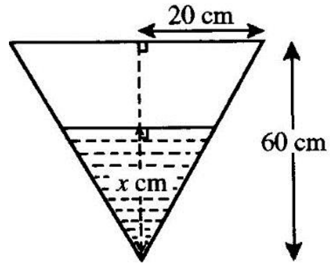
$$= -\frac{1}{6\pi}\,\text{cms}^{-1} \text{ or } -0.0531\,\text{cms}^{-1}$$

(b) When r =15, $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \dfrac{\mathrm{d}V}{\mathrm{d}r} \times \dfrac{\mathrm{d}r}{\mathrm{d}t}$

$$= 4\pi(15)^2 \times \left(-\frac{1}{6\pi}\right)$$
$$= -150\,\text{cm}^3\text{s}^{-1}$$

Example 5

The diagram shows a vertical cross-section of a container in the form of an inverted cone of height 60 cm and base radius 20 cm. The circular base is held horizontal and uppermost. Water is poured into the container at a constant rate of 40 cm^3 s^{-1}.



(i) Show that, when the depth of water in the container is x cm, the volume of water in the container is $\dfrac{\pi x^3}{27}$ cm^3.

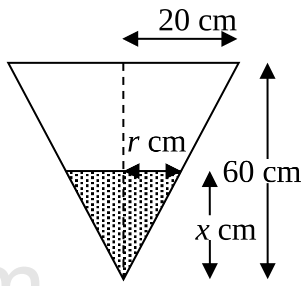
(ii) Find the rate of increase of x at the instant when $x = 2$.

(iii) State, with a reason, whether this rate will increase or decrease as t increases.

Let the radius of the water surface and volume of the water be r cm and V cm^3 respectively.

$$\therefore \frac{dV}{dt} = 40\,\text{cm}^3\text{s}^{-1}$$

(i) $\dfrac{r}{20} = \dfrac{x}{60}$



$$r = \frac{x}{3}$$

$$V = \frac{1}{3}\pi r^2 x$$

$$= \frac{1}{3}\pi\left(\frac{x}{3}\right)^2 x$$

$$= \frac{\pi x^3}{27}$$

(ii) $\dfrac{dV}{dr} = \dfrac{\pi x^2}{9}\,\text{cm}^2$

Since $\dfrac{dV}{dt} = \dfrac{dV}{dx} \times \dfrac{dx}{dt}$, $\quad \therefore \dfrac{dx}{dt} = \dfrac{dV}{dt} \div \dfrac{dV}{dr}$

$$= 40 \times \frac{9}{\pi x^2}$$

$$= \frac{360}{\pi x^2}\,\text{cms}^{-1}$$

When $x = 2$, $\dfrac{dx}{dt} = \dfrac{360}{\pi(2)^2}$

$$= 28.6\,\text{cms}^{-1}$$

(iii) When t increases, x increases.

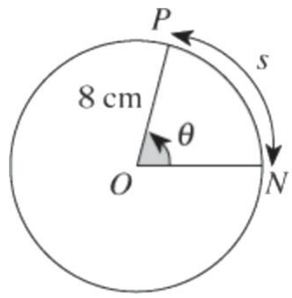
As $\dfrac{dx}{dt} = \dfrac{360}{\pi x^2}\,\text{cms}^{-1}$ is inversely proportional to x^2, $\dfrac{dx}{dt}$ decreases as t increases.

Example 6

N is a fixed point on the circumference of a circle, centre O, radius 8 cm. A variable point P moves round the circumference such that θ (or $\angle PON$) increases at a constant rate of $\frac{\pi}{2}$ rad. per second. Find

(a) the rate of increase of s (arc length from N to P),

(b) the rate of increase of A cm^2, the area of the sector NOP. **(Pg396Q8)**



(a) Arc length from N to P, $s = 8\theta$ cm

$$\frac{ds}{dt} = \frac{ds}{d\theta} \times \frac{d\theta}{dt}$$
$$= 8 \times \frac{\pi}{2}$$
$$= 4\pi \text{ cm s}^{-1}$$

(b) Area of the sector NOP, $A = \frac{1}{2} \times 8^2 \times \theta$

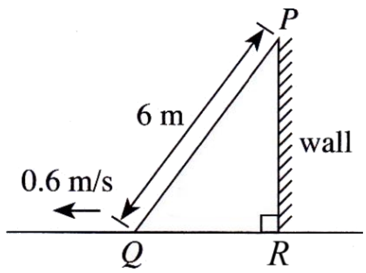
$$= 32\theta \text{ cm}^2$$

$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$$
$$= 32 \times \frac{\pi}{2}$$
$$= 16\pi \text{ cm}^2 \text{ s}^{-1}$$

Example 7

A ladder PQ of length 6 m is leaning against a vertical wall. The bottom end of the ladder is sliding away from the wall at a constant rate of 0.6 ms^{-1}.

Find the rate at which the top of the ladder is sliding down the wall when P is 4.8 m from the ground. **(Pg392Q13)**



Let $PR = y$ cm and $QR = x$ cm.

$$y = \sqrt{6^2 - x^2}$$
$$= (36 - x^2)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(36 - x^2)^{-\frac{1}{2}}(-2x)$$
$$= -\frac{x}{\sqrt{36 - x^2}}$$

When $y = 4.8$, $x^2 = 36 - 4.8^2$

$$x = \sqrt{36 - 4.8^2}$$
$$= 3.6$$

When $x = 3.6$, $\frac{dx}{dt} = 0.6$, $\frac{dy}{dt} = -\frac{3.6}{\sqrt{36 - 3.6^2}} \times 0.6$

$$= -0.45 \text{ ms}^{-1}$$

The top of the ladder is sliding down the wall at a rate of 0.45 ms^{-1}.

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Assignment 20: Connected Rates of Change (II)

1. The area of a circle increases at a rate of 2π cm^{2}s^{-1}. Find the rate of increase of the radius when the radius is 6 cm.

   Let A cm^2 and r cm be the area and radius of the circle respectively.

   $A = \pi r^2$

   $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

   $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \times \frac{\mathrm{d}r}{\mathrm{d}t}$

   When $r = 6$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi$, $\quad 2\pi = 2\pi(6) \times \frac{\mathrm{d}r}{\mathrm{d}t}$

   $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{6}$ cm s^{-1}

---

2. A circular ripple spreads across a lake. If the area of the ripple increases at a uniform rate of 10π m^{2}s^{-1}, find the radius of the ripple when the radius is increasing at a rate of $2.5\,\mathrm{m\,s}^{-1}$.

   Let A cm^2 and r cm be the area and radius of the circle respectively.

   $A = \pi r^2$

   $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

   $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \times \frac{\mathrm{d}r}{\mathrm{d}t}$

   When $\frac{\mathrm{d}r}{\mathrm{d}t} = 2.5$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 10\pi$, $\quad 10\pi = 2\pi r \times 2.5$

   $r = 2$

   Radius of the ripple = 2 cm

3. The area of a square increases at a rate of 10 cm^2s^{-1}. Find the rate of change in the length of its side when the area is 4 cm^2. **(Pg391Q7(a))**

Let x and A represent the length and the area of the square.

$$A = x^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 2x \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

When $A = 4$, $\quad x^2 = 4$

$$x = 2 \ \ (x > 0)$$

When $x = 2$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 10$, $\quad 10 = 4\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 2.5 \ \text{cm s}^{-1}$$

4. The surface area of a cube increases at 0.2 cm^2s^{-1}. Find the rate of increase of the volume when the length of a side is 1 cm. **(Pg391Q7(b))**

Let x, A and V represent the length, area and the volume of the cube.

$$A = 6x^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 12x$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 12x \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

When $r = 1$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 0.2$, $\quad 0.2 = 12(1) \times \frac{\mathrm{d}r}{\mathrm{d}t}$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{0.2}{12}$$

$$= \frac{1}{60} \text{cm s}^{-1}$$

$$V = x^3$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 3x^2$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = 3x^2 \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

When $r = 1$, $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{60}$, $\quad \frac{\mathrm{d}V}{\mathrm{d}t} = 3(1) \times \frac{1}{60}$

$$= 0.05 \, \text{cm}^3\text{s}^{-1}$$

5. A viscous liquid is poured onto a flat surface. It forms a circular patch which grows at a steady rate of $4\,\text{cm}^2\text{s}^{-1}$. Find, in terms of π,

(a) the radius of the patch 16 seconds after pouring has begun,

(b) the rate of increase of the radius at this instant. **(Pg396Q7)**

(a) Let the radius and the area of the circular patch be r cm and A cm^2 respectively.

The circular patch grows at a steady rate of $4\,\text{cm}^2\text{s}^{-1} \Rightarrow A = 4t$

When $t = 16$,

$$\begin{aligned} A &= 4(16) \\ &= 64\,\text{cm}^2 \\ A &= \pi r^2 \\ \pi r^2 &= 64 \\ r &= \sqrt{\frac{64}{\pi}} \\ &= \frac{8}{\sqrt{\pi}} \end{aligned}$$

(b) $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

When $r = \dfrac{8}{\sqrt{\pi}}$, $\dfrac{\mathrm{d}A}{\mathrm{d}t} = 4\,\text{cm}^2\text{s}^{-1}$,

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} \\ 4 &= 2\pi\left(\frac{8}{\sqrt{\pi}}\right) \times \frac{\mathrm{d}r}{\mathrm{d}t} \\ 4 &= 16\sqrt{\pi} \times \frac{\mathrm{d}r}{\mathrm{d}t} \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{4}{16\sqrt{\pi}} \\ &= \frac{1}{4\sqrt{\pi}}\,\text{cm s}^{-1} \end{aligned}$$

6. A water tank with a rectangular base 30 cm by 40 cm contains water to a depth of 20 cm. The water is then poured at a steady rate into an inverted conical container with base radius equal to its height and whose axis is vertical.

After t seconds, the volume of the water that has been transferred is given by $V = \dfrac{\pi r^3}{3}$, r cm is the radius of the horizontal surface of the liquid.

Given that all the water is transferred in 6 seconds, find at the instant when the depth of the liquid is 12 cm, the rate of increase

(i) of the depth of the liquid,
(ii) the area of the horizontal surface of the 1iquid.

Solution

Volume of water transferred in 6 s = $30 \times 40 \times 20$ cm^3

Rate of change of volume, $\dfrac{dV}{dt} = \dfrac{30 \times 40 \times 20}{6}$ cm^{3}s^{-1}

$= 4000$ cm^{3}s^{-1}

$$V = \frac{\pi r^3}{3} \Rightarrow \frac{dV}{dr} = \pi r^2$$

(i) $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$

$4000 = \pi r^2 \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{4000}{\pi r^2}$ cms^{-1}

When $r = 12$ cm, $\dfrac{dr}{dt} = \dfrac{4000}{144\pi}$ cm s^{-1}

Since $r = h$, $\dfrac{dr}{dt} = \dfrac{dh}{dt} = 8.84$ cm s^{-1}

(ii) $A = \pi r^2 \Rightarrow \dfrac{dA}{dr} = 2\pi r$

$\dfrac{dA}{dt} = \dfrac{dA}{dr} \times \dfrac{dr}{dt}$

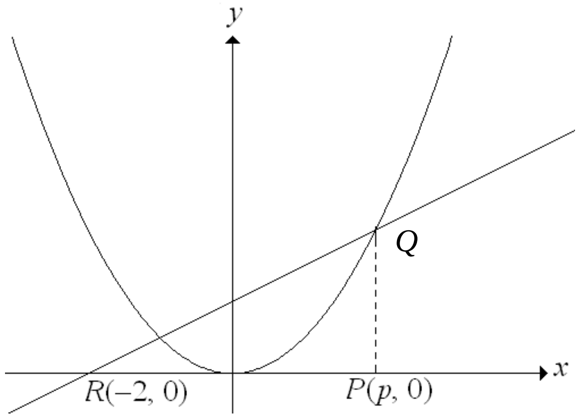
$= 2\pi r \dfrac{dr}{dt}$

When $r = 12$ cm, $\dfrac{dr}{dt} = \dfrac{4000}{144\pi}$ cms^{-1}

$\dfrac{dA}{dt} = 2\pi \times 12 \times \dfrac{4000}{144\pi}$

$= 666\dfrac{2}{3}$ cm^{2}s^{-1}

7. The diagram shows part of the graph of $y = \frac{x^2}{2}$. The point $P(p, 0)$ is on the x-axis and the point Q is on the curve. **(Pg397Q7)**



(i) Given that PQ is parallel to the y-axis and the coordinates of R is $(-2, 0)$, express the area A of the triangle PQR in terms of p. Hence, show that $\frac{dA}{dp} = \frac{3}{4}p^2 + p$.

(ii) If p is increasing at a rate of 0.1 units per second, find the rate at which A is increasing at the instant when $p = 4$.

(iii) State, with a reason, whether $\frac{\mathrm{d}A}{\mathrm{d}t}$ will increase or decrease as t increases.

(i) Coordinates of $Q = \left(p, \frac{p^2}{2}\right)$

Area of triangle PQR $= \frac{1}{2}(p+2)\left(\frac{p^2}{2}\right)$ units

$= \frac{1}{4}(p^3 + 2p^2)$ units

$\therefore \frac{\mathrm{d}A}{\mathrm{d}p} = \frac{3}{4}p^2 + p$

(ii) $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}p} \times \frac{\mathrm{d}p}{\mathrm{d}t}$

$= \left(\frac{3}{4}p^2 + p\right) \times 0.1$ unit^2 per second

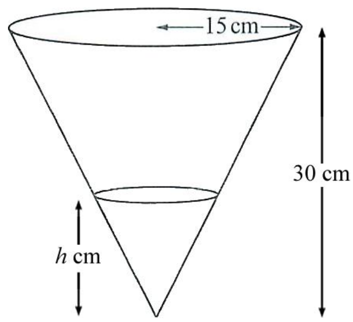
When $p = 4$, $\frac{\mathrm{d}A}{\mathrm{d}t} = \left(\frac{3}{4} \times 4^2 + 4\right) \times 0.1$

$= 1.6$ unit^2 per second

(iii) When t increases, both p and p^2 will increase.

Hence, $\frac{\mathrm{d}A}{\mathrm{d}t}$ which is jointly proportional to p and p^2 increases as t increases.

8. [The volume of a cone of height h and base radius R is given by $\frac{1}{3}\pi R^2 H$.]



The diagram shows a hollow conical tank of height 30 cm and radius 15 cm. The tank is held fixed with its circular rim horizontal. Water is then poured into the empty tank at a constant rate of $20\,\text{cm}^3\text{s}^{-1}$. After t seconds the depth of water is h cm.

(i) Show that the volume of water in the tank, $V\,\text{cm}^3$, at time t is given by $V = \dfrac{\pi h^3}{12}$.

(ii) Find the rate of change of the depth when $h = 5$.

(iii) State, with a reason, whether this rate will increase or decrease as t increases.

Specimen Paper

(i) $$\frac{V}{\frac{1}{3}\pi(15^2)(30)} = \frac{h^3}{30^3}$$

$$V = \frac{h^3}{30^3} \times \frac{1}{3}\pi(15^2)(30)$$

$$= \frac{\pi h^3}{12}$$

(ii) $$\frac{dV}{dh} = \frac{\pi h^2}{4}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$20 = \frac{\pi h^2}{4} \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{80}{\pi h^2}$$

When $h = 5$, $\dfrac{dh}{dt} = \dfrac{80}{25\pi}\,\text{cms}^{-1}$ or $1.02\,\text{cms}^{-1}$

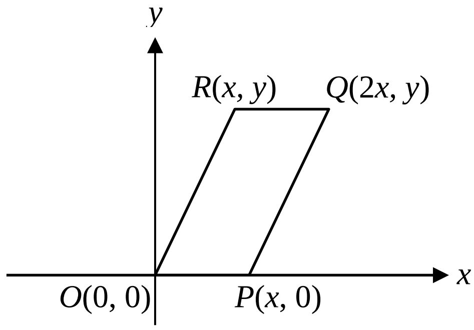
(iii) When t increases, h will increase.

As $\dfrac{dh}{dt} = \dfrac{80}{\pi h^2}$ is inversely proportional to h^2, this rate will decrease as t increases.

9.* The figure shows a parallelogram *OPQR*. Given that *Q* lies on the line $y = 2x + 1$ and x increases at a rate of 1.2 units per second, find

(a) the rate of change of area of the parallelogram *OPQR* when $x = 1.5$,

(b) the rate of change of the length of the diagonal *OQ* at this instant. **(Pg392Q11)**



(a) Let the area of the parallelogram *OPQR* be *A*.
Since $Q(2x, y)$ lies on the line $y = 2x + 1$,
y-coordinates of $Q = 2(2x) + 1$
$= 4x + 1$

$$A = x(4x+1)$$
$$= 4x^2 + x$$
$$\frac{dA}{dx} = 8x + 1$$

When $x = 1.5$, $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$
$$= (8 \times 1.5 + 1) \times 1.2$$
$$= 15.6 \text{ unit}^2 \text{ s}^{-1}$$

(b) Let $OQ = s$.
$$s^2 = (2x)^2 + (4x+1)^2$$
$$= 4x^2 + 16x^2 + 8x + 1$$
$$= 20x^2 + 8x + 1$$
$$s = \sqrt{20x^2 + 8x + 1}$$

$$\frac{ds}{dx} = \frac{1}{2}(20x^2 + 8x + 1)^{-\frac{1}{2}}(40x + 8)$$
$$= \frac{20x + 4}{\sqrt{20x^2 + 8x + 1}}$$

When $x = 1.5$, $\frac{ds}{dt} = \frac{20x+4}{\sqrt{20x^2+8x+1}} \times \frac{ds}{dx}$
$$= \frac{20(1.5)+4}{\sqrt{20(1.5)^2 + 8(1.5) + 1}} \times 1.2$$
$$\approx 5.36 \text{ unit s}^{-1}$$

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 13

Name: ______________________________ (____) Date: ________________

Class: Sec 4______

Revision 13 Rate of change

Increasing and Decreasing Functions

1. A function $y = \mathrm{f}(x)$ is increasing on an interval $a < x < b$, if $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$.

2. A function $y = \mathrm{f}(x)$ is decreasing on an interval $a < x < b$ if $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$.

Rates of Change

1. If a variable x varies with time t, then $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ is the rate of change of x with respect to time.

2. If $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ is a constant, x changes at a constant rate.

3. If $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ is a function of t, x changes at a non-uniform rate. The value of $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ at $t = a$, is the instantaneous rate of change of x at the instance $t = a$.

4. If $\dfrac{\mathrm{d}x}{\mathrm{d}t} < 0$, then a decrease in magnitude of x is observed as the value of t increases.

5. If $\dfrac{\mathrm{d}x}{\mathrm{d}t} > 0$, then an increase in magnitude of x is observed as the value of t increases.

Connected Rates of Change

If $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ is the rate of change of x with respect to time t and $y = \mathrm{f}(x)$, then the rate of change of y with respect to t is given by $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}y}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}t}$.

Example

1. Given that $y = \dfrac{e^{x^2}}{x}$ for $x > 0$, find the range of values of x for which y is a decreasing function.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x(2xe^{x^2}) - e^{x^2}}{x^2}$$

$$= \frac{e^{x^2}(2x^2-1)}{x^2}$$

- Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$

Since $x^2 > 0,\ e^{x^2} > 0,\ \dfrac{\mathrm{d}y}{\mathrm{d}x} < 0,$

$$2x^2 - 1 < 0$$

$$\left(x + \frac{1}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}}\right) < 0$$

$$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

$$-\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$$

Since $x > 0,\ 0 < x < \dfrac{\sqrt{2}}{2}$

---

2. It is given that $y = 18 + px + qx^2 - x^3$ where p and q are integers. The only values of x for which y is a decreasing function of x are those values for which $x < \dfrac{7}{3}$ or $x > 3$. Find the value of p and of q.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p + 2qx - 3x^2$$

When y is a decreasing function, $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0,$ $\quad p + 2qx - 3x^2 < 0$

$$3x^2 - 2qx - p > 0$$

$x < \dfrac{7}{3}$ or $x > 3$ $\quad \Rightarrow \left(x - \dfrac{7}{3}\right)(x-3) > 0$

$$\Rightarrow (3x-7)(x-3) > 0$$

$$\Rightarrow 3x^2 - 16x + 21 > 0$$

Comparing coefficient of x, $\quad -2q = -16$

$$q = 8$$

Comparing constant term, $\quad p = 21$

3. A curve has the equation $y=(x-12)\sqrt{x+6}$.

(a) Express $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in the form $\dfrac{kx}{\sqrt{x+6}}$ where k is a constant.

(b) Find the value of x for which $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{3}{2}\dfrac{\mathrm{d}x}{\mathrm{d}t}$.

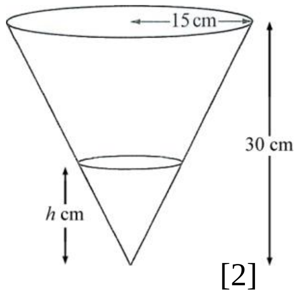
(a)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=(x+6)^{\frac{1}{2}}+\frac{1}{2}(x-12)(x+6)^{-\frac{1}{2}}\\&=(x+6)^{-\frac{1}{2}}\left[(x+6)+\frac{1}{2}(x-12)\right]\\&=\frac{(2x+12+x-12)}{2\sqrt{x+6}}\\&=\frac{3x}{2\sqrt{x+6}}\end{aligned}$$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\mathrm{d}y}{\mathrm{d}x}\times\dfrac{\mathrm{d}x}{\mathrm{d}t}$

When $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{3}{2}\dfrac{\mathrm{d}x}{\mathrm{d}t}$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{3}{2}\\\frac{3x}{2\sqrt{x+6}}&=\frac{3}{2}\\\frac{x}{\sqrt{x+6}}&=1\\x^2&=x+6\\x^2-x-6&=0\\(x-3)(x+2)&=0\\\therefore x&=3,\ -2\ (\text{NA})\end{aligned}$$

4. [The volume of a cone of height h and base radius R is given by $\frac{1}{3}\pi R^2 H$.]

The diagram shows a hollow conical tank of height 30 cm and radius 15 cm. The tank is held fixed with its circular rim horizontal. Water is then poured into the empty tank at a constant rate of $20\,\text{cm}^3 s^{-1}$.
After t seconds the depth of water is h cm.



(i) Show that the volume of water in the tank, V cm^3, at time t is given by $V = \frac{\pi h^3}{12}$. [2]

(ii) Find the rate of change of the depth when $h = 5$.

(iii) State, with a reason, whether this rate will increase or decrease as t increases.

Solution

(i) Let the radius of the water surface be r cm.

$$\frac{r}{15} = \frac{h}{30}$$

$$r = \frac{h}{2}$$

- Apply the properties of similar triangles to establish a relationship between r and h.

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$= \frac{\pi h^3}{12}$$

(ii) $\frac{dV}{dh} = \frac{1}{4}\pi h^2$

- Apply Chain Rule $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$20 = \frac{\pi h^2}{4} \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{80}{\pi h^2}$$

When $h = 5$, $\frac{dV}{dt} = 20$, $\quad \frac{dh}{dt} = \frac{80}{25\pi}$

$$= 1.02\,\text{cms}^{-1}$$

(iii) Since $\frac{dh}{dt} = \frac{80}{\pi h^2}$, h is inversely proportional to h^2, the rate will decrease as t increases.

- Apply the concept of inverse proportion.

Exercise

1. The equation of a curve is $y = x^3 + \frac{5}{2}x^2 - 2x + 1$. Find the values of x for which y is a decreasing function.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 5x - 2$$

y is a decreasing function , $\therefore \frac{\mathrm{d}y}{\mathrm{d}x} < 0.$

$$3x^2 + 5x - 2 < 0$$
$$(3x-1)(x+2) < 0$$
$$-2 < x < \frac{1}{3}$$

---

2. Find the range of values of x for which the curve $y = \frac{2x^2 - 3}{2x^2 + 3}$ is an increasing.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2x^2+3)(4x) - (2x^2-3)(4x)}{(2x^2+3)2}$$
$$= \frac{24x}{(2x^2+3)^2}$$

y is an increasing function , $\therefore \frac{\mathrm{d}y}{\mathrm{d}x} > 0.$

$$\frac{24x}{(2x^2+3)^2} > 0$$

Since $(2x^2+3)^2 > 0, \quad x > 0$

---

3. A curve has the equation $y = \frac{\ln x}{x^2}$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) Hence, find the range of values of x, such that y is increasing.

(i) $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2\left(\frac{1}{x}\right) - 2x\ln x}{x^4}$$
$$= \frac{1 - 2\ln x}{x^3}$$

(ii) y is an increasing function , $\therefore \frac{\mathrm{d}y}{\mathrm{d}x} > 0.$

Since $x > 0, \quad 1 - 2\ln x > 0$

$$\ln x < \frac{1}{2}$$
$$\therefore, < \sqrt{e}$$

4. The equation of a curve is $y = \ln(5-2x)$, where $x < \dfrac{5}{2}$.

(i) Find the coordinates of the point on the curve at which the normal to the curve is parallel to $2y = x+3$.

(ii) Show that as x increases, y is a decreasing function.

(i) $$\frac{dy}{dx} = -\frac{2}{5-2x} = \frac{2}{2x-5}$$

When $\dfrac{dy}{dx} = -2$, $$-2 = \frac{2}{2x-5}$$
$$4x - 10 = -2$$
$$4x = 8$$
$$x = 2$$
$$y = 0$$

The required point is (2, 0).

(ii) $$x < \frac{5}{2} \Rightarrow 2x - 5 < 0$$
$$\Rightarrow \frac{2}{2x-5} < 0$$

Hence, y is a decreasing function.

---

5. (i) Given that $y = (2x-1)\sqrt{4x+1}$. Obtain an expression for $\dfrac{dy}{dx}$.

(ii) The variables x and y are related by the equation $y = \dfrac{(2x-5)^6}{3}$. At the instant when $x = 3$, y is decreasing at the rate of 2 units per second. Find the rate of change of x.

(i) $$\frac{dy}{dx} = 2(4x+1)^{\frac{1}{2}} + (2x-1)\left(\frac{1}{2}\right)(4x+1)^{-\frac{1}{2}}(4)$$
$$= 2(4x+1)^{-\frac{1}{2}}(4x+1+2x-1)$$
$$= 12x(4x+1)^{-\frac{1}{2}} \text{ or } \frac{12x}{\sqrt{4x+1}}$$

(ii) $$\frac{dy}{dx} = \frac{6(2x-5)^5(2)}{3} = 4(2x-5)^5$$

When $x = 3$, $\dfrac{dy}{dt} = -2$, $$\frac{dx}{dt} = \frac{dy}{dt} \div \frac{dy}{dx}$$
$$= -2 \div 4$$
$$= -0.5 \text{ units per second}$$

6. A tub contains liquid with height x metres. The volume, V m^3, of the tub is given by

$$V = 0.05[(3x+2)^3 - 8].$$

If liquid is poured into the tub at a constant rate of 0.081 m^3 per second, find the rate at which the height of the liquid is increasing when $V = 0.95$.

$$\frac{dV}{dx} = 0.05(3)(3x+2)^2(3)$$
$$= 0.45(3x+2)^2$$

When $V = 0.95$, $0.95 = 0.05[(3x+2)^3 - 8]$

$$19 = (3x+2)^3 - 8$$
$$(3x+2)^3 = 27$$
$$3x+2 = 3$$
$$3x = 1$$
$$x = \frac{1}{3}$$

When $x = \frac{1}{3}$, $\frac{dV}{dt} = 0.081$, $\frac{dx}{dt} = \frac{dV}{dt} \div \frac{dV}{dx}$

$$= 0.081 \div (0.45 \times 9)$$
$$= 0.02 \text{ units per second}$$

---

7. A metal ball is heated to a temperature of $225\,°\mathrm{C}$ before being dropped into a liquid. As the ball cools, its temperature, $T\,°\mathrm{C}$, t minutes after it enters the liquid is given by

$$T = P + 190e^{-kt},$$

where P and k are constants.

(i) Explain why $P = 35$.

When $t = 4$, the temperature of the ball reaches $120\,°\mathrm{C}$.

(ii) Find the value of k correct to 3 significant figures.

(iii) Find the rate at which the temperature of the ball is decreasing at the instant when $t = 10$.

(iv) From the equation of T given above, explain why the temperature of the ball can never fall below $35\,°\mathrm{C}$.

(i) When $t = 0$, $T = 225$, $225 = P + 190$

$$P = 35$$

(ii) When $t = 4$, $T = 120$, $120 = 35 + 190e^{-4k}$

$$190e^{-4k} = 85$$
$$e^{-4k} = \frac{85}{190}$$
$$-4k = \ln\left(\frac{17}{38}\right)$$
$$k = -\frac{1}{4}\ln\left(\frac{17}{38}\right)$$
$$= 0.201$$

(iii) $\dfrac{dT}{dt} = -190\,ke^{-kt}$

When $t = 10$, $\dfrac{dT}{dt} = -190\left[-\dfrac{1}{4}\ln\left(\dfrac{17}{38}\right)\right]e^{-10\times\left[-\frac{1}{4}\ln\left(\frac{17}{38}\right)\right]}$

$= -5.11\,°\text{C min}^{-1}$

The temperature is decreasing at a rate of $5.11\,°\text{C min}^{-1}$.

(iv) Since $e^{-kt} > 0$, as $t \to \infty$, $e^{-kt} \to 0$

$35 + 190e^{-kt} \to 35$

$\therefore T \to 35$

---

8. The voltage V, in volts, of an electrical signal in an electrical system is given by the formula

$$V = 4\sin \pi t$$

where t is in seconds.

(i) Find the exact rate of change of voltage after $\dfrac{1}{4}$ seconds have elapsed.

(ii) Find the exact times when the rate of change of voltage is $2\pi\sqrt{3}$ volts per second for $0 < t < 4$.

(iii) Given that current (I in amperes) supplied to the system is governed by the equation $I = \dfrac{V}{5}$, find the rate of change of current when the rate of change of voltage is 2 volts per second.

(i) $\dfrac{dV}{dt} = 4\pi\cos \pi t$

When $t = \dfrac{1}{4}$, $\dfrac{dV}{dt} = 4\pi\cos\dfrac{\pi}{4}$

$= \dfrac{4\pi}{\sqrt{2}}$

$= 2\pi\sqrt{2}\ \text{Vs}^{-1}$

(ii) When $\dfrac{dV}{dt} = 2\pi\sqrt{3}$, $4\pi\cos \pi t = 2\pi\sqrt{3}$

$\cos \pi t = \dfrac{\sqrt{3}}{2}$

$\pi t = \dfrac{\pi}{6}, \dfrac{11\pi}{6}, \dfrac{13\pi}{6}, \dfrac{23\pi}{6}$

$t = \dfrac{1}{6}, \dfrac{11}{6}, \dfrac{13}{6}, \dfrac{23}{6}$

(iii) $\dfrac{dI}{dt} = \dfrac{1}{5}\dfrac{dV}{dt}$

When $\dfrac{dV}{dt} = 2$, $\dfrac{dI}{dt} = \dfrac{2}{5}\ \text{As}^{-1}$

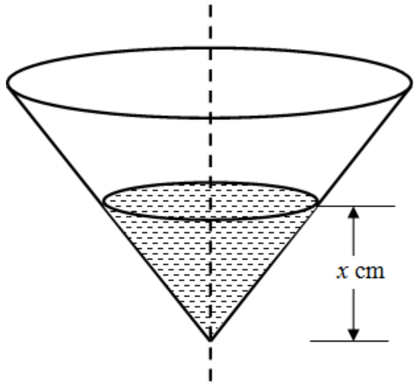
9. A container is in the shape of an inverted right circular cone which has a vertical axis and a base radius that is equal to its height. Water is poured into the vessel at a constant rate of $50\,\text{cm}^3\text{s}^{-1}$. The depth of the water is x cm.

(i) Show that the volume of water in the container is $V = \dfrac{1}{3}\pi x^3\,\text{cm}^3$.

Calculate, at the instant when depth of water is 20 cm, the rate of increase of

(ii) the depth of the water, in terms of π,

(iii) the area of the horizontal surface of the water.



(i) Let the base radius, height of the cone be R cm.
Let the radius of the water surface be r cm.

$$\frac{r}{R} = \frac{x}{R}$$
$$r = x$$
$$V = \frac{1}{3}\pi r^2 x$$
$$= \frac{1}{3}\pi x^3$$

(ii) $\dfrac{\mathrm{d}V}{\mathrm{d}x} = \pi x^2$

When $x = 20$, $\dfrac{\mathrm{d}V}{\mathrm{d}x} = \pi x^2$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}x}$$
$$= 50 \div 400\pi$$
$$= \frac{1}{8\pi}$$
$$\approx 0.0398\,\text{cms}^{-1}$$

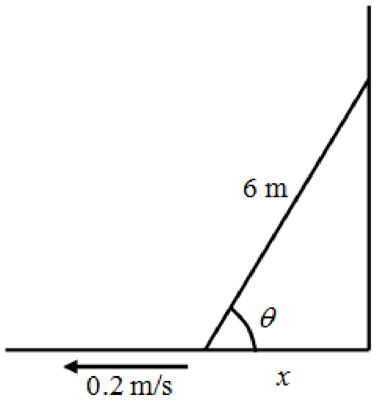
(iii) Let the area of the horizontal surface of the water be A cm^2.

$$A = \pi x^2$$
$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2\pi x$$

When $x = 20$, $\dfrac{\mathrm{d}A}{\mathrm{d}x} = \pi x^2$,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$= 2\pi(20) \times \frac{1}{8\pi}$$
$$= 5\,\text{cm}^2\text{s}^{-1}$$

10. In the diagram, a ladder, 6 m, is leaning against a vertical wall. The distance between the base of the ladder and the wall is x m. A force is exerted, pulling the base of the ladder away from the wall at a constant rate of 0.2 m/s.



(a) (i) Show that $x = 6\cos\theta$.

(ii) Find $\dfrac{d\theta}{dt}$ at the instant when $\theta = 0.367$ radians.

(b) If it took 20 seconds for the ladder to be flat on the floor, find

(i) the initial value, in radians, of θ,

(ii) the time taken for $\dfrac{d\theta}{dt}$ to reach a value of –0.0385.

(a) (i) $\dfrac{x}{6} = \cos\theta$

$x = 6\cos\theta$

(ii) $\dfrac{dx}{d\theta} = -6\sin\theta$

When $\theta = 0.367$, $\dfrac{d\theta}{dt} = \dfrac{dx}{dt} \div \dfrac{dx}{d\theta}$

$= 0.2 \div (-6\sin 0.367)$

$= -0.092897$

≈ -0.0929 rad per second

(b) (i) When $t = 0$, $x = 6 - 0.2 \times 20$

$= 2$

$\cos\theta = \dfrac{2}{6}$

$\theta = 1.23$

(ii) When $\dfrac{d\theta}{dt} = -0.0385$, $-0.0385 = \dfrac{0.2}{-6\sin\theta}$

$\sin\theta = \dfrac{0.2}{6(0.0385)}$

$\theta = 1.0467$

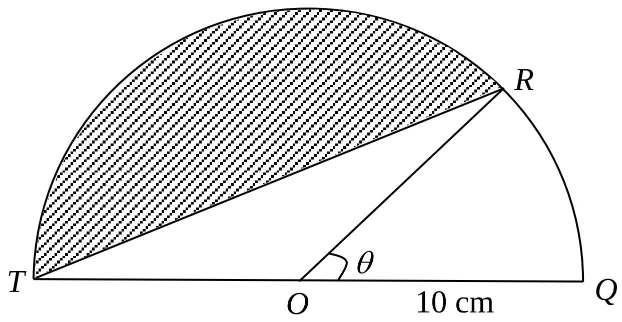
When $\theta = 0.52404$, $x = 6\cos 1.0467$

$= 3.0025$

Time taken $= \dfrac{3.0025 - 2}{0.2}$

$= 5.01$ s

11. The diagram shows a semi-circle with centre O and radius 10 cm. TOQ is a horizontal line. Beginning from Q, the point R moves along the semicircle at a rate of 0.132 radian per second.



Given that $\angle ROQ = \theta$ radians at any instant,

(i) Show that at any instant, the area of the shaded region, A, is given by

$$A = 50(\pi - \theta - \sin\theta).$$

(ii) Find the rate of change of A after 5 seconds.

(i) $A = \frac{1}{2} \times 10^2 \times [\pi - \theta - \sin(\pi - \theta)]$

$= 50(\pi - \theta - \sin\theta)$

(ii) $\frac{dA}{d\theta} = 50(-1 - \cos\theta)$

When $t = 5$, $\theta = 5(0.132)$

$= 0.66$

When $\theta = 0.66$, $\frac{d\theta}{dt} = 0.132$, $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= 50(-1 - \cos 0.66) \times 0.132$

$= -11.8\ \text{cm}^2\ \text{s}^{-1}$

Singapore Chinese Girls' School
Secondary 4
Additional Mathematics
Revision 4

Name: ______________________________ (   )   Date: ______________

Class: Sec 4______

Revision 4: Remainder and Factor Theorem

1. **The Remainder Theorem** states that if a polynomial $\mathrm{f}(x)$ is divided by $(x-a)$, the remainder $\mathrm{R} = \mathrm{f}(a)$.

2. **The Factor Theorem** states that if a polynomial $\mathrm{f}(x)$ is exactly divisible by $(x-a)$, the remainder $R = \mathrm{f}(a) = 0$.
Conversely, if $R = \mathrm{f}(a) = 0$, then $(x-a)$ is a factor of $\mathrm{f}(x)$.
Hence, $(x-a)$ is a factor of $\mathrm{f}(x) \Leftrightarrow \mathrm{f}(a) = 0$ and $\mathrm{f}(x)$ is divisible by $(x-a)$.

Example

1. The term containing the highest power of x in the polynomial $\mathrm{f}(x)$ is $2x^4$. Two of the roots of the equation $\mathrm{f}(x) = 0$ are -1 and 2. Given that $x^2 - 3x + 1$ is a quadratic factor of $\mathrm{f}(x)$, find
(i) an expression for $\mathrm{f}(x)$ in descending powers of x,
(ii) the number of real roots of the equation $\mathrm{f}(x) = 0$, justifying your answer,
(iii) the remainder when $\mathrm{f}(x)$ is divided by $2x - 1$. **2008**

(i) $(x+1)$ and $(x-2)$ are linear factors of $\mathrm{f}(x)$.

$$\begin{aligned}\mathrm{f}(x) &= 2(x+1)(x-2)(x^2-3x+1)\\ &= 2(x^2-x-2)(x^2-3x+1)\\ &= 2(x^4-3x^3+x^2-x^3+3x^2-x-2x^2+6x-2)\\ &= 2(x^4-4x^3+2x^2+5x-2)\\ &= 2x^4-8x^3+4x^2+10x-4\end{aligned}$$

- Do not let $\mathrm{f}(x)$ be represented by $2x^4 + ax^3 + bx^2 + cx + d$ and attempt to find the value of a, of b, of c and of d. You would need four equations to solve for the four unknowns.

(ii) Consider the equation $x^2 - 3x + 1$,
Discriminant = 5 > 0
$\mathrm{f}(x) = 0$ has four real roots.

(iii) $$\mathrm{f}\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 - 8\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^2 + 10\left(\frac{1}{2}\right) - 4$$
$$= 1\frac{1}{8}$$

- Apply Remainder Theorem Do not use Long or Synthetic Division

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2. When $f(x) = x^3 + kx^2 - 7x + 2$ is divided by $x - k$, the remainder is $2k$.
Find the possible values of k, leaving your answer to two decimal places where necessary.

$f(x) = 2k$

$k^3 + k^3 - 7k + 2 = 2k$

$2k^3 - 9k + 2 = 0$

Let $g(k) = 2k^3 - 9k + 2$

$g(2) = 16 - 18 + 2 = 0$

$(k-2)$ is a factor of $g(k)$.

Let $g(k) = (k-2)(2k^2 + bk - 1)$

Comparing coefficients, $-2b - 1 = -9$

$b = 4$

$\therefore g(k) = (k-2)(2k^2 + 4k - 1)$

$g(k) = 0$

$k = 2, \dfrac{-4 \pm \sqrt{24}}{2}$

$= 2, 0.22, \ 2.22$

1. Apply Remainder Theorem to form the required cubic equation.
2. Find the **first root using trial and error**.
3. Show the **method** for finding the corresponding **quadratic factor.**
4. Factorise **completely**.
5. **Equate g(k) to zero**, otherwise answer marks will not be awarded.

Exercise

1. (a) Solve the equation $2x^3 - 9x^2 + 3x + 4 = 0$.

(b) Find the value of p, of q and of r in the following identity

$$x^3 + 2x^2 + 5x - 10 = (x-1)(x+2)(x-p) + qx + r.$$

(a) Let $f(x) = 2x^3 - 9x^2 + 3x + 4$

$f(1) = 2 - 9 + 3 + 4 = 0$

$(x-1)$ is a factor of $f(x)$.

Let $f(x) = (x-1)(2x^2 + bx - 4)$

Comparing coefficient of x, $-b - 4 = 3$

$b = -7$

$f(x) = (x-1)(2x^2 - 7x - 4)$

$= (x-1)(2x+1)(x-4)$

$f(x) = 0 \Rightarrow x = -\dfrac{1}{2}, \ 1, \ 4$

(b) $x^3+2x^2+5x-10=(x-1)(x+2)(x-p)+qx+r$

Let $x=1$, $\quad 1+2+5-10=q+r$

$q+r=-2 \quad \ldots\ldots(1)$

Let $x=-2$, $\quad 8-8-10-10=-2q+r$

$-2q+r=-20 \quad \ldots\ldots(2)$

(1)–(2) $\quad 3q=18$

$q=6$

$r=-8$

Let $x=0$, $\quad -10=2p+r$

$-10=2p-8$

$p=-1$

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2. (a) (i) Given that $3x^3+x^2-4x+3=(Ax+B)(x-1)(x+2)+Cx-1$ for all real values of x, find the values of A, B and C.

(ii) Hence, deduce the remainder when $3x^3+x^2-4x+3$ is divided by x^2+x-2.

(i) Let $x=1$, $\quad 3+1-4+3=C-1$

$C=4$

Let $x=0$, $\quad 3=-2B-1$

$2B=-4$

$B=-2$

Comparing coefficient of x^3, $\quad A=3$

(ii) $3x^3+x^2-4x+3 = (3x-2)(x-1)(x+2)+4x-1$

$= (3x-2)(x^2+x-2)+4x-1$

Remainder = $4x-1$

(b) Solve the equation $2x^3+5x^2-28x-15=0$.

Let $f(x)=2x^3+5x^2-28x-15$

$f(-5)=-250+125+140-15=0$

$(x+5)$ is a factor of $f(x)$.

Let $f(x)=(x+5)(2x^2+bx-3)$

Comparing coefficient of x, $\quad 5b-3=-28$

$5b=-25$

$b=-5$

$f(x)=(x+5)(2x^2-5x-3)$

$=(x+5)(2x+1)(x-3)$

$f(x)=0 \Rightarrow x=-5,\ -\frac{1}{2},\ 3$

3. (a) The expression $2x^3 - 3x^2 + ax + 2$ is divisible by $2x - 1$ but leaves a remainder $2b$ when divided by $x + 2$. Find the value of a and of b. Hence factorise the expression completely.

Let $f(x) = 2x^3 - 3x^2 + ax + 2$

$f\left(\frac{1}{2}\right) = 0$

$2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + a\left(\frac{1}{2}\right) + 2 = 0$

$\frac{1}{4} - \frac{3}{4} + \frac{a}{2} + 2 = 0$

$\frac{a}{2} = -\frac{3}{2}$

$a = -3$

$f(-2) = 2b$

$2b = 2(-2)^3 - 3(-2)^2 - 3(-2) + 2$

$= -20$

$b = -10$

Let $2x^3 - 3x^2 - 3x + 2 = (2x-1)(x^2 + bx - 2)$

Comparing coefficient of x, $-b - 4 = -3$

$b = -1$

$f(x) = (2x-1)(x^2 - x - 2)$

$= (2x-1)(x-2)(x+1)$

(b) Given that $Ax^3 - 12x^2 + 2x + 5 = (2x+1)(2x-1)(x+B) + 3x + C$ for all values of x, determine the values of A, B and C. Hence, state the remainder when $Ax^3 - 12x^2 + 2x + 5$ is divided by $4x^2 - 1$.

Comparing coefficient of x^3, $A = 4$

Let $x = \frac{1}{2}$, $2\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) + 5 = 3\left(\frac{1}{2}\right) + C$

$C = 2$

Let $x = 0$, $5 = (1)(-1)(B) + 2$

$B = -3$

$\therefore 2x^3 - 12x^2 + 2x + 5 = (2x+1)(2x-1)(x-3) + 3x + 2$

Remainder = $3x + 2$

4. Find the values of a and b for which the function $\mathrm{f}(x) = 2x^4 - 7x^3 + ax^2 + bx - 21$ is exactly divisible by $x^2 - 2x - 3$. Hence determine, showing all necessary working, the number of real roots of the equation $\mathrm{f}(x) = 0$.

$x^2 - 2x - 3 = (x-3)(x+1)$
$x-3$ and $x+1$ are factors of $\mathrm{f}(x)$.

$\mathrm{f}(3) = 0$
$2(3)^4 - 7(3)^3 + a(3)^2 + b(3) - 21 = 0$
$-48 + 9a + 3b = 0$
$3a + b = 16 \qquad \ldots\ldots(1)$

$\mathrm{f}(-1) = 0$
$2(-1)^4 - 7(-1)^3 + a(-1)^2 + b(-1) - 21 = 0$
$-12 + a - b = 0$
$a - b = 12 \qquad \ldots\ldots(2)$

(1)+(2) $\quad 4a = 28$
$a = 7$
$b = -5$

Let $2x^4 - 7x^3 + 7x^2 - 5x - 21 = (x^2 - 2x - 3)(2x^2 + Bx + 7)$
Comparing coefficient of x, $\quad -5 = 14 - 3B$
$3B = 9$
$B = 3$

$\therefore 2x^4 - 7x^3 + 7x^2 - 5x - 21 = (x^2 - 2x - 3)(2x^2 + 3x + 7)$

$\mathrm{f}(x) = 0 \Rightarrow x^2 - 2x - 3 = 0 \quad$ or $\quad 2x^2 + 3x + 7 = 0$
Discriminant $= 3^2 - 4(2)(7)$
$= -33$
No real roots.
$\mathrm{f}(x) = 0$ has only two real roots.

5. The polynomial $ax^3 - 4x^2 - 5x + b$ is exactly divisible by $x - 2$ and has a remainder of –4 when divided by $x - 1$.

(i) Find the value of a and of b.

(ii) Hence, factorise $ax^3 - 4x^2 - 5x + b$ completely.

(i) Let $\mathrm{f}(x) = ax^3 - 4x^2 - 5x + b$

$\mathrm{f}(2) = 0$

$8a - 16 - 10 + b = 0$

$8a + b = 26 \quad \ldots\ldots(1)$

$\mathrm{f}(1) = 0$

$a - 4 - 5 + b = -4$

$a + b = 5 \quad \ldots\ldots(2)$

(1)–(2) $\quad 7a = 21$

$a = 3$

$b = 2$

(ii) Let $\mathrm{f}(x) = 3x^3 - 4x^2 - 5x + 2 = (x-2)(3x^2 + Bx - 1)$

Comparing coefficient of x, $\quad -5 = -2B - 1$

$2B = 4$

$B = 2$

$$\begin{aligned}\mathrm{f}(x) &= 3x^3 - 4x^2 - 5x + 2 \\ &= (x-2)(3x^2 + 2x - 1) \\ &= (x-2)(3x-1)(x+1)\end{aligned}$$

6. The expression $px^3 - 3qx^2 + qx + 2p$ is exactly divisible by $(2x+1)$ and has a remainder of –10 when divided by $(x+1)$.

(i) Show that the values of p and q are 2 and 3 respectively.

Hence, or otherwise,
(ii) factorise the above expression completely, and
(iii) determine the remainder when the expression above is divided by $(x-3)$.

(i) Let $\mathrm{f}(x) = px^3 - 3qx^2 + qx + 2p$

$$\mathrm{f}\left(-\frac{1}{2}\right) = 0$$

$$p\left(-\frac{1}{2}\right)^3 - 3q\left(-\frac{1}{2}\right)^2 + q\left(-\frac{1}{2}\right) + 2p = 0$$

$$-\frac{p}{8} - \frac{3q}{4} - \frac{q}{2} + 2p = 0$$

$$15p - 10q = 0$$

$$p = \frac{2q}{3} \qquad \ldots\ldots(1)$$

$$\mathrm{f}(-1) = -10$$

$$p(-1)^3 - 3q(-1)^2 + q(-1) + 2p = -10$$

$$-p - 3q - q + 2p = -10$$

$$p = 4q - 10 \quad \ldots\ldots(2)$$

Sub (1) into (2)

$$\frac{2q}{3} = 4q - 10$$

$$\frac{10q}{3} = 10$$

$$q = 3$$

$$p = 2$$

(ii) Let $\mathrm{f}(x) = 2x^3 - 9x^2 + 3x + 4 = (2x+1)(x^2 + Bx + 4)$
Comparing coefficient of x, $3 = B + 8$
$B = -5$

$$\mathrm{f}(x) = 2x^3 - 9x^2 + 3x + 4$$

$$= (2x+1)(x^2 - 5x + 4)$$

$$= (2x+1)(x-1)(x-4)$$

7. The cubic polynomial $f(x)$ is such that the coefficient of x^3 is 2 and the roots of $f(x)=0$ are -1, 2 and $3k$. It is given that $f(x)$ has a remainder of –36 when divided by $(x+4)$.
Find the value of k.

$f(x)=2(x+1)(x-2)(x-3k)$
$f(-4)=-36$
$2(-3)(-6)(-4-3k)=-36$
$-4-3k=-1$
$3k=-3$
$k=-1$

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8. The expressions x^3-2x^2-px+6 and $x^3+x^2+(8-p)x+10$ leave the same remainder when they are divided by $(x+a)$, where a is an integer.

(i) Let $f(x)=x^3-2x^2-px+6$ and $g(x)=x^3+x^2+(8-p)x+10$.
$f(-a)=g(-a)$
$a^3-2a^2-pa+6=a^3+a^2+(8-p)a+10$
$\therefore 3a^2+8a+4=0$

(ii) $3a^2+8a+4=0$
$(3a+2)(a+2)=0$
$a=-\frac{2}{3}$ (NA), -2

(iii) $f(-a)=-20$
$(-2)^3-2(-2)^2-p(-2)+6=-20$
$-8-8+2p+6=-20$
$2p=-10$
$p=-5$

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9. The expressions $f(x)=6x^3+ax^2+bx-6$ has a factor $x+2$ but leaves a remainder of –12 when divided by $x-1$.

(i) Find the value of a and of b.

(ii) Factorise $f(x)$ completely and hence solve the equation $48x^3+4ax^2=6-2bx$.

(i) Let $f(x)=6x^3+ax^2+bx-6$

$f(-2)=0$

$6(-2)^3+a(-2)^2+b(-2)-6=0$

$-48+4a-2b-6=0$

$4a-2b=54$

$b=2a-27 \quad \ldots\ldots(1)$

$f(1)=-12$

$6(1)^3+a(1)^2+b(1)-6=-12$

$a+b=-12 \quad \ldots\ldots(2)$

Sub (1) into (2) $\quad a+2a-27=-12$

$3a=15$

$a=5$

$b=-17$

(ii) Let $f(x)=6x^3+5x^2-17x-6=(x+2)(6x^2+Bx-3)$

Comparing coefficient of x, $\quad -17=2B-3$

$2B=-14$

$B=-7$

$f(x)=(x+2)(6x^2-7x-3)$

$\quad =(x+2)(3x+1)(2x-3)$

$48x^3+20x^2-34x-6=0$

$6(2x)^3+5(-2x)^2-17(2x)-6=0$

$(2x+2)(6x+1)(4x-3)=0$

$x=-1,-\frac{1}{6},\frac{3}{4}$

10. The function $\mathrm{f}(x)$ is defined by $\mathrm{f}(x) = 2x^3 - 2x^2 - 7x - 3$.

(i) Show that $(x+1)$ is the factor of $\mathrm{f}(x)$.

(ii) Solve $\mathrm{f}(x) = 0$. Leave your answers in exact form.

(i) $\mathrm{f}(x) \quad = 2(-1)^3 - 2(-1)^2 - 7(-1) - 3$

$= 0$

$\therefore$ $(x+1)$ is the factor of $\mathrm{f}(x)$.

(ii) Let $\mathrm{f}(x) = 2x^3 - 2x^2 - 7x - 3 = (x+1)(2x^2 + bx - 3)$

Comparing coefficient of x, $\quad -7 = b - 3$

$b = -4$

$\mathrm{f}(x) = (x+1)(2x^2 - 4x - 3)$

$\mathrm{f}(x) = 0 \quad \Rightarrow x + 5 = 0 \qquad$ or $\qquad 2x^2 - 4x - 3 = 0$

$\Rightarrow x = -5 \qquad x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(2)(-3)}}{4}$

$= \dfrac{4 \pm 4\sqrt{10}}{4}$

$= \dfrac{2 \pm \sqrt{10}}{2}$

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11. (a) The cubic polynomial is such that the coefficient of x^3 is –1 and the roots of the equation $\mathrm{f}(x) = 0$ are –2, 3 and k. Given that $\mathrm{f}(x)$ has a remainder of 12 when divided by $x - 4$, find the value of k.

$\mathrm{f}(x) = -(x+2)(x-3)(x-k)$

$\mathrm{f}(4) = 12$

$-(6)(1)(4-k) = 12$

$k - 4 = 2$

$k = 6$

(b) (i) Solve the equation $4x^3 - 7x^2 - 21x + 18 = 0$.

(ii) Hence solve the equation $\frac{1}{2}x^3 - \frac{7}{4}x^2 = \frac{21}{2}x - 18$.

(i) Let $f(x) = 4x^3 - 7x^2 - 21x + 18$

$f(-2) = 4(-2)^3 - 7(-2)^2 - 21(-2) + 18$

$= 0$

$(x+2)$ is a factor of $f(x)$.

Let $f(x) = 4x^3 - 7x^2 - 21x + 18 = (x+2)(4x^2 + bx + 9)$

Comparing coefficient of x, $-21 = 2b + 9$

$b = -15$

$$f(x) = 4x^3 - 7x^2 - 21x + 18$$
$$= (x+2)(4x^2 - 15x + 9)$$
$$= (x+2)(4x-3)(x-3)$$

$f(x) = 0 \Rightarrow x = -2,\ \frac{3}{4},\ 3$

(ii) $\frac{1}{2}x^3 - \frac{7}{4}x^2 = \frac{21}{2}x - 18$

$\frac{1}{2}x^3 - \frac{7}{4}x^2 - \frac{21}{2}x + 18 = 0$

$4\left(\frac{x}{2}\right)^3 - 7\left(\frac{x}{2}\right)^2 - 21\left(\frac{x}{2}\right) + 18 = 0$

$\left(\frac{x}{2} + 2\right)\left(\frac{4x}{2} - 3\right)\left(\frac{x}{2} - 3\right) = 0$

$x = -4,\ \frac{3}{2},\ 6$

12. Solve the equation $3x^3+2=8x^2-3x$. Hence or otherwise, find the values of y such that $3(y-1)^3-8(y-1)^2+3y-1=0$.

$3x^3+2=-3x$

$3x^3-8x^2+3x+2=0$

Let $\mathrm{f}(x)=3x^3-8x^2+3x+2$

$\mathrm{f}(1)=3-8+3+2$

$=0$

$(x-1)$ is a factor of $\mathrm{f}(x)$.

Let $\mathrm{f}(x)=3x^3-8x^2+3x+2=(x-1)(3x^2+bx-2)$

Comparing coefficient of x, $\quad 3=-b-2$

$b=-5$

$\mathrm{f}(x)=(x-1)(3x^2-5x-2)$

$=(x-1)(3x+1)(x-2)$

$\mathrm{f}(x)=0 \Rightarrow x=1,\ -\frac{1}{3},\ 2$

$3(y-1)^3-8(y-1)^2+3y-1=0$

$y-1=1,\ -\frac{1}{3},\ 2$

$y=2,\ \frac{2}{3},\ 3$

13. The expression $2x^3 + ax^2 + bx - 2$ is exactly divisible by $x - 1$ and by $x + 2$.

(i) Calculate the value of a and of b.
(ii) Find the third factor of the expression.
(iii) Hence solve the equation $2x^3 + ax^2 + bx - 2 = 0$.

(i) Let $f(x) = 2x^3 + ax^2 + bx - 2$.

$f(1) = 0$

$2 + a + b - 2 = 0$

$a = -b$ ......(1)

$f(-2) = 0$

$-16 + 4a - 2b - 2 = 0$

$4a - 2b = 18$

$2a - b = 9$ ......(2)

Sub (1) into (2) $3a = 9$

$a = 3$

$b = -3$

(ii) The third factor is $(2x + 1)$.

(iii) $2x^3 + 3x^2 - 3x - 2 = 0$

$(x - 1)(x + 2)(2x + 1) = 0$

$x = 1, \ -2, \ -\frac{1}{2}$

Alternative Method for (i)

(i) By inspection, the third factor is $(2x + 1)$.

$2x^3 + ax^2 + bx - 2$

$= (x - 1)(x + 2)(2x + 1)$

$= (x^2 + x - 2)(2x + 1)$

$= 2x^3 + 2x^2 - 4x + x^2 + x - 2$

$= 2x^3 + 3x^2 - 3x - 2$

$a = 3$

$b = -3$